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1-SOLITON SOLUTION OF THE THREE COMPONENT SYSTEM OF WU-ZHANG EQUATIONS

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Abstract

In this paper, the 1-soliton solution is obtained for the three-component Wu-Zhang equation. The soliton components comprises both topological as well as non-topological soliton solutions. The ansatz method is employed to carry out the integration of this coupled system of nonlinear evolution equations.

Keywords: Solitons, Integrability, Ansatz method.

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1. Introduction

In the theoretical investigation of the dynamics of nonlinear waves, coupled nonlinear partial differential equations (NLPDEs) are of great importance, due to their very wide applications in many fields of physics. As a matter of fact, coupled NLPDEs are used to model motions of waves in a great array of contexts, including plasma physics, fluid mechanics, optical fibers, hydrodynamics, quantum mechanics and many other nonlinear dispersive systems.

To understand the complex dynamics underlying coupled NLPDEs, it is instructive to investigate the propagation behaviour of traveling waves and their stability in the presence of perturbations. Traveling waves appear in many physical structures in solitary wave theory such as solitons, kinks, peakons, and cuspons [12].

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Among the travelling waves solutions, envelope solitons are of general physical interest because the soliton approach is universal in different fields of modern nonlinear science. Solitons are defined as localized waves that propagate without change of its shape and velocity properties and stable against mutual collisions [13]. Two different types of envelope solitons, bright and dark, can propagate in nonlinear dispersive media [8]. Compared with the bright soliton which is a pulse on a zero-intensity background, the dark soliton appears as an intensity dip in an infinitely extended constant background [6]. The formation of solitons has been regarded as a consequence of the delicate balance between dispersion (or diffraction) and nonlinearity under certain conditions.

In recent years, various powerful methods of integrability have been established and developed. These techniques are applied left and right to various NLPDEs to solve them and obtain closed form of solutions of physical relevance. There are various solutions that are obtained by incorporating these techniques of integrability. They are soliton solutions, cnoidal waves, kinks and anti-kinks, peakons, cuspons and stumpons just to name a few. In this paper, the focus is going to be on obtaining the soliton solutions only. Some of these techniques are the Adomian decomposition method, the homotopy analysis method, the solitary wave ansatz method, the subsidiary ordinary differential equations method, Fan's F-expansion method, the exponential function method, the G'/G expansion method and many others.

It is very interesting to note that the solitary wave ansatz method has been successfully applied to many kinds of NLPDEs with constant and varying coefficients, such as, for example, the $K(m,n)$ equation [2, 11], the BBM equation [10, 14], the $B(m,n)$ equation [4], the nonlinear Schrödinger's equation [5, 7, 1], the Kawahara equation [3] and many others. This new method has been proved by many to be reliable, effective and powerful.

Recent works in the literature analyze various evolution equations having important applications. Examples include the Wu-Zhang equation equations, which are used for modeling (2+1)-dimensional dispersive long wave [9]:

$$u_t + uu_x + vu_y + w_x = 0, \tag{1}$$

$$v_t + uv_x + vv_y + w_y = 0, \tag{2}$$

$$w_t + (uw)_x + (vw)_y + \frac{1}{3}\left(u_{xxx} + u_{xyy} + v_{xxy} + v_{yyy}\right) = 0, \tag{3}$$

where w is the elevation of the water, u is the surface velocity of water along the x-direction, and v is the surface velocity of water along the y-direction.

A problem of our interest consists in solving the following form of Wu-Zhang equation:

$$u_t + uu_x + vu_y + w_x + \alpha_1 w = 0, \tag{4}$$

$$v_t + uv_x + vv_y + w_y + \alpha_2 w = 0, \tag{5}$$

$$w_t + (uw)_x + (vw)_y + \frac{1}{3}\left(u_{xxx} + u_{xyy} + v_{xxy} + v_{yyy}\right) + \alpha_3 w_x = 0, \tag{6}$$

where α_1 and α_2 are nonzero real constants related to the presence of the linear damping, while α_3 is a nonzero constant corresponding to the effect of the first-order dispersion of the wave component w. If setting $\alpha_1 = \alpha_2 = \alpha_3 = 0$, equations (4)-(6) reduces to the model equations (1)-(3).

