

# FIXED POINT THEOREMS FOR $R$ -WEAKLY COMMUTING MAPPINGS ON NORMED VECTOR SPACES

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## Abstract

Common fixed points of  $R$ -weakly commuting mappings on a normed vector space are obtained. This fixed point is guaranteed under a generalized and natural contraction condition and the convergence of a certain inductively defined sequence of points in the normed vector space. The uniqueness of this common fixed point is also a notable feature. The off-shoot corollaries of the main result obtained here happen to be special cases which cover many results discovered by different authors.

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## 1. Introduction

Given any binary operation, one always tries to check whether it possesses some standard properties or not. Composition of mappings is a binary operation, which, in general, lacks the commutativity property. It is a welcome situation when two mappings can show this nice harmony amongst themselves by commuting with each other. This can be defined for two self-mappings on the most general structure, viz., a set.

**1.1. Definition.** (Commuting Mappings) Any two self-mappings  $A$  and  $B$  on a set  $X$  are said to be commuting at a point  $x \in X$  if, and only if,  $ABx = BAx$ .

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Though this property is essentially local in nature, a few pairs of mappings can commute at almost all points of the set.

A normed vector space is a richer structure as, in addition to the distance function, viz., metric induced by the norm, it is equipped with two binary operations and also can use scalars, which are elements from an outer structure, viz., the field, over which it is defined.

Of course, commutativity of any binary operation, as also of the composition of mappings, is always desirable but not guaranteed. So, in an effort to search for a generalized property which is close to commutativity for normed vector spaces,  $R$ -weakly commuting mappings were defined.

Thus if  $X$  is a normed vector space, then for two self-mappings  $A$  and  $B$  on  $X$  and at a point  $x \in X$ , we desire,

$$\|ABx - BAx\| \leq R \|Ax - Bx\|, \text{ for some } R > 0.$$

But then this condition immediately suggests that each pair of self-mappings on a normed vector space should satisfy it at every point except in two cases, viz., either probably at coincidence points of the mappings where at those points  $x$  (with  $Ax = Bx$ ), the right hand side of the inequality becomes zero and left hand side may not be zero, or at those points where either  $\|Ax - Bx\|$  is finite and  $\|ABx - BAx\|$  is not or where both are not finite. In all other cases, one can choose sufficiently large  $R$ , which would make  $\|ABx - BAx\|$  smaller than required  $R \|Ax - Bx\|$ . Being something trivial, it may seem to be an uninteresting condition. But if the positive real number  $R$ , which balances the inequality, is constrained to be common (and of course finite) at all the points, it remains trivial no more.

**1.2. Definition.** ( $R$ -Weakly Commuting Mappings) If  $X$  is a normed vector space, two self-mappings  $A$  and  $B$  on  $X$  are said to be  $R$ -weakly commuting if, and only if, there exists a real number  $R > 0$  such that  $\|ABx - BAx\| \leq R \|Ax - Bx\|$  for all  $x \in X$ .

The definitions make it clear that commuting mappings on a normed vector space are  $R$ -weakly commuting, but the converse need not hold.

## 2. $R$ -Weakly commuting mappings and their fixed points

Taking up pairs of  $R$ -weakly commuting mappings on normed vector space and a very general and natural contraction condition, under the requirement of the convergence of certain inductively defined sequences, we prove our main result for the existence and uniqueness of common fixed point of those mappings.

**2.1. Theorem.** Let  $X$  be a normed vector space,  $S, I$  and  $T, J$  two pairs of respectively  $R_1$ -weakly and  $R_2$ -weakly commuting mappings on  $X$ . Also let  $C$  be a closed, convex subset of  $X$  such that

$$(2.1) \quad I(C) \supseteq (1 - k)I(C) + kS(C),$$

$$(2.2) \quad J(C) \supseteq (1 - k')J(C) + k'T(C),$$

where  $0 < k, k' \leq 1$  and suppose that

$$(2.3) \quad \begin{aligned} \|Sx - Ty\|^p \leq \phi & \left( \max \{ \|Sx - Ix\|^p + \|Ix - Jy\|^p + \|Jy - Ty\|^p, \right. \\ & \|Sx - Ix\|^p + \|Ix - Ty\|^p, \\ & \left. \|Sx - Jy\|^p + \|Jy - Ty\|^p \} \right), \end{aligned}$$

