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ON A TYPE OF K-CONTACT MANIFOLDS

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Abstract

In this paper we generalize the results of Tanno and Zhen. Then we study the cyclic parallel Ricci tensor on a K-contact manifold.

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1. Introduction

In [12], S. Tanno studied Ricci symmetric (i.e. $\nabla S = 0$) K-contact manifolds. Also in a recent paper [13], G. Zhen studied conformally symmetric (i.e. $\nabla C = 0$) K-contact manifolds. It is known that, if a Riemannian manifold is conformally symmetric, then the manifold satisfies the harmonic Weyl conformal curvature tensor. But the converse result is not necessarily true. In this paper we generalize the results of Tanno and Zhen.

2. Preliminaries

Let (M^n, g) be a contact Riemannian manifold with contact form η, associated vector field ξ, $(1,1)$-tensor field ϕ and associated Riemannian metric g. If ξ is a Killing vector field, then M^n is called a K-contact manifold [1, 10]. A K-contact manifold is called a Sasakian manifold [1], if the relation

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.1}$$

holds on M, where ∇ denotes the operator of covariant differentiation with respect of g.

In a K-contact manifold the following relations hold [1, 10, 11]:

$$a)\ \phi\xi = 0, \qquad b)\ \eta(\xi) = 1, \qquad c)\ g(X,\xi) = \eta(X), \tag{2.2}$$

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.4}$$

$$\nabla_X \xi = -\phi X. \tag{2.5}$$

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Also we have,

$$
\begin{aligned}
&g(R(\xi,X)Y,\xi)=\eta(R(\xi,X)Y)=g(X,Y)-\eta(X)\eta(Y),\\
&R(\xi,X)\xi=-X+\eta(X)\xi,\\
&S(X,\xi)=(n-1)\eta(X),\\
&(\nabla_X\phi)Y=R(\xi,X)Y,
\end{aligned} \tag{2.6}
$$

for any vector fields. Further, since ξ is a Killing vector, S and τ remain invariant under it, that is,

$$\pounds_\xi S=0, \tag{2.7}$$

and

$$\pounds_\xi \tau=0, \tag{2.8}$$

where $\pounds$ denotes Lie derivation.

K-contact manifolds have been studied by several authors, such as, De and Guha [9], De and Biswas [4], De and Ghosh [5], Cabrerizo *et al.* [3].

The curvature tensor R satisfies the second Bianchi identity if

$$(\nabla_X R)(Y,Z,W)+(\nabla_Y R)(X,Z,W)+(\nabla_Z R)(X,Y,W)=0. \tag{2.9}$$

2.1. Proposition. *Let R and S be the curvature tensor and Ricci tensor of the manifold M. Then we get from second Bianchi identity*

$$(\operatorname{div} R)(X,Y,Z)=(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z), \tag{2.10}$$

where "div" denotes divergence. □

2.2. Definition. The curvature tensor R is *harmonic* if

$$(divR)(X,Y,Z)=0. \tag{2.11}$$

It is obvious that every Ricci symmetric manifold is harmonic. On the other hand, the Weyl conformal curvature tensor on M^n is given by

$$
\begin{aligned}
C(X,Y)Z&=R(X,Y)Z\\
&-\frac{1}{n-2}[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY]\\
&+\frac{\tau}{(n-1)(n-2)}[g(Y,Z)X-g(X,Z)Y],
\end{aligned} \tag{2.12}
$$

where τ is the scalar curvature of M^n.

2.3. Definition. A Riemannian manifold M^n is *of harmonic conformal curvature tensor* if

$$(\operatorname{div} C)(X,Y,Z)=0. \tag{2.13}$$

It is known, [6, 7], that

$$
\begin{aligned}
(divC)(X,Y)Z=\frac{n-3}{n-2}\{&(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z)\\
&-\frac{1}{2(n-1)}[g(Y,Z)d\tau(X)-g(X,Z)d\tau(Y)]\}.
\end{aligned}
$$

A K-contact manifold M is said to be an *Einstein manifold* if

$$S=ag,$$

holds on M, where a is a smooth function on M.