The aim of this paper is to extend the solitary wave ansatz method of finding new soliton solutions for the three component-nonlinear Wu-Zhang equations (4)-(6). We will see that the proposed method is concise and effective in solving coupled NLPDEs. We also find the necessary constraints of their existence in the case of nonzero linear damping and first-order dispersion coefficients.

2. Soliton solutions

To begin with let us assume the following solitary wave ansatze:

$$u(x,y,t) = A_1 \tanh^{p_1} \tau, \tag{7}$$
$$v(x,y,t) = A_2 \tanh^{p_2} \tau, \tag{8}$$
$$w(x,y,t) = \frac{A_3}{\cosh^{p_3} \tau}, \tag{9}$$

where

$$\tau = B_1 x + B_2 y - vt \tag{10}$$

and

$$p_1 > 0,\ p_2 > 0,\ p_3 > 0 \tag{11}$$

for solitons to exist.

From (7)-(9), we obtain

$$u_t = p_1 v A_1 \left(\tanh^{p_1+1} \tau - \tanh^{p_1-1} \tau\right), \tag{12}$$
$$uu_x = p_1 A_1^2 B_1 \left\{\tanh^{2p_1-1} \tau - \tanh^{2p_1+1} \tau\right\}, \tag{13}$$
$$\begin{aligned} u_{xxx} = p_1 A_1 B_1^3 \Big\{ &(p_1-1)(p_1-2) \tanh^{p_1-3} \tau \\ &- (p_1+1)(p_1+2) \tanh^{p_1+3} \tau \\ &- \left\{2p_1^2 + (p_1-1)(p_1-2)\right\} \tanh^{p_1-1} \tau \\ &+ \left\{2p_1^2 + (p_1+1)(p_1+2)\right\} \tanh^{p_1+1} \tau \Big\}, \end{aligned} \tag{14}$$
$$\begin{aligned} u_{xyy} = p_1 A_1 B_1 B_2^2 \Big\{ &(p_1-1)(p_1-2) \tanh^{p_1-3} \tau \\ &- (p_1+1)(p_1+2) \tanh^{p_1+3} \tau \\ &- \left\{2p_1^2 + (p_1-1)(p_1-2)\right\} \tanh^{p_1-1} \tau \\ &+ \left\{2p_1^2 + (p_1+1)(p_1+2)\right\} \tanh^{p_1+1} \tau \Big\}, \end{aligned} \tag{15}$$
$$v_t = p_2 v A_2 \left(\tanh^{p_2+1} \tau - \tanh^{p_2-1} \tau\right), \tag{16}$$
$$vv_y = p_2 A_2^2 B_2 \left\{\tanh^{2p_2-1} \tau - \tanh^{2p_2+1} \tau\right\}, \tag{17}$$
$$\begin{aligned} v_{xxy} = p_2 A_2 B_1^2 B_2 \Big\{ &(p_2-1)(p_2-2) \tanh^{p_2-3} \tau \\ &- (p_2+1)(p_2+2) \tanh^{p_2+3} \tau \\ &- \left\{2p_2^2 + (p_2-1)(p_2-2)\right\} \tanh^{p_2-1} \tau \\ &+ \left\{2p_2^2 + (p_2+1)(p_2+2)\right\} \tanh^{p_2+1} \tau \Big\}, \end{aligned} \tag{18}$$
$$\begin{aligned} v_{yyy} = p_2 A_2 B_2^3 \Big\{ &(p_2-1)(p_2-2) \tanh^{p_2-3} \tau \\ &- (p_2+1)(p_2+2) \tanh^{p_2+3} \tau \\ &- \left\{2p_2^2 + (p_2-1)(p_2-2)\right\} \tanh^{p_2-1} \tau \\ &+ \left\{2p_2^2 + (p_2+1)(p_2+2)\right\} \tanh^{p_2+1} \tau \Big\}, \end{aligned} \tag{19}$$