for all  $x, y \in C$ , where  $p > 0$  and  $\phi$  is a function which is upper semi-continuous from the right such that  $\phi(t) < t$  for each  $t > 0$ . If for some  $x_0 \in C$ , the sequence  $\{x_n\}$  in  $X$  defined inductively for  $n = 0, 1, 2, 3, \dots$ , by

$$(2.4) \quad Ix_{2n+1} = (1 - a_{2n})Ix_{2n} + a_{2n}Sx_{2n},$$

$$(2.5) \quad Jx_{2n+2} = (1 - a_{2n+1})Jx_{2n+1} + a_{2n+1}Tx_{2n+1},$$

where  $0 < a_n \leq 1$  for all  $n \geq 0$  and  $\liminf a_n > 0$ , converges to a point  $z \in C$  and for this  $\{x_n\}$ , if  $Ix_n \rightarrow Iz$  and  $Jx_n \rightarrow Jz$  as  $n \rightarrow \infty$ , then  $S, I, J$  and  $T$  have the unique common fixed point  $Tz$  in  $C$ .

*Proof.* We prove the theorem in three steps.

In the first step, we prove that  $Iz = Jz = Sz = Tz = w$ . From (2.4) and (2.5), we have

$$\begin{aligned} a_{2n}Sx_{2n} &= Ix_{2n+1} - (1 - a_{2n})Ix_{2n}, \\ a_{2n+1}Tx_{2n+1} &= Jx_{2n+2} - (1 - a_{2n+1})Jx_{2n+1}. \end{aligned}$$

Since  $\lim Ix_n = Iz$ ,  $\lim Jx_n = Jz$  and  $0 < a_n \leq 1$  for all  $n$  with  $\liminf a_n > 0$ , then letting  $n \rightarrow \infty$ , in both the above equations, we get respectively,

$$\begin{aligned} \lim Sx_{2n} &= \lim Ix_{2n} = Iz, \\ \lim Tx_{2n+1} &= \lim Jx_{2n+1} = Jz. \end{aligned}$$

If possible, suppose that  $Iz \neq Jz$ . Then for large  $n$ ,  $Sx_{2n} \neq Tx_{2n+1}$  and so using (2.3) with  $x = x_{2n}$  and  $y = x_{2n+1}$ , we have

$$\begin{aligned} \|Sx_{2n} - Tx_{2n+1}\|^p &\leq \phi(\max\{\|Sx_{2n} - Ix_{2n}\|^p + \|Ix_{2n} - Jx_{2n+1}\|^p \\ &\quad + \|Jx_{2n+1} - Tx_{2n+1}\|^p, \\ &\quad \|Sx_{2n} - Ix_{2n}\|^p + \|Ix_{2n} - Tx_{2n+1}\|^p, \\ &\quad \|Sx_{2n} - Jx_{2n+1}\|^p + \|Jx_{2n+1} - Tx_{2n+1}\|^p\}). \end{aligned}$$

Letting  $n$  tend to infinity, we have

$$\begin{aligned} \|Iz - Jz\|^p &\leq \phi(\max\{\|Iz - Iz\|^p + \|Iz - Jz\|^p + \|Jz - Jz\|^p, \\ &\quad \|Iz - Iz\|^p + \|Iz - Jz\|^p, \\ &\quad \|Iz - Jz\|^p + \|Jz - Jz\|^p\}) \\ &= \phi(\max\{\|Iz - Jz\|^p, \|Iz - Jz\|^p, \|Iz - Jz\|^p\}) \\ &= \phi(\|Iz - Jz\|^p) \\ &< \|Iz - Jz\|^p, \end{aligned}$$

a contradiction and so  $Iz = Jz$ .

