Hacettepe Journal of Mathematics and Statistics
Volume 41 (4) (2012), 477 – 485

HOLOMORPHIC SOLUTION FOR A CLASS OF KORTEWEG-DE VRIES EQUATIONS INVOLVING FRACTIONAL ORDER

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Received 20 : 10 : 2010 : Accepted 13 : 12 : 2011

Abstract

In this article, we consider some classes of Korteweg-de Vries equations of fractional order in a complex domain. The existence and uniqueness of holomorphic solution are established. We illustrate our theoretical result by examples.

Keywords: Fractional calculus, Fractional differential equation, Holomorphic solution, Unit disk, Riemann-Liouville operators, Nonlinear, Korteweg-de Vries equation (KdV), Tsunami, Forced (KdV).

2000 AMS Classification: 34 A 12.

1. Introduction

Fractional differential equations have emerged as a new branch of applied mathematics which has been used for many mathematical models in science and engineering. In fact, fractional differential equations are considered as an alternative model to nonlinear differential equations. Various types play important roles and are tools not only in mathematics but also in physics, control systems, dynamical systems and engineering in creating mathematical models of many physical phenomena. Naturally, such equations need to be solved. Many studies on fractional calculus and fractional differential equations, involving different operators such as Riemann-Liouville operators, ErdÚlyi-Kober operators, Weyl-Riesz operators, Caputo operators and Grünwald-Letnikov operators, have appeared during the past three decades with applications in other fields [1, 2, 4, 11, 12, 15]. Recently, the existence of analytic solutions for fractional differential equations in a complex domain are posed [5, 6, 7, 8].

The study of nonlinear problems is of crucial importance in all areas of mathematics, mechanics and physics. Some of the most interesting features of physical systems are hidden in their nonlinear behavior, and can only be studied with appropriate methods

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designed to tackle and process nonlinear problems. One of the most important nonlinear problem is the Kortewegûde Vries equation, it is used in many sections of nonlinear mechanics and physics. Recently a numerical method has been proposed for solving the KdVB equation. Zaki has used the collocation method with quintic Bûspline finite elements [16], Soliman has employed the collocation solution of the KdV equation using septic splines as the elemental shape function [13], Kaya has implemented the Adomian decomposition method for solving the KdVB equation [10] and Jafari and Firoozjaee have developed a homotopy approach [9] for the numerical study of the Korteweg-de Vries (KdV) and the Korteweg-de Vries Burgers (KdVB) equations with initial conditions.

It is well known that the solution to the Cauchy problem of the KdV equation with an analytic initial profile is analytic in the space variable for a fixed time. However, analyticity in the time variable fails (see [3]). The present paper deals with a nonlinear fractional differential equation, in the sense of the Srivastava-Owa operators (see [14])

1.1. Definition. The fractional derivative of order α is defined, for a function $f(z)$ by

$$D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha}d\zeta;\ 0 \le \alpha < 1,$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

1.2. Definition. The fractional integral of order α is defined, for a function $f(z)$, by

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)}\int_0^z f(\zeta)(z-\zeta)^{\alpha-1}d\zeta;\ \alpha > 0,$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z-plane ($\mathbb{C}$) containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

1.3. Remark. From Definition 1.1 and Definition 1.2, we have

$$D_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}z^{\mu-\alpha},\ \mu > -1;\ 0 < \alpha < 1$$

and

$$I_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}z^{\mu+\alpha},\ \mu > -1;\ \alpha > 0.$$

In the present work we consider a fractional differential equation which takes the form

$$A(z)u_{zzz} + B(z)u_{zz} + C(z)uD_z^\alpha u + u_t = 0, \tag{1}$$

subject to the initial condition $u(t,0) = 0$, where $t \in J := [0,1]$, $z \in U := \{z \in \mathbb{C} : |z| < 1\}$, $u(t,z)$ is an unknown function and

$$\begin{aligned} &A(z) := z^3 a(z),\ a(z) \neq 0,\ z \in U \\ &B(z) := z^2 b(z),\ z \in U \\ &C(z) := z^\alpha c(z),\ c(z) \neq 0,\ z \in U. \end{aligned}$$

2. Existence of a unique solution

In this section, we pose the existence and uniqueness of a holomorphic solution for problem (1).

