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REMARKS ON INEQUALITIES FOR THE TANGENT FUNCTION

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Abstract

In the paper, the authors analyze and compare two double inequalities for bounding the tangent function, reorganize the proof in C.-P. Chen and F. Qi (*A double inequality for remainder of power series of tangent function*, Tamkang J. Math. **34** (4), 351–355, 2003) by using the usual definition of Bernoulli numbers, and correct some errors on page 6, (1.29) and (1.30) of F. Qi, D.-W. Niu, and B.-N. Guo (*Refinements, generalizations, and applications of Jordan's inequality and related problems*, J. Inequal. Appl. **2009** (2009), Article ID 271923, 52 pages, 2009). Moreover, the authors propose a sharp double inequality as a conjecture.

Keywords: Inequality, Tangent function, Comparison, Bernoulli number, Induction, Reorganization, Conjecture.

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1. Introduction

Usually Bernoulli numbers B_i may be defined by

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}\frac{B_i}{i!}x^i=1-\frac{x}{2}+\sum_{j=1}^{\infty}B_{2j}\frac{x^{2j}}{(2j)!},\quad |x|<2\pi. \tag{1.1}$$

In [2, p. 16 and p. 56], it is listed that for $q\ge 1$

$$\zeta(2q)=(-1)^{q-1}\frac{(2\pi)^{2q}}{(2q)!}\frac{B_{2q}}{2}, \tag{1.2}$$

where ζ is the Riemann zeta function defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}. \tag{1.3}$$

From (1.2), it follows that

$$(-1)^{n-1}B_{2n}=|B_{2n}|. \tag{1.4}$$

The tangent function $\tan x$ and cotangent function $\cot x$ can be expanded into power series with coefficients involving Bernoulli numbers respectively as

$$\tan x=\sum_{i=1}^{\infty}\frac{2^{2i}\left(2^{2i}-1\right)|B_{2i}|}{(2i)!}x^{2i-1} \tag{1.5}$$

for $|x|<\frac{\pi}{2}$ and

$$\cot x=\frac{1}{x}-\sum_{i=1}^{\infty}\frac{2^{2i}|B_{2i}|}{(2i)!}x^{2i-1} \tag{1.6}$$

for $|x|<\pi$. See [1, p. 75, 4.3.67 and 4.3.70].

Let $S_n(x)$ denote

$$S_n(x)=\sum_{i=1}^{n}\frac{2^{2i}\left(2^{2i}-1\right)|B_{2i}|}{(2i)!}x^{2i-1} \tag{1.7}$$

for $0<x<\frac{\pi}{2}$. In [4], a double inequality for the difference $\tan x-S_n(x)$ on $\left(0,\frac{\pi}{2}\right)$ was established by using induction and an alternative definition of Bernoulli numbers different from (1.1). This result may be reformulated as the following theorem.

1.1. Theorem. *For $x\in(0,\frac{\pi}{2})$ and $n\in\mathbb{N}$, we have*

$$\frac{2^{2(n+1)}\left[2^{2(n+1)}-1\right]|B_{2(n+1)}|}{(2n+2)!}<\frac{\tan x-S_n(x)}{x^{2n}\tan x}<\left(\frac{2}{\pi}\right)^{2n}, \tag{1.8}$$

where the scalars

$$\frac{2^{2(n+1)}\left[2^{2(n+1)}-1\right]|B_{2(n+1)}|}{(2n+2)!}\quad\text{and}\quad\left(\frac{2}{\pi}\right)^{2n} \tag{1.9}$$

in (1.8) *are the best possible.*

Recently, a double inequality for $\frac{\tan x}{x}$ was obtained in [14], which may be rearranged as the following theorem:

1.2. Theorem. *For $0<x<\frac{\pi}{2}$ and $n\ge 0$, let $P_{2n}=a_0+a_1x^2+\dots+a_nx^{2n}$ and*

$$a_n=\frac{2^{2(n+1)}\left[2^{2(n+1)}-1\right]\pi^2}{(2n+2)!}|B_{2(n+1)}|-\frac{2^{2(n+1)}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|. \tag{1.10}$$

Then

$$\frac{P_{2n}(x)+\alpha_n x^{2(n+1)}}{\pi^2-4x^2}<\frac{\tan x}{x}<\frac{P_{2n}(x)+\beta_n x^{2(n+1)}}{\pi^2-4x^2}, \tag{1.11}$$

where

$$\alpha_n=\frac{8-P_{2n}(\pi/2)}{(\pi/2)^{2(n+1)}} \tag{1.12}$$

and $\beta_n=a_{n+1}$ are the best constants.