If possible, now suppose that  $Jz \neq Sz$ . Then for large  $n$ ,  $Tx_{2n+1} \neq Sz$  and so using (2.3) with  $x = z$  and  $y = x_{2n+1}$ ,

$$\begin{aligned} \|Sz - Tx_{2n+1}\|^p &\leq \phi(\max\{\|Sz - Iz\|^p + \|Iz - Jx_{2n+1}\|^p \\ &\quad + \|Jx_{2n+1} - Tx_{2n+1}\|^p, \\ &\quad \|Sz - Iz\|^p + \|Iz - Tx_{2n+1}\|^p, \\ &\quad \|Sz - Jx_{2n+1}\|^p + \|Jx_{2n+1} - Tx_{2n+1}\|^p\}). \end{aligned}$$

Letting  $n$  tend to infinity, we have

$$\begin{aligned} \|Sz - Jz\|^p &\leq \phi(\max\{\|Sz - Iz\|^p + \|Iz - Jz\|^p + \|Jz - Jz\|^p, \\ &\quad \|Sz - Iz\|^p + \|Iz - Jz\|^p, \\ &\quad \|Sz - Jz\|^p + \|Jz - Jz\|^p\}) \\ &= \phi(\max\{\|Sz - Iz\|^p, \|Sz - Iz\|^p, \|Sz - Jz\|^p\}) \\ &= \phi(\|Sz - Jz\|^p) \\ &< \|Sz - Jz\|^p, \end{aligned}$$

a contradiction and so  $Sz = Jz$ .

If possible, suppose that  $Iz \neq Tz$ . Then for large  $n$ ,  $Sx_{2n} \neq Tz$  and so using (2.3) with  $x = x_{2n}$  and  $y = z$ , we have

$$\begin{aligned} \|Sx_{2n} - Tz\|^p &\leq \phi(\max\{\|Sx_{2n} - Ix_{2n}\|^p + \|Ix_{2n} - Jz\|^p \\ &\quad + \|Jz - Tz\|^p, \\ &\quad \|Sx_{2n} - Ix_{2n}\|^p + \|Ix_{2n} - Tz\|^p, \\ &\quad \|Sx_{2n} - Jz\|^p + \|Jz - Tz\|^p\}). \end{aligned}$$

Letting  $n$  tend to infinity, we have

$$\begin{aligned} \|Iz - Tz\|^p &\leq \phi(\max\{\|Iz - Iz\|^p + \|Iz - Jz\|^p + \|Jz - Tz\|^p, \\ &\quad \|Iz - Iz\|^p + \|Iz - Tz\|^p, \\ &\quad \|Iz - Jz\|^p + \|Jz - Tz\|^p\}) \\ &= \phi(\max\{\|Jz - Tz\|^p, \|Iz - Tz\|^p, \|Jz - Tz\|^p\}) \\ &= \phi(\|Iz - Tz\|^p) \\ &< \|Iz - Tz\|^p, \end{aligned}$$

a contradiction and so  $Iz = Tz$ .

We have therefore proved that  $Iz = Jz = Sz = Tz = w$ , that is,  $z$  is a coincidence point of  $I, J, S$ , and  $T$ .

In the second step, we prove the existence of the common fixed point.

Since  $I$  and  $S$  are  $R_1$ -weakly commuting,

$$\|Iw - Sw\| = \|ISz - SIz\| \leq R_1 \|Sz - Iz\| = 0,$$

and so  $Sw = Iw$ .

Now suppose that  $Sw \neq w$ . Then using (2.3) with  $x = Sz$  and  $y = z$ , we have

$$\begin{aligned} \|Sw - w\|^p &= \|Sw - Tz\|^p \\ &\leq \phi(\max\{\|Sw - Iw\|^p + \|Iw - Jz\|^p + \|Jz - Tz\|^p, \\ &\quad \|Sw - Iw\|^p + \|Iw - Tz\|^p, \\ &\quad \|Sw - Jz\|^p + \|Jz - Tz\|^p\}) \\ &= \phi(\max\{\|Sw - Iw\|^p + \|Iw - w\|^p + \|w - w\|^p, \\ &\quad \|Sw - Iw\|^p + \|Iw - w\|^p, \\ &\quad \|Sw - w\|^p + \|w - w\|^p\}) \\ &= \phi(\|Sw - w\|^p) \\ &< \|Sw - w\|^p, \end{aligned}$$

a contradiction and so  $Sw = w = Iw$ , proving that  $w$  is a fixed point of  $S$  and  $I$ .

Similarly, since  $J$  and  $T$  are  $R_2$ -weakly commuting,

$$\|Jw - Tw\| = \|JTz - TJz\| \leq R_2 \|Tz - Jz\| = 0,$$

and so  $Tw = Jw$ .