2.1. Definition. The *majorant relations* are described as: if $a(x) = \sum a_i x^i$ and $A(x) = \sum A_i x^i$, then we say that $a(x) \ll A(x)$ if and only if $|a_i| \le A_i$ for each i.

We have the following result:

2.2. Theorem. *Assume the problem (1). If*

$$\Re(\beta_k(z)) := k(k-1)\Re\Big\{(k-2)a(z) + b(z)\Big\} > 0,\ \ k \in \mathbb{N}^* \tag{2}$$

then the equation (1) has a unique holomorphic solution $u(t, z)$ *near* $(0, 0) \in J \times U$.

Proof. We realize that equation (1) has a formal solution

$$u(t, z) = \sum_{k=1}^{\infty} u_k(t) z^k,\ \ (z \in U). \tag{3}$$

Then we substitute the series (3) into the equation (1) and compare the coefficients of z^k in two sides of the equation to yield

$$\begin{aligned}
&\frac{u_0^2 c(z)}{\Gamma(1-\alpha)} + u_0' = 0 \\
&u_0 u_1 \Big(\frac{\Gamma(1)}{\Gamma(1-\alpha)} + \frac{\Gamma(2)}{\Gamma(2-\alpha)}\Big) c(z) + u_1' = 0 \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
&n(n-1)[(n-2)a(z) + b(z)]u_n + u_n' = -c(z) \sum_{j=0}^{n} \frac{\Gamma(n-j+1)}{\Gamma(n-j+1-\alpha)} u_j u_{n-j} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad := \phi_n(t, z, u).
\end{aligned} \tag{4}$$

Thus we obtain the following formula

$$u_n'(t) + n(n-1)[(n-2)a(z) + b(z)]u_n(t) = \phi_n(t, z, u). \tag{5}$$

Then by the assumption (2), the equation (5) has a unique holomorphic solution $u_k(t)$ near $t = 0$. Moreover, $u_k(t)$ is bounded for all $k \in \mathbb{N}$ such that

$$\|u_k\| \le \frac{\|\phi\|}{\|\beta_k\|}, \tag{6}$$

where $\|\phi\| = \max_{t\in J} |\phi(\cdot)|$ and

$$\beta_k(z) := k(k-1)[(k-2)a(z) + b(z)],\ \ k \in \mathbb{N}^*.$$

To prove inequality (6); From equation (5) we have

$$\frac{d}{dt}\Big[e^{\int_0^t \beta_k(s)ds} \times u_k(t)\Big] = e^{\int_0^t \beta_k(s)\,ds} \times \phi(t).$$

Thus

$$\int_0^t \frac{d}{dy}\Big[e^{\int_0^y \beta_k(s)ds} \times u_k(y)\Big]\,dy = \int_0^t e^{\int_0^y \beta_k(s)\,ds} \times \phi(y)\,dy,$$

that is

$$e^{\int_0^t \beta_k(s)ds} \times u_k(t) - u_k(0) = \int_0^t e^{\int_0^y \beta_k(s)\,ds} \times \phi(y)\,dy$$

which is equivalent to

$$u_k(t) = e^{-\int_0^t \beta_k(s)ds} \times \int_0^t e^{\int_0^y \beta_k(s)ds} \times \phi(y)\,dy.$$

Therefore,

$$\|u_k\| \le \max_{t\in J} \frac{\left|e^{-\int_0^t \beta_k(s)ds} \times \int_0^t e^{\int_0^y \beta_k(s)\,ds} \times \beta_k(y)\,dy\right|}{|\beta_k(t)|} \times \|\phi\|$$

$$\le \frac{\|\phi\|}{\|\beta_k\|}.$$

Now we proceed to prove that the formal series solution (3) is convergent near $(0,0) \in (J,U)$. We expand ϕ into its Taylor series with respect to z, u, i.e.

$$\phi(t,z,u) = \sum_{m+p\ge 2}^{\infty} a_m(t) z^m u^p$$

such that

(i) $a_{m,p}(t)$ is holomorphic in J.
(ii) $|a_{m,p}(t)| \le A_{m,p},\ A_{m,p} > 0$ on J.
(iii) $\sum_{m,p\ge 2}^{\infty} A_{m,p} V^{m+p}$ converges in (t,V), where $V > 0$ satisfies $|u| \le V$ and $|z| \le V$.