The aims of this paper are to analyze and compare Theorems 1.1 and 1.2, and to reorganize the proof of Theorem 1.1 by adopting the usual definition (1.1) of Bernoulli numbers.

2. Comparison of Theorems 1.1 and 1.2

In what follows, we analyze and compare Theorem 1.1 and Theorem 1.2.

It is well known that the first six Bernoulli numbers are

$$B_0=1,\quad B_1=-\frac{1}{2},\quad B_2=\frac{1}{6},\quad B_4=-\frac{1}{30},\quad B_6=\frac{1}{42},\quad \text{and}\quad B_8=-\frac{1}{30}.$$

The inequality (1.8) may be rewritten as

$$\frac{S_n(x)}{x-\left\{2^{2(n+1)}[2^{2(n+1)}-1]|B_{2(n+1)}|/(2n+2)!\right\}x^{2n+1}}<\frac{\tan x}{x}<\frac{S_n(x)}{x-(2/\pi)^{2n}x^{2n+1}} \tag{2.1}$$

and

$$\frac{1}{1-\left\{2^{2(n+1)}[2^{2(n+1)}-1]|B_{2(n+1)}|/(2n+2)!\right\}x^{2n}}<\frac{\tan x}{S_n(x)}<\frac{1}{1-(2/\pi)^{2n}x^{2n}} \tag{2.2}$$

for $0<x<\frac{\pi}{2}$ and $n\in\mathbb{N}$, where the constants

$$\frac{2^{2(n+1)}\left[2^{2(n+1)}-1\right]|B_{2(n+1)}|}{(2n+2)!}\quad\text{and}\quad\left(\frac{2}{\pi}\right)^{2n} \tag{2.3}$$

in (2.1) and (2.2) are the best possible.

If we take $n=1$ in (2.1), then

$$\frac{3}{3-x^2}<\frac{\tan x}{x}<\frac{\pi^2}{\pi^2-4x^2},\ 0<x<\frac{\pi}{2}. \tag{2.4}$$

If we let $n=2$ in (2.1), then

$$\frac{5\left(x^2+3\right)}{15-2x^4}<\frac{\tan x}{x}<\frac{\pi^4\left(x^2+3\right)}{3(\pi^4-16x^4)},\quad 0<x<\frac{\pi}{2}. \tag{2.5}$$

If we set $n=3$ in (2.1), then

$$\frac{21\left(15+5x^2+2x^4\right)}{315-17x^6}<\frac{\tan x}{x}<\frac{\pi^6(2x^4+5x^2+15)}{15(\pi^6-64x^6)},\ 0<x<\frac{\pi}{2}. \tag{2.6}$$

If we let $n = 4$ in (2.1), then

$$\frac{9\left(315+105x^2+42x^4+17x^6\right)}{2835-62x^8} < \frac{\tan x}{x} < \frac{\pi^8\left(315+105x^2+42x^4+17x^6\right)}{315\left(\pi^8-256x^8\right)}, \quad 0 < x < \frac{\pi}{2}. \tag{2.7}$$

If we take $n = 0$ in (1.11), we have

$$\frac{\pi^4+4\left(8-\pi^2\right)x^2}{\pi^2\left(\pi^2-4x^2\right)} < \frac{\tan x}{x} < \frac{3\pi^2+\left(\pi^2-12\right)x^2}{3\left(\pi^2-4x^2\right)}, \quad 0 < x < \frac{\pi}{2}. \tag{2.8}$$

If we set $n = 1$ in (1.11), we obtain

$$\frac{3\pi^6+\pi^4\left(\pi^2-12\right)x^2+4\left(96-\pi^4\right)x^4}{3\pi^4\left(\pi^2-4x^2\right)} < \frac{\tan x}{x} < \frac{2\left(\pi^2-10\right)x^4+5\left(\pi^2-12\right)x^2+15\pi^2}{15\left(\pi^2-4x^2\right)}, \quad 0 < x < \frac{\pi}{2}. \tag{2.9}$$

Further letting $n = 2$ in (1.11) gives

$$\frac{8\left(960-\pi^6\right)x^6+2\pi^6\left(\pi^2-10\right)x^4+5\pi^6\left(\pi^2-12\right)x^2+15\pi^8}{15\pi^6\left(\pi^2-4x^2\right)} < \frac{\tan x}{x} < \frac{\left(17\pi^2-168\right)x^6+42\left(\pi^2-10\right)x^4+3\left(35\pi^2-420\right)x^2+315\pi^2}{315\left(\pi^2-4x^2\right)} \tag{2.10}$$

for $0 < x < \frac{\pi}{2}$.