Now suppose that  $Tw \neq w$ . Then using (2.3) with  $x = z$  and  $y = Tz$ , we have

$$\begin{aligned} \|Tw - w\|^p &= \|w - Tw\|^p = \|Sz - Tw\|^p \\ &\leq \phi(\max\{\|Sz - Iz\|^p + \|Iz - Jw\|^p + \|Jw - Tw\|^p, \\ &\quad \|Sz - Iz\|^p + \|Iz - Tw\|^p, \\ &\quad \|Sz - Jw\|^p + \|Jw - Tw\|^p\}) \\ &= \phi(\max\{\|w - w\|^p + \|w - Tw\|^p + \|Tw - Tw\|^p, \\ &\quad \|w - w\|^p + \|w - Tw\|^p, \\ &\quad \|w - Tw\|^p + \|Tw - Tw\|^p\}) \\ &= \phi(\|Tw - w\|^p) \\ &< \|Tw - w\|^p, \end{aligned}$$

a contradiction and so  $Tw = w = Jw$ , proving that  $w$  is a fixed point of  $T$  and  $J$ . Thus,  $w = Iz = Jz = Sz = Tz$  is a common fixed point of all the four mappings  $I, J, S$  and  $T$ .

In the third and final step, we establish the uniqueness of the common fixed point  $w$ . If possible, suppose that there exist two distinct common fixed points  $w$  and  $w'$ .

Using (2.3) with  $x = w$  and  $y = w'$ , we have

$$\begin{aligned} \|Sw - Tw'\|^p &\leq \phi(\max\{\|Sw - Iw\|^p + \|Iw - Jw'\|^p + \|Jw' - Tw'\|^p, \\ &\quad \|Sw - Iw\|^p + \|Iw - Tw'\|^p, \\ &\quad \|Sw - Jw'\|^p + \|Jw' - Tw'\|^p\}) \\ \|w - w'\|^p &\leq \phi(\max\{\|w - w\|^p + \|w - w'\|^p + \|w' - w'\|^p, \\ &\quad \|w - w\|^p + \|w - w'\|^p, \\ &\quad \|w - w'\|^p + \|w' - w'\|^p\}) \\ &= \phi(\max\{\|w - w'\|^p, \|w - w'\|^p, \|w - w'\|^p\}) \\ &= \phi(\|w - w'\|^p) \\ &< \|w - w'\|^p, \end{aligned}$$

a contradiction and so  $w = w'$ . The common fixed point is therefore unique, completing the proof of the theorem.  $\square$

### 3. Implied results as particular cases for two or fewer pairs

Theorem 2.1 is rich enough to give many other results as mere particular cases. We list an interesting collection of these and just specify how these are readily obtainable from it.

**3.1. Corollary.** ([4, Theorem 2.1]) *Let  $S, I$  and  $T, J$  be two pairs of mappings of a normed vector space  $X$  into itself which commute pairwise at their coincidence points, let  $C$  be a closed, convex subset of  $X$  such that*

$$(2.1) \quad I(C) \supseteq (1 - k)I(C) + kS(C),$$

$$(2.2) \quad J(C) \supseteq (1 - k')J(C) + k'T(C),$$

where  $0 < k, k' < 1$  and suppose that

$$(2.3) \quad \begin{aligned} \|Sx - Ty\|^p &\leq \phi(\max\{\|Sx - Ix\|^p + \|Ix - Jy\|^p + \|Jy - Ty\|^p, \\ &\|Sx - Ix\|^p + \|Ix - Ty\|^p, \\ &\|Sx - Jy\|^p + \|Jy - Ty\|^p\}), \end{aligned}$$

for all  $x, y \in C$ , where  $p > 0$  and  $\phi$  is a function which is upper semi-continuous from the right such that  $\phi(t) < t$  for each  $t > 0$ . If for some  $x_0 \in C$ , the sequence  $\{x_n\}$  in  $X$  defined inductively for  $n = 0, 1, 2, 3, \dots$ , by

$$(2.4) \quad Ix_{2n+1} = (1 - a_{2n})Ix_{2n} + a_{2n}Sx_{2n},$$

$$(2.5) \quad Jx_{2n+2} = (1 - a_{2n+1})Jx_{2n+1} + a_{2n+1}Tx_{2n+1},$$

where  $a_0 = 1, 0 < a_n \leq 1$  for all  $n > 0$  and  $\liminf a_n > 0$ , converges to a point  $z \in C$  and for this  $\{x_n\}$ , if  $Ix_n \rightarrow Iz$  and  $Jx_n \rightarrow Jz$  as  $n \rightarrow \infty$ , then  $S, I, J$  and  $T$  have the unique common fixed point  $Tz$  in  $C$ .