From the equation (5), we observe that

$$\begin{aligned}
&\Big[\frac{d}{dt} + \beta_1\Big] u_1(t) \\
&\qquad = \phi_1, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\Big[\frac{d}{dt} + \beta_k\Big] u_k(z) \\
&\qquad = -\sum_{m+p\ge 2}\Big[\sum_{k_1+\cdots+k_m+l_1+\cdots+l_p=k} a_{m,p} \times z_{k_1} \times \cdots \times z_{k_m} \times u_{l_1} \times \cdots \times u_{l_p}\Big].
\end{aligned} \tag{7}$$

Without loss of generality we may assume that there exists a constant $0 < K < 1$ such that

$$|u_1(t)| \le K \text{ and } |z| \le K.$$

Setting $C := \frac{1}{\|\beta_k\|}$, then we pose the following formula:

$$V(z) = Kz + \frac{C}{1-r} \sum_{m+p\ge 2} \frac{A_{m,p}}{(1-r)^{m+p-2}} V^{m+p}, \tag{8}$$

where r is a parameter with $0 < r < 1$. Since equation (9) is an analytic functional equation in V then, in view of the implicit function theorem, equation (9) has a unique holomorphic solution $V(z)$ in a neighborhood of $z = 0$ with $V(0) = 0$. Expanding $V(z)$ into its Taylor series in z we have

$$V(z) = \sum_{k\ge 1} V_k z^k, \tag{9}$$

where

$$\begin{aligned} V_k &= \frac{C}{1-r}\sum_{m+p\geq 2}\Bigg[\sum_{k_1+\cdots+k_m+l_1+\cdots+l_p=k}\frac{A_{m,p}}{(1-r)^{m+p-2}} \\ &\qquad\qquad \times V_{k_1}\times\cdots\times V_{k_m}\times V_{l_1}\times\cdots\times V_{l_p}\Bigg] \\ &:= \frac{C_k}{(1-r)^{k-1}},\ k\in\mathbb{N} \\ &> 0, \end{aligned} \tag{10}$$

with $C_1 = K$.

Next our aim is to show that the series $\sum_{k\geq 1} V_k z^k$ is a majorant series for the formal series solution $\sum_{k\geq 1} u_k(t)z^k$ near $z=0$. For this purpose we will show that

$$|u_k(t)| \leq V_k \text{ on } J. \tag{11}$$

Since $(1-r) < 1$ implies

$$\frac{1}{(1-r)^{m+p-2}} \geq 1,\ r<1$$

then we have

$$\begin{aligned} |u_k(t)| &\leq C\sum_{m+p\geq 2}\Bigg[\sum_{k_1+\cdots+k_m+l_1+\cdots+l_p=k} A_{m,p}\times|u_{l_1}(t)|\times\cdots\times|u_{l_p}(t)|\Bigg] \\ &\leq C\sum_{m+p\geq 2}\Bigg[\sum_{k_1+cdots+k_m+l_1+\cdots+l_p=k} A_{m,p} \\ &\qquad\qquad \times V_{k_1}\times\cdots\times V_{k_m}\times V_{l_1}\times\cdots\times V_{l_p}\Bigg] \\ &\leq C\sum_{m+p\geq 2}\Bigg[\sum_{k_1+\cdots+k_m+l_1+\cdots+l_p=k}\frac{A_{m,p}}{(1-r)^{m+p-2}} \\ &\qquad\qquad \times V_{k_1}\times\cdots\times V_{k_m}\times V_{l_1}\times\cdots\times V_{l_p}\Bigg] \\ &\leq \frac{C_k}{(1-r)^{k-2}} \leq \frac{C_k}{(1-r)^{k-1}} = V_k. \end{aligned}$$