As $x > 0$ becomes small, then the left-hand side inequalities in (2.4), (2.5) and (2.6) are better than those in (2.8), (2.9) and (2.10) respectively. On the other hand, the right-hand side inequality in (2.7) is better than that in (2.9) as $x > 0$ becomes smaller. So the inequalities in Theorems 1.1 and 1.2 are not included in each other.

3. Reorganization of the proof of Theorem 1.1

To conform with more readers and to correct errors caused by wrongly citing (1.29) and (1.30) in [10, Page 6], we here reorganize the proof of Theorem 1.1 in [4] by using the usual definition (1.1) of Bernoulli numbers B_n for $n \in \mathbb{N}$.

For proving Theorem 1.1, we need the following lemma.

3.1. Lemma. *For $0 < x < \frac{\pi}{2}$ and $n \in \mathbb{N}$, let*

$$h_n(x) = \frac{\tan x - S_n(x)}{x^{2n}\tan x}. \tag{3.1}$$

Then

$$h_n(x) = \sum_{j=1}^{n} \frac{2^{2(n-j+1)}\left[2^{2(n-j+1)}-1\right]\left|B_{2(n-j+1)}\right|}{[2(n-j+1)]!} \sum_{k=j}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2(k-j)}. \tag{3.2}$$

Proof. We prove this lemma by induction on n.

For $n = 1$, we have

$$\begin{aligned} h_1(x) &= \frac{\tan x - S_1(x)}{x^2 \tan x} = \frac{1}{x^2} - \frac{\cot x}{x} \\ &= \frac{1}{x^2} - \frac{1}{x}\left[\frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2k-1}\right] \\ &= \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2(k-1)}, \end{aligned}$$

so the formula (3.2) holds for $n = 1$.

For $n = 2$, we have

$$\begin{aligned} h_2(x) &= \frac{\tan x - S_2(x)}{x^4 \tan x} = \frac{1}{x^4} - \frac{\cot x}{x^3} - \frac{\cot x}{3x} \\ &= \frac{1}{x^4} - \frac{1}{x^3}\left[\frac{1}{x} - \frac{1}{3}x - \sum_{k=2}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2k-1}\right] - \frac{1}{3x}\left[\frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2k-1}\right] \\ &= \sum_{k=2}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2(k-2)} + \frac{1}{3}\sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2(k-1)}, \end{aligned}$$

so the formula (3.2) holds for $n = 2$.

Suppose that the formula (3.2) holds for $n = m$. Then for $n = m + 1$, we have

$$\begin{aligned} h_{m+1} &= \frac{\tan x - S_{m+1}(x)}{x^{2(m+1)} \tan x} \\ &= \frac{\tan x - S_m(x) - 2^{2(m+1)}[2^{2(m+1)} - 1]|B_{2(m+1)}|x^{2m+1}/[2(m+1)]!}{x^{2(m+1)} \tan x} \\ &= \frac{1}{x^2} \cdot \frac{\tan x - S_m(x)}{x^{2m} \tan x} - \frac{2^{2(m+1)}\left[2^{2(m+1)} - 1\right]|B_{2(m+1)}|}{[2(m+1)]!} \cdot \frac{\cot x}{x} \\ &= \frac{1}{x^2}\sum_{j=1}^{m} \frac{2^{2(m-j+1)}\left[2^{2(m-j+1)} - 1\right]|B_{2(m-j+1)}|}{[2(m-j+1)]!} \sum_{k=j}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2(k-j)} \\ &\quad - \frac{2^{2(m+1)}\left[2^{2(m+1)} - 1\right]|B_{2(m+1)}|}{[2(m+1)]!} \frac{1}{x}\left[\frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2k-1}\right] \\ &= \frac{1}{x^2}\sum_{j=1}^{m} \frac{2^{2j}\left(2^{2j} - 1\right)|B_{2j}|}{(2j)!} \cdot \frac{2^{2(m-j+1)}|B_{2(m-j+1)}|}{[2(m-j+1)]!} \\ &\quad + \sum_{j=2}^{m+1} \frac{2^{2(m-j+2)}\left[2^{2(m-j+2)} - 1\right]|B_{2(m-j+2)}|}{[2(m-j+2)]!} \sum_{k=j}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2(k-j)} \\ &\quad - \frac{2^{2(m+1)}\left[2^{2(m+1)} - 1\right]|B_{2(m+1)}|}{[2(m+1)]!} \frac{1}{x^2} \\ &\quad + \frac{2^{2(m+1)}\left[2^{2(m+1)} - 1\right]|B_{2(m+1)}|}{[2(m+1)]!} \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2(k-1)} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{m+1} \frac{2^{2(m-j+2)}\left[2^{2(m-j+2)}-1\right]|B_{2(m-j+2)}|}{[2(m-j+2)]!} \sum_{k=j}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2(k-j)} \\
&\quad + \frac{1}{x^2} \sum_{j=1}^{m} \frac{2^{2j}\left(2^{2j}-1\right)|B_{2j}|}{(2j)!} \frac{2^{2(m-j+1)}|B_{2(m-j+1)}|}{[2(m-j+1)]!} \\
&\quad - \frac{2^{2(m+1)}\left[2^{2(m+1)}-1\right]|B_{2(m+1)}|}{[2(m+1)]!} \cdot \frac{1}{x^2}.
\end{aligned} \tag{3.3}
$$