*Proof.* In the proof of Theorem 2.1, it is clear that the locations where we need to use the  $R$ -weakly commuting nature of the pairs of mappings  $S, I$  and  $T, J$  are their coincidence points, and the result is more naturally obtainable under the strong property of commutativity, when this happens to be a special case of it. Of course, the hypothesis of the corollary is not that much stronger either, as it does not demand the global commutativity of the concerned pairs of mappings, (like our Theorem 2.1 demands); it just demands those mappings to be commuting at coincidence points.

Again, of course, under the strong condition of the continuity of mappings  $I$  and  $J$ , in both Theorem 2.1 and Corollary 3.1, as well as in all the special cases following ahead, for the sequence  $\{x_n\}$  defined inductively by (2.4) and (2.5) converging to some point  $z$  in  $X$ , the requirements of  $Ix_n \rightarrow Iz$  and  $Jx_n \rightarrow Jz$  become redundant; but being weaker than continuity, we continue to state all the results under these conditions.  $\square$

Now, strengthening the contraction condition (2.3) gives the following.

**3.2. Corollary.** Let  $X$  be a normed vector space, let  $S, I$  and  $T, J$  be two pairs of respectively  $R_1$ -weakly and  $R_2$ -weakly commuting mappings on  $X$ , let  $C$  be a closed, convex subset of  $X$  such that

$$(2.1) \quad I(C) \supseteq (1 - k)I(C) + kS(C),$$

$$(2.2) \quad J(C) \supseteq (1 - k')J(C) + k'T(C),$$

where  $0 < k, k' \leq 1$  and suppose that

$$(3.1) \quad \begin{aligned} \|Sx - Ty\|^p &\leq \phi(\max\{\|Ix - Jy\|^p, \|Sx - Ix\|^p + \|Ty - Jy\|^p, \\ &\frac{1}{2}(\|Sx - Jy\|^p + \|Ix - Ty\|^p)\}), \end{aligned}$$

for all  $x, y \in C$ , where  $p > 0$  and  $\phi$  is a function which is upper semi-continuous from the right such that  $\phi(t) < t$  for each  $t > 0$ . If for some  $x_0 \in C$ , the sequence  $\{x_n\}$  in  $X$  defined inductively for  $n = 0, 1, 2, 3, \dots$ , by

$$(2.4) \quad Ix_{2n+1} = (1 - a_{2n})Ix_{2n} + a_{2n}Sx_{2n},$$

$$(2.5) \quad Jx_{2n+2} = (1 - a_{2n+1})Jx_{2n+1} + a_{2n+1}Tx_{2n+1},$$

where  $0 < a_n \leq 1$  for all  $n \geq 0$  and  $\liminf a_n > 0$ , converges to a point  $z \in C$  and for this  $\{x_n\}$ , if  $Ix_n \rightarrow Iz$  and  $Jx_n \rightarrow Jz$  as  $n \rightarrow \infty$ , then  $S, I, J$  and  $T$  have the unique common fixed point  $Tz$  in  $C$ .

*Proof.* For brevity, we let  $\|Ix - Jy\|^p = a$ ,  $\|Sx - Ix\|^p = b$ ,  $\|Ty - Jy\|^p = c$ ,  $\|Sx - Jy\|^p = d$ , and  $\|Ix - Ty\|^p = e$ . Since  $\|\cdot\| \geq 0$  and  $p > 0$ ;  $a, b, c, d, e \geq 0$ , it is clear that

$$\max \left\{ a, b + c, \frac{1}{2}[d + e] \right\} \leq \max \{ a + b + c, b + e, c + d \}.$$

So, the contraction condition (3.1) guarantees the natural one (2.3). Hence by Theorem 2.1,  $S$ ,  $I$ ,  $J$  and  $T$  have the unique common fixed point  $Tz$  in  $C$ . This completes the proof of the theorem.  $\square$

Corollary 3.2 has just narrowed down the contraction condition in Theorem 2.1 and it is obvious that the result is guaranteed under it. Then the reason behind stating it explicitly is that it is a generalization of the famous result [6, Theorem 1], in view of replacing commutativity by  $R$ -weak commutativity, and continuity by the local convergence condition for the images of the sequences  $Ix_n$  and  $Jx_n$ .