Hence we obtain the inequality (11). This completes the proof of Theorem 2.2. □

2.3. Example. Assume the following equation

$$\begin{cases} u_t + \frac{u(t,z)z^{0.5}D_z^{0.5}u(t,z)}{1.8} + u_{zzz} = 0, & z\in U, \\ u(t,0) = 0, & \text{in an interval of } t=0, \end{cases} \tag{12}$$

where $u(t,z)$ is the unknown function. By putting

$$u(t,z) = \mu(t)z^3 + v(t,z)\ (v(t,z) = O(z^4))$$

as a formal solution we see that $\mu(z)$ satisfies

$$z^6\mu(t)^2 + 6\mu(t) + z^3\mu'(t) = g(t,z),\ g(0,0)=0.$$

Now by assuming

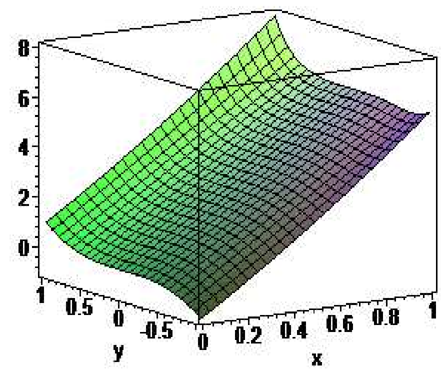
$$\mu(t) := q + \psi(t),$$

where q is a constant and $\psi(t) = O(t)$ we obtain that $q = 0$. Hence we impose the following equation:

$$(13) \qquad \begin{cases} z^3\psi'(t) + 6\psi(t) + z^6\psi^2(t) = g(t,z), \\ \psi(0) = 0, \end{cases}$$

where the holomorphic solution $\psi(t)$ exists uniquely and converges in a neighborhood of the origin (Theorem 2.2).

Figure 1. Solution of problem (13)



In the same manner as Example 2.3, we pose the following example

2.4. Example. Assume the following equation

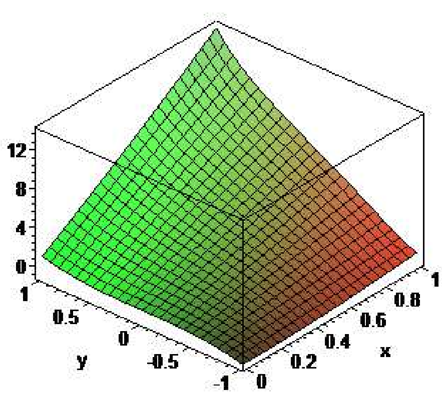
$$(14) \qquad \begin{cases} u_t + \frac{u(t,z)z^{0.5}D_z^{0.5}u(t,z)}{1.8} + u_{zz} + u_{zzz} = 0, & z \in U, \\ u(t,0) = 0, & \text{in an interval of } t = 0, \end{cases}$$

where $u(t,z)$ is the unknown function. Thus yields the following equation:

$$(15) \qquad \begin{cases} z^3\psi'(t) + 6(1+z)\psi(t) + z^6\psi^2(t) = h(t,z), \\ \psi(0) = 0, \end{cases}$$

where $h(0,0) = 0$. Then the holomorphic solution $\psi(t)$ exists uniquely and converges in a neighborhood of the origin (Theorem 2.2).

Figure 2. Solution of problem (15)



3. Holomorphic solutions to forced KdV equation of fractional order

The Korteweg-de Vries equation with a forcing term is provided by recent studies as a mathematical model describing the physics of a shallow layer of fluid subject to external forcing. Also, the basic hydrodynamic model of tsunami generation by atmospheric disturbances is based on the well-known Korteweg-de Vries equation with a forcing term (see [17]). In this section, we establish the existence and uniqueness of a holomorphic solution for the forced (KdV) equation containing fractional derivative

$$a(z)u_{zzz} + b(z)u_z + c(z)uD_z^\alpha u + u_t = f_z, \tag{16}$$

where u refers to the elevation of the free water surface with $u(t,0) = 0, a(z) = az^3, b(z) = bz, c(z) = cz^\alpha$ are complex valued functions for $z \in U$ and f represents to the force on the solid bottom (t,z) which is holomorphic in $(J \times U)$ then it has the following series:

$$f(t,z) = \sum_{i,j\geq 1}^{\infty} f_{i,j}t^i z^j, \;\; f(0,0) = 0.$$

We have the following result:

3.1. Theorem. *Consider problem* (16)*. If*

$$\gamma_k := k[(k-1)(k-2)a + b] \neq 0, \;\; k \in \mathbb{N}, \tag{17}$$

then equation (16) has a unique holomorphic solution $u(t,z)$ *near* $(0,0) \in J \times U$.