$$vu_y = p_1 A_1 A_2 B_2 \left\{\tanh^{p_1+p_2-1}\tau - \tanh^{p_1+p_2+1}\tau\right\}, \tag{20}$$

$$uv_x = p_2 A_1 A_2 B_1 \left\{\tanh^{p_1+p_2-1}\tau - \tanh^{p_1+p_2+1}\tau\right\}, \tag{21}$$

$$(uw)_x = A_1 A_3 B_1 \left\{\frac{p_1 \tanh^{p_1-1}\tau}{\cosh^{p_3}\tau} - \frac{(p_1+p_3)\tanh^{p_1+1}\tau}{\cosh^{p_3}\tau}\right\}, \tag{22}$$

$$(vw)_y = A_2 A_3 B_2 \left\{\frac{p_2 \tanh^{p_2-1}\tau}{\cosh^{p_3}\tau} - \frac{(p_2+p_3)\tanh^{p_2+1}\tau}{\cosh^{p_3}\tau}\right\}, \tag{23}$$

$$w_t = \frac{p_3 v A_3 \tanh\tau}{\cosh^{p_3}\tau}, \tag{24}$$

$$w_x = -\frac{p_3 A_3 B_1 \tanh\tau}{\cosh^{p_3}\tau}, \tag{25}$$

$$w_y = -\frac{p_3 A_3 B_2 \tanh\tau}{\cosh^{p_3}\tau}. \tag{26}$$

Substituting (12)-(26) into (4)-(6) yields

$$\begin{aligned} &p_1 v A_1 \left(\tanh^{p_1+1}\tau - \tanh^{p_1-1}\tau\right) + p_1 A_1^2 B_1 \left\{\tanh^{2p_1-1}\tau - \tanh^{2p_1+1}\tau\right\} \\ &\quad + p_1 A_1 A_2 B_2 \left\{\tanh^{p_1+p_2-1}\tau - \tanh^{p_1+p_2+1}\tau\right\} \\ &\qquad - \frac{p_3 A_3 B_1 \tanh\tau}{\cosh^{p_3}\tau} + \frac{\alpha_1 A_3}{\cosh^{p_3}\tau} = 0, \end{aligned} \tag{27}$$

$$\begin{aligned} &p_2 v A_2 \left(\tanh^{p_2+1}\tau - \tanh^{p_2-1}\tau\right) + p_2 A_1 A_2 B_1 \left\{\tanh^{p_1+p_2-1}\tau - \tanh^{p_1+p_2+1}\tau\right\} \\ &\quad + p_2 A_2^2 B_2 \left\{\tanh^{2p_2-1}\tau - \tanh^{2p_2+1}\tau\right\} - \frac{p_3 A_3 B_2 \tanh\tau}{\cosh^{p_3}\tau} + \frac{\alpha_2 A_3}{\cosh^{p_3}\tau} = 0\ , \end{aligned} \tag{28}$$