The particular cases that follow now are obtained by identifying some of the mappings amongst  $S, I, J$  and  $T$ .

**3.3. Corollary.** *Let  $X$  be a normed vector space, let  $S$  and  $I$  be a pair of  $R$ -weakly commuting mappings on  $X$ , let  $C$  be a closed, convex subset of  $X$  such that*

$$(2.1) \quad I(C) \supseteq (1 - k)I(C) + kS(C),$$

$$(3.2) \quad T(C) \supseteq (1 - k')T(C) + k'T(C),$$

where  $0 < k, k' \leq 1$  and suppose that

$$(3.3) \quad \|Sx - Ty\|^p \leq \phi(\|Sx - Ix\|^p + \|Ix - Ty\|^p),$$

for all  $x, y \in C$ , where  $p > 0$  and  $\phi$  is a function which is upper semi-continuous from the right such that  $\phi(t) < t$  for each  $t > 0$ . If for some  $x_0 \in C$ , the sequence  $\{x_n\}$  in  $X$  defined inductively for  $n = 0, 1, 2, 3, \dots$ , by

$$(2.4) \quad Ix_{2n+1} = (1 - a_{2n})Ix_{2n} + a_{2n}Sx_{2n},$$

$$(3.4) \quad Tx_{2n+2} = Tx_{2n+1},$$

where  $0 < a_n \leq 1$  for all  $n \geq 0$  and  $\liminf a_n > 0$ , converges to a point  $z \in C$  and for this  $\{x_n\}$ , if  $Ix_n \rightarrow Iz$  and  $Tx_n \rightarrow Tz$  as  $n \rightarrow \infty$ , then  $S$ ,  $I$ , and  $T$  have the unique common fixed point  $Tz$  in  $C$ .

*Proof.* Putting  $J = T$  in Theorem 2.1, the contraction condition (2.3) becomes

$$\begin{aligned} \|Sx - Ty\|^p &\leq \phi(\max \{ \|Sx - Ix\|^p + \|Ix - Ty\|^p + \|Ty - Ty\|^p, \\ &\quad \|Sx - Ix\|^p + \|Ix - Ty\|^p, \\ &\quad \|Sx - Ty\|^p + \|Ty - Ty\|^p \}) \\ &= \phi(\max \{ \|Sx - Ix\|^p + \|Ix - Ty\|^p \\ &\quad \|Sx - Ix\|^p + \|Ix - Ty\|^p, \\ &\quad \|Sx - Ty\|^p \}) \\ &= \phi(\max \{ \|Sx - Ix\|^p + \|Ix - Ty\|^p, \|Sx - Ty\|^p \}) \\ &= \phi(\max \{ \|Sx - Ix\|^p + \|Ix - Ty\|^p, \}) . \end{aligned}$$

In the step prior to the last one, the second term in the maximum is neglected due to the nature of  $\phi$ ;  $\|Sx - Ty\|^p \leq \phi(\|Sx - Ty\|^p)$  would mean that  $\|Sx - Ty\| = 0$ , under which again the last step holds. All the conditions in Theorem 2.1 are satisfied and hence  $S$ ,  $I$ , and  $T$  have the unique common fixed point  $Tz$  in  $C$ .  $\square$

This has generalized [4, Corollary 3.3]. Instead of equating  $J$  and  $T$ , if we opt for equating  $I$  and  $S$ , we get the same result;  $J$  being replaced by  $I$  and  $T$  by  $S$ .