Proof. Equation (16) has a formal solution

$$u(t,z) = \sum_{k=1}^{\infty} u_k(t)z^k, \;\; (z \in U). \tag{18}$$

Then substituting the series (18) into equation (16) and comparing the coefficients of z^k on both sides of the equation yields

$$\begin{aligned} &bu_1 + u_0u_1\left(\frac{\Gamma(1)}{\Gamma(1-\alpha)} + \frac{\Gamma(2)}{\Gamma(2-\alpha)}\right)c(z) + u_1' = \sum_{i=1}^{\infty} f_{i,1}t^i \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &k[(k-1)(k-2)a + b]u_k + u_k' = \sum_{i=1}^{\infty} kf_{i,k}t^i - c\sum_{q=0}^{k} \frac{\Gamma(k-q+1)}{\Gamma(k-q+1-\alpha)}u_q u_{k-q} \\ &\qquad := \Phi(t,z,u). \end{aligned} \tag{19}$$

Thus we obtain the following formula:

$$k[(k-1)(k-2)a + b]u_k(t) + u_k'(t) = \Phi(t,z,u). \tag{20}$$

Then by the assumption (17), equation (20) has a unique holomorphic solution $u_k(t)$ near $t = 0$. In the same manner as for Theorem 2.2, we can complete the Proof of Theorem 3.1. $\square$

3.2. Example. Assume the following equation

$$\begin{cases} u_t + \frac{u(t,z)z^{0.5}D_z^{0.5}u(t,z)}{1.8} + zu_z + z^3u_{zzz} = z^6\text{sech}(t), & z \in U, \\ u(t,0) = 0, & \text{in an interval of } t = 0, \end{cases} \tag{21}$$

where $u(t,z)$ is the unknown function. By putting

$$u(t,z) = z + \mu(t)z^3 + v(t,z) \;\; (v(t,z) = O(z^4))$$

as a formal solution we find that $\mu(z)$ satisfies

$$z^6\mu(t)^2 + z + 9z^3\mu(t) + z^3\mu'(t) - z^6\mathrm{sech}(t) = g(t,z).$$

Now by assuming

$$\mu(t) := q + \psi(t),$$

where q is a constant and $\psi(t) = O(t)$, we obtain that $q = \pm 1$. Hence we impose the following equations:

$$\begin{cases} z^3\psi'(t) + z^6(1 - 2\psi(t) + \psi^2(t)) + 9z^3(\psi(t) - 1) - z^6\mathrm{sech}(t) = g(t,z), \quad q = -1 \\ \psi(0) = 0, \end{cases} \tag{22}$$

$$\begin{cases} z^3\psi'(t) + z^6(1 + 2\psi(t) + \psi^2(t)) + 9z^3(1 + \psi(t)) - z^6\mathrm{sech}(t) = g(t,z), \quad q = 1 \\ \psi(0) = 0, \end{cases} \tag{23}$$

where the holomorphic solution $\psi(t)$ exists uniquely and converges in a neighborhood of the origin (Theorem 3.1).

Figure 3. Solution of problem (22)

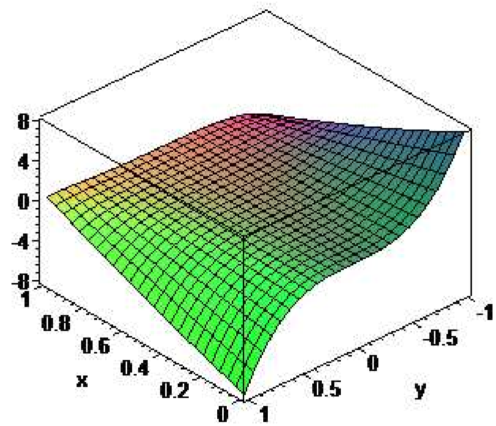
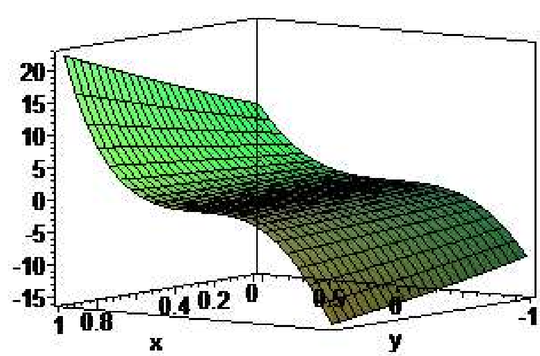


Figure 4. Solution of problem (23)



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