Since $\tan x \cot x = 1$, we have

$$
\left[\sum_{i=1}^{\infty} \frac{2^{2i}\left(2^{2i}-1\right)|B_{2i}|}{(2i)!} x^{2i-1}\right] \left[\frac{1}{x} - \sum_{i=1}^{\infty} \frac{2^{2i}|B_{2i}|}{(2i)!} x^{2i-1}\right] = 1,
$$

which is equivalent to

$$
\sum_{i=2}^{\infty} \frac{2^{2i}\left(2^{2i}-1\right)|B_{2i}|}{(2i)!} x^{2i-2} = \left[\sum_{i=1}^{\infty} \frac{2^{2i}\left(2^{2i}-1\right)|B_{2i}|}{(2i)!} x^{2i-1}\right] \left[\sum_{i=1}^{\infty} \frac{2^{2i}|B_{2i}|}{(2i)!} x^{2i-1}\right], \tag{3.4}
$$

equating coefficients of the term x^{2m} on both sides of (3.4) yields

$$
\frac{2^{2(m+1)}\left[2^{2(m+1)}-1\right]|B_{2(m+1)}|}{[2(m+1)]!} = \sum_{j=1}^{m} \frac{2^{2j}\left(2^{2j}-1\right)|B_{2j}|}{(2j)!} \frac{2^{2(m-j+1)}|B_{2(m-j+1)}|}{[2(m-j+1)]!}. \tag{3.5}
$$

Substituting (3.5) into (3.3) and simplifying give

$$
h_{m+1}(x) = \sum_{j=1}^{m+1} \frac{2^{2(m-j+2)}\left[2^{2(m-j+2)}-1\right]|B_{2(m-j+2)}|}{[2(m-j+2)]!} \sum_{k=j}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2(k-j)}. \tag{3.6}
$$

By induction, the proof of Lemma 3.1 is complete. □

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Utilizing the formulas

$$
B_{2n}(x) = \frac{(-1)^{n-1} 2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos(2k\pi x)}{k^{2n}} \tag{3.7}
$$

for $n \in \mathbb{N}$ and $0 \le x \le 1$ and

$$
B_n(0) = (-1)^n B_n(1) = B_n \tag{3.8}
$$

for $n \ge 0$, where the Bernoulli polynomials $B_n(x)$ are defined by

$$
\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi, \tag{3.9}
$$

see [1, p. 805, 23.1.18 and 23.1.20] and [1, p. 804, 23.1.1], and differentiating (3.2) easily reveal that that $h_n'(x) > 0$, and so the function $h_n(x)$ is strictly increasing on $(0, \frac{\pi}{2})$. A standard computation using (3.1) shows that

$$
\lim_{x \to 0^+} h_n(x) = \frac{2^{2n+2}\left(2^{2n+2}-1\right)|B_{2(n+1)}|}{(2n+2)!}
$$

and

$$
\lim_{x \to (\pi/2)^-} h_n(x) = \left(\frac{2}{\pi}\right)^{2n}.
$$

Therefore, we have

$$2^{2n+2}\left(2^{2n+2}-1\right)|B_{2(n+1)}| \over (2n+2)! < h_n(x) < \left(\frac{2}{\pi}\right)^{2n}, \tag{3.10}$$

which is equivalent to the double inequality (1.8). Theorem 1.1 is proved. □

3.2. Remark. By the way, we point out that the formula (3.2) implies that the function

$$\frac{\tan x - S_n(x)}{x^{2n}\tan x} \tag{3.11}$$

is absolutely monotonic on $\left(0, \frac{\pi}{2}\right)$, that is,

$$\left[\frac{\tan x - S_n(x)}{x^{2n}\tan x}\right]^{(i)} \ge 0 \tag{3.12}$$

for all $i \in \mathbb{N}$ on $\left(0, \frac{\pi}{2}\right)$. For more information on absolutely monotonic functions, please refer to [5] and closely related references therein.