**3.4. Corollary.** *Let  $X$  be a normed vector space, let  $I, J$  and  $T$  be mappings on  $X$  such that  $T, I$  and  $T, J$  are respectively  $R_1$ -weakly and  $R_2$ -weakly commuting, let  $C$  be a closed, convex subset of  $X$  such that*

$$(3.5) \quad I(C) \supseteq (1 - k)I(C) + kT(C),$$

$$(2.2) \quad J(C) \supseteq (1 - k')J(C) + k'T(C),$$

where  $0 < k, k' \leq 1$  and suppose that

$$(3.6) \quad \begin{aligned} \|Tx - Ty\|^p \leq \phi(\max\{\|Tx - Ix\|^p + \|Ix - Jy\|^p + \|Jy - Ty\|^p, \\ \|Tx - Ix\|^p + \|Ix - Ty\|^p, \\ \|Tx - Jy\|^p + \|Jy - Ty\|^p\}), \end{aligned}$$

for all  $x, y \in C$ , where  $p > 0$  and  $\phi$  is a function which is upper semi-continuous from the right such that  $\phi(t) < t$  for each  $t > 0$ . If for some  $x_0 \in C$ , the sequence  $\{x_n\}$  in  $X$  defined inductively for  $n = 0, 1, 2, 3, \dots$ , by

$$(3.7) \quad Ix_{2n+1} = (1 - a_{2n})Ix_{2n} + a_{2n}Tx_{2n},$$

$$(2.5) \quad Jx_{2n+2} = (1 - a_{2n+1})Jx_{2n+1} + a_{2n+1}Tx_{2n+1},$$

where  $0 < a_n \leq 1$  for all  $n \geq 0$  and  $\liminf a_n > 0$ , converges to a point  $z \in C$  and for this  $\{x_n\}$ , if  $Ix_n \rightarrow Iz$  and  $Jx_n \rightarrow Jz$  as  $n \rightarrow \infty$ , then  $I, J$  and  $T$  have the unique common fixed point  $Tz$  in  $C$ .

*Proof.* When  $S = T$ , Theorem 2.1 gives this result. □

**3.5. Corollary.** *Let  $X$  be a normed vector space, let  $S, I$  and  $T$  be mappings on  $X$  such that  $S, I$  and  $T, I$  are respectively  $R_1$ -weakly and  $R_2$ -weakly commuting, let  $C$  be a closed, convex subset of  $X$  such that*

$$(2.1) \quad I(C) \supseteq (1 - k)I(C) + kS(C),$$

$$(3.8) \quad I(C) \supseteq (1 - k')I(C) + k'T(C),$$

where  $0 < k, k' \leq 1$  and suppose that

$$(3.9) \quad \begin{aligned} \|Sx - Ty\|^p \leq \phi(\max\{\|Sx - Ix\|^p + \|Ix - Iy\|^p + \|Iy - Ty\|^p, \\ \|Sx - Ix\|^p + \|Ix - Ty\|^p, \\ \|Sx - Iy\|^p + \|Iy - Ty\|^p\}), \end{aligned}$$

for all  $x, y \in C$ , where  $p > 0$  and  $\phi$  is a function which is upper semi-continuous from the right such that  $\phi(t) < t$  for each  $t > 0$ . If for some  $x_0 \in C$ , the sequence  $\{x_n\}$  in  $X$  defined inductively for  $n = 0, 1, 2, 3, \dots$ , by

$$(2.4) \quad Ix_{2n+1} = (1 - a_{2n})Ix_{2n} + a_{2n}Sx_{2n},$$

$$(3.10) \quad Ix_{2n+2} = (1 - a_{2n+1})Ix_{2n+1} + a_{2n+1}Tx_{2n+1},$$

where  $0 < a_n \leq 1$  for all  $n \geq 0$  and  $\liminf a_n > 0$ , converges to a point  $z \in C$  and for this  $\{x_n\}$ , if  $Ix_n \rightarrow Iz$  as  $n \rightarrow \infty$ , then  $S, I$  and  $T$  have the unique common fixed point  $Tz$  in  $C$ .

*Proof.* When  $I = J$ , Theorem 2.1 gives this result. □

Now we combine both approaches of Corollary 3.4 and Corollary 3.5 to get the following result.