and

$$\begin{aligned} &\frac{p_3 v A_3 \tanh\tau}{\cosh^{p_3}\tau} + A_1 A_3 B_1 \left\{\frac{p_1 \tanh^{p_1-1}\tau}{\cosh^{p_3}\tau} - \frac{(p_1+p_3)\tanh^{p_1+1}\tau}{\cosh^{p_3}\tau}\right\} \\ &\quad + A_2 A_3 B_2 \left\{\frac{p_2 \tanh^{p_2-1}\tau}{\cosh^{p_3}\tau} - \frac{(p_2+p_3)\tanh^{p_2+1}\tau}{\cosh^{p_3}\tau}\right\} \\ &\qquad \frac{p_1 A_1 B_1 \left(B_1^2 + B_2^2\right)}{3} \Big\{ (p_1-1)(p_1-2)\tanh^{p_1-3}\tau \\ &\qquad\quad - (p_1+1)(p_1+2)\tanh^{p_1+3}\tau \\ &\qquad\quad - \left\{2p_1^2 + (p_1-1)(p_1-2)\right\}\tanh^{p_1-1}\tau \\ &\qquad\quad + \left\{2p_1^2 + (p_1+1)(p_1+2)\right\}\tanh^{p_1+1}\tau \Big\} \\ &\qquad + \frac{p_2 A_2 B_2 \left(B_1^2 + B_2^2\right)}{3} \Big\{ (p_2-1)(p_2-2)\tanh^{p_2-3}\tau \\ &\qquad\quad - (p_2+1)(p_2+2)\tanh^{p_2+3}\tau \\ &\qquad\quad - \left\{2p_2^2 + (p_2-1)(p_2-2)\right\}\tanh^{p_2-1}\tau \\ &\qquad\quad + \left\{2p_2^2 + (p_2+1)(p_2+2)\right\}\tanh^{p_2+1}\tau \Big\} \\ &\qquad - \frac{\alpha_3 p_3 A_3 B_1 \tanh\tau}{\cosh^{p_3}\tau} = 0. \end{aligned} \tag{29}$$

From (27), equating the exponents $2p_1 + 1$ and $p_1 + p_2 + 1$ gives

$$2p_1 + 1 = p_1 + p_2 + 1, \tag{30}$$

so that

$$p_1 = p_2, \tag{31}$$

which is also obtained by equating the exponents $2p_1 - 1$ and $p_1 + p_2 - 1$. It should be noted that the relation (31) is also yielded when the exponents pairs $2p_2 - 1$ and $p_1 + p_2 - 1$, $2p_2 + 1$ and $p_1 + p_2 + 1$ are equated with each other in (28).

The linearly independent functions in (27)-(29) are $\tanh^{p+j}\tau$ where $j = -3, -1, 1, 3$. So, from (27)-(29), each of the coefficients of these linearly independent functions must be zero. Now, setting the coefficient of $\tanh^{p_1-3}\tau$ in (29) to zero yields (with $p_1 = p_2$):

$$p_1 = 1 \tag{32}$$

and

$$p_1 = 2. \tag{33}$$

Let us firstly consider the case $p_1 = 1$. This gives the same value of $p_2 = 1$ as seen from (31). If we put $p_1 = 1$ in (27), the equation reduces to

$$\begin{aligned} &vA_1\left(\tanh^2\tau - 1\right) + A_1^2B_1\left\{\tanh\tau - \tanh^3\tau\right\} \\ &\qquad + A_1A_2B_2\left\{\tanh\tau - \tanh^3\tau\right\} - \frac{p_3A_3B_1\tanh\tau}{\cosh^{p_3}\tau} + \frac{\alpha_1A_3}{\cosh^{p_3}\tau} = 0 \end{aligned} \tag{34}$$

Expanding the tanh terms to the $1/\cosh\tau$ terms, the equation (34) becomes

$$-\frac{vA_1}{\cosh^2\tau} + \frac{A_1^2B_1\tanh\tau}{\cosh^2\tau} + \frac{A_1A_2B_2\tanh\tau}{\cosh^2\tau} - \frac{p_3A_3B_1\tanh\tau}{\cosh^{p_3}\tau} + \frac{\alpha_1A_3}{\cosh^{p_3}\tau} = 0. \tag{35}$$