4. A conjecture

Stimulated by the right hand side of the inequality (2.1), we raise the following conjecture.

4.1. Conjecture. *The function*

$$\frac{\tan x}{S_n(x)}\left[1-\left(\frac{2}{\pi}\right)^{2n} x^{2n}\right] \tag{4.1}$$

for $n \in \mathbb{N}$ *is strictly decreasing and concave on* $\left(0, \frac{\pi}{2}\right)$. *Consequently, the double inequality*

$$\frac{4n/\pi S_n(\pi/2)}{1-(2/\pi)^{2n}x^{2n}} < \frac{\tan x}{S_n(x)} < \frac{1}{1-(2/\pi)^{2n}x^{2n}} \tag{4.2}$$

holds for $0 < x < \frac{\pi}{2}$ *and* $n \in \mathbb{N}$, *where the constants* $\frac{4n}{\pi S_n(\pi/2)}$ *and* 1 *in the numerators of the very ends of* (4.2) *are the best possible.*

4.2. Remark. We finally remark that, for the history, background, generalizations, and applications of inequalities relating to $\tan x$, please refer to [3, 7, 8, 12] and closely related references therein, and that, for more information on inequalities of trigonometric functions and inverse trigonometric functions, please refer to [6, 9, 11, 13], the expository and survey article [10] and plenty of references therein.

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References

[1] Abramowitz, M. and Stegun, I. A. (Eds) *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (National Bureau of Standards, Applied Mathematics Series **55**, 9th printing, Washington, 1970).

[2] Andrews, G. E., Askey, R. and Roy, R. *Special Functions* (Encyclopedia of Mathematics and its Applications **71**, Cambridge University Press, Cambridge, 1999).

[3] Becker, M. and Strak, E. L. *On a hierarchy of polynomial inequalities for* $\tan x$, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **No. 602-633**, 133–138, 1978.

[4] Chen, C.-P. and Qi, F. *A double inequality for remainder of power series of tangent function*, Tamkang J. Math. **34** (4), 351–355, 2003; Available online at `http://dx.doi.org//10.5556/j.tkjm.34.2003.351-355`

[5] Guo, B. -N. and Qi, F. *A property of logarithmically absolutely monotonic functions and the logarithmically complete monotonicity of a power-exponential function*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **72** (2), 21–30, 2010.

[6] Guo, B.-N. and Qi, F. *Sharpening and generalizations of Carlson's inequality for the arc cosine function*, Hacet. J. Math. Stat. **39** (3), 403–409, 2010.

[7] Kuang, J. -C. *Chángyòng Bùděngshì* (*Applied Inequalities*), 3rd ed. (Shandong Science and Technology Press, Ji'nan City, Shandong Province, China, 2004). (in Chinese)

[8] Mitrinović, D. S. *Analytic Inequalities* (Springer-Verlag, New York, Heidelberg, Berlin, 1970).

[9] Qi, F. and Guo, B. -N. *Sharpening and generalizations of Shafer's inequality for the arc sine function*, Integral Transforms Spec. Funct. **23** (2), 129–134, 2012; Available online at `http://dx.doi.org/10.1080/10652469.2011.564578`.

[10] Qi, F., Niu, D. -W. and Guo, B. -N. *Refinements, generalizations, and applications of Jordan's inequality and related problems*, J. Inequal. Appl. **2009** (2009), Article ID 271923, 52 pages, 2009; Available online at `http://dx.doi.org/10.1155/2009/271923`.

[11] Qi, F., Zhang, S. -Q. and Guo, B. -N. *Sharpening and generalizations of Shafer's inequality for the arc tangent function*, J. Inequal. Appl. **2009** (2009), Article ID 930294, 9 pages, 2009; Available online at `http://dx.doi.org/10.1155/2009/930294`.

[12] Steckin, S. B. *Some remarks on trigonmetric polynomials*, Uspehi Mat. Nauk. (N.S.) **10** **1** (63), 159–166, 1955. (in Russian)

[13] Zhao, J. -L., Wei, C. -F., Guo, B. -N. and Qi, F. *Sharpening and generalizations of Carlson's double inequality for the arc cosine function*, Hacet. J. Math. Stat. **41** (2), 201–209, 2012.

[14] Zhu, L. and Hua, J. -K. *Sharpening the Becker-Stark inequalities*, J. Inequal. Appl. **2010** (2010), Article ID 931275, 4 pages, 2010; Available online at `http://dx.doi.org/10.1155/2010/931275`.