**3.6. Corollary.** *Let  $X$  be a normed vector space, let  $T$  and  $J$  be  $R$ -weakly commuting mappings on  $X$ , let  $C$  be a closed, convex subset of  $X$  such that*

$$(2.2) \quad J(C) \supseteq (1 - k')J(C) + k'T(C),$$

where  $0 < k' \leq 1$  and suppose that

$$(3.11) \quad \begin{aligned} \|Tx - Ty\|^p \leq \phi(\max \{ &\|Tx - Jx\|^p + \|Jx - Jy\|^p + \|Jy - Ty\|^p, \\ &\|Tx - Jx\|^p + \|Jx - Ty\|^p, \\ &\|Tx - Jy\|^p + \|Jy - Ty\|^p \}), \end{aligned}$$

for all  $x, y \in C$ , where  $p > 0$  and  $\phi$  is a function which is upper semi-continuous from the right such that  $\phi(t) < t$  for each  $t > 0$ . If for some  $x_0 \in C$ , the sequence  $\{x_n\}$  in  $X$  defined inductively for  $n = 0, 1, 2, 3, \dots$ , by

$$(3.12) \quad Jx_{n+1} = (1 - a_n)Jx_n + a_nTx_n,$$

where  $0 < a_n \leq 1$  for all  $n \geq 0$  and  $\liminf a_n > 0$ , converges to a point  $z \in C$  and for this  $\{x_n\}$ , if  $Jx_n \rightarrow Jz$  as  $n \rightarrow \infty$ , then  $J$  and  $T$  have the unique common fixed point  $Tz$  in  $C$ .

*Proof.* When  $S = T$  and  $I = J$ , Theorem 2.1 gives this result. □

This has additionally generalized [4, Corollary 3.5].

#### 4. Particular case for a sequence of maps

Most of the time, the approach of the earlier section was to reduce the number of mappings by equating them in various combinations. In the other way round, we can consider many pairs of maps simultaneously for common fixed point. In fact, a whole sequence of mappings can be considered for this purpose.

**4.1. Corollary.** *Let  $X$  be a normed vector space, let  $I, J$  and  $T_n$  be mappings of  $X$  into itself for  $n = 1, 2, 3, \dots$ . Further suppose that the pairs of mappings  $T_{2n-1}, I$  and  $T_{2n}, J$ ,  $R_{2n-1}$ -weakly and  $R_{2n}$ -weakly commute respectively and that  $C$  is a closed, convex subset of  $X$  such that*

$$(4.1) \quad I(C) \supseteq (1 - k)I(C) + kT_{2n-1}(C),$$

$$(4.2) \quad J(C) \supseteq (1 - k')J(C) + k'T_{2n}(C),$$

where  $0 < k, k' \leq 1$  and suppose that

$$(4.3) \quad \begin{aligned} \|T_ix - T_{i+1}y\|^p \leq \phi(\max \{ &\|T_ix - Ix\|^p + \|Ix - Jy\|^p + \|Jy - T_{i+1}y\|^p, \\ &\|T_ix - Ix\|^p + \|Ix - T_{i+1}y\|^p, \\ &\|T_ix - Jy\|^p + \|Jy - T_{i+1}y\|^p \}), \end{aligned}$$

for all  $x, y \in C$ , where  $p > 0$  and  $\phi$  is a function which is upper semi-continuous from the right such that  $\phi(t) < t$  for each  $t > 0$ . If for some  $x_0 \in C$ , the sequence  $\{x_n\}$  in  $X$  defined inductively for  $n = 0, 1, 2, 3, \dots$ , by

$$(4.4) \quad Ix_{2n+1} = (1 - a_{2n})Ix_{2n} + a_{2n}T_{2n-1}x_{2n},$$

$$(4.5) \quad Jx_{2n+2} = (1 - a_{2n+1})Jx_{2n+1} + a_{2n+1}T_{2n}x_{2n+1},$$

where  $0 < a_n \leq 1$  for all  $n \geq 0$  and  $\liminf a_n > 0$ , converges to a point  $z \in C$  and for this  $\{x_n\}$ , if  $Ix_n \rightarrow Iz$  and  $Jx_n \rightarrow Jz$  as  $n \rightarrow \infty$ , then  $S, I, J$  and  $T_n$  have the unique common fixed point  $Iz$  in  $C$ .

*Proof.* This is a direct application of Theorem 2.1. □

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