From (35), equating the exponents of the functions $\tanh\tau/\cosh^{p_3}\tau$ and $\tanh\tau/\cosh^2\tau$ gives

$$p_3 = 2. \tag{36}$$

Now, from (35), the linearly independent functions are $1/\cosh^2\tau$ and $\tanh\tau/\cosh^2\tau$. Therefore, setting their respective coefficients to zero yields

$$v = \frac{\alpha_1A_3}{A_1}, \tag{37}$$

$$A_1^2B_1 + A_1A_2B_2 - 2A_3B_1 = 0. \tag{38}$$

If we put $p_1 = p_2 = 1$ and $p_3 = 2$ in (28), the equation reduces to

$$-\frac{vA_2}{\cosh^2\tau} + \frac{A_1A_2B_1\tanh\tau}{\cosh^2\tau} + \frac{A_2^2B_2\tanh\tau}{\cosh^2\tau} - \frac{2A_3B_2\tanh\tau}{\cosh^2\tau} + \frac{\alpha_2A_3}{\cosh^2\tau} = 0. \tag{39}$$

In (39), the linearly independent functions are $1/\cosh^2\tau$ and $\tanh\tau/\cosh^2\tau$, which gives

$$v = \frac{\alpha_2A_3}{A_2}, \tag{40}$$

$$A_2^2B_2 + A_1A_2B_1 - 2A_3B_2 = 0. \tag{41}$$

On setting $p_1 = p_2 = 1$ and $p_3 = 2$ in (29), and after expanding the $1/\cosh\tau$ terms to the tanh terms, the equation reduces to

$$\begin{aligned} &2vA_3\left(\tanh\tau - \tanh^3\tau\right) + \left(A_1A_3B_1 + A_2A_3B_2\right)\left\{1 - 4\tanh^2\tau + 3\tanh^4\tau\right\} \\ &\qquad - \frac{2}{3}\left(B_1^2 + B_2^2\right)\left(A_1B_1 + A_2B_2\right)\left\{1 - 4\tanh^2\tau + 3\tanh^4\tau\right\} \\ &\qquad\qquad + 2\alpha_3A_3B_1\left(\tanh\tau - \tanh^3\tau\right) = 0. \end{aligned} \tag{42}$$

Now from (42), setting the coefficients of the linearly independent functions $\tanh^j\tau$ for $j = 0, 2, 3, 4$ to zero yields

$$v = -\alpha_3B_1, \tag{43}$$

$$A_3 = \frac{2}{3}\left(B_1^2 + B_2^2\right). \tag{44}$$

We clearly see from (44) that the amplitude A_3 of the w-soliton is only dependent on B_1 and B_2.

Also, from (38) and (41), it is possible to recover

$$\frac{A_1^2 - A_2^2}{A_1 A_2} = \frac{B_1^2 - B_2^2}{B_1 B_2}, \tag{45}$$

which gives the constraint relation between the free parameters $A_{1,2}$ and $B_{1,2}$ of the u-soliton and v-soliton.

Finally equating the three values of the solitons v from (37), (40) and (43), gives

$$\frac{\alpha_1}{A_1} = \frac{\alpha_2}{A_2} = -\frac{\alpha_3 B_1}{A_3}, \tag{46}$$

which serves as a constraint relation between the coefficients and the soliton parameters. Notice that the second case $p_1 = 2$ is not considered here as it does not give unique values of the soliton parameters.

Thus, the soliton solutions to the Wu-Zhang- type equation (4)-(6) are given by

$$u(x, y, t) = A_1 \tanh \tau, \tag{47}$$

$$v(x, y, t) = A_2 \tanh \tau, \tag{48}$$

$$w(x, y, t) = \frac{A_3}{\cosh^2 \tau}, \tag{49}$$

where the solitons velocity is given by (37) or (40) or (43), while the amplitude A_3 of the w-soliton is given by (44). Finally the constraint relations between the soliton parameters and the model coefficient are displayed in (45) and (46).

3. Conclusions

In this paper the three component Wu-Zhang equations are studied. The 1-soliton solution solution is obtained for this coupled system of nonlinear evolution equation. The ansatz method was employed to carry out the integration. The solution is comprised of topological as well as non-topological soliton solutions.

In future, this study will be explored further. This equation will be studied with time-dependent coefficients. The perturbation terms will also be considered. The conservation laws will also be established for this coupled system. Finally, the quasi-stationary soliton solution will also be obtained for the perturbed equation. These results will be reported in future publications.

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