3. Main results

3.1. Theorem. *If a K-contact manifold is of harmonic conformal curvature tensor, that is,* $\operatorname{div} C = 0$*, then the manifold is an Einstein manifold.*

Proof. Let M be a K-contact manifold that satisfies $\operatorname{div} C = 0$. Then we have

$$(3.1)\qquad (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \frac{1}{2(n-1)}[g(Y,Z)d\tau(X) - g(X,Z)d\tau(Y)].$$

From (2.7), it follows that

$$(3.2)\qquad (\nabla_\xi S)(Y,Z) = -S(\nabla_Y \xi, Z) - S(Y, \nabla_Z \xi),$$

and from (2.8), we get $d\tau(\xi) = 0$. Putting $X = \xi$ in (3.1), we have

$$(3.3)\qquad (\nabla_\xi S)(Y,Z) - (\nabla_Y S)(\xi,Z) = \frac{1}{2(n-1)}[g(Y,Z)d\tau(\xi) - \eta(Z)d\tau(Y)].$$

Using (3.2) in (3.3), we have

$$(3.4)\qquad -S(\nabla_Y \xi, Z) - S(Y, \nabla_Z \xi) - (\nabla_Y S)(\xi,Z) = \frac{1}{2(n-1)}[g(Y,Z)d\tau(\xi) - \eta(Z)d\tau(Y)].$$

Again using $d\tau(\xi) = 0$ in (3.4), we get

$$(3.5)\qquad -S(Y, \nabla_Z \xi) - \nabla_Y S(\xi, Z) + S(\xi, \nabla_Y Z) = -\frac{1}{2(n-1)}\eta(Z)d\tau(Y).$$

Using (2.5) and (2.6) in (3.5) yields

$$(3.6)\qquad S(Y,\phi Z) - (n-1)\nabla_Y \eta(Z) + (n-1)\eta(\nabla_Y Z) = -\frac{1}{2(n-1)}\eta(Z)d\tau(Y),$$

which implies that

$$(3.7)\qquad S(Y,\phi Z) - (n-1)(\nabla_Y \eta)(Z) = -\frac{1}{2(n-1)}\eta(Z)d\tau(Y),$$

or,

$$S(Y,\phi Z) - S(\nabla_Y Z, \xi) = -\gamma\eta(Z)d\tau(Y).$$

After some calculations, we obtain

$$(3.8)\qquad S(Y,\phi Z) + (n-1)g(\phi Y, Z) = -\frac{1}{2(n-1)}\eta(Z)d\tau(Y).$$

If we put $Z = \phi Z$ in (3.8), then we have

$$(3.9)\qquad S(Y,\phi^2 Z) + (n-1)g(\phi Y, \phi Z) = 0.$$

Using (2.3) and (2.4) in (3.9), we get

$$S(Y,Z) = (n-1)g(Y,Z),$$

which implies that the manifold is an Einstein manifold. □

Now we have the following:

3.2. Corollary. *Since* $\nabla C = 0$ *implies* $\operatorname{div} C = 0$*, therefore a conformally symmetric K-contact manifold is an Einstein manifold.* □

The above Corollary has been proved by Zhen [13].

Also in a Riemannian manifold $\operatorname{div} R = 0$ implies $\operatorname{div} C = 0$, therefore from Theorem 3.1 we can state the following:

3.3. Corollary. *If a K-contact Riemannian manifold is of harmonic curvature tensor then it is an Einstein manifold.* □

From Theorem 3.1 and Corollary 3.3, we can state the following:

3.4. Theorem. *In a K-contact Riemannian manifold the following conditions are equivalent:*

(i) $\operatorname{div} R = 0$,
(ii) $\operatorname{div} C = 0$,
(iii) *Einstein manifold.* □

4. K-contact manifolds with cyclic parallel Ricci tensor

A. Gray [8] introduced two classes of Riemannian manifolds determined by the covariant derivative of the Ricci tensor; the class A consisting of all Riemannian manifolds whose Ricci tensor S is a *Codazzi tensor*, i.e.,

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z).$$

The class B consisting of all Riemannian manifolds whose Ricci tensor is *cyclic parallel*, i.e.

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(X,Z) + (\nabla_Z S)(X,Y) = 0. \tag{4.1}$$

If we take $X = \xi$ in (4.1), we have

$$(\nabla_\xi S)(Y,Z) + (\nabla_Y S)(\xi,Z) + (\nabla_Z S)(\xi,Y) = 0. \tag{4.2}$$

Since ξ is a Killing vector field, we get

$$\begin{aligned} &- S(\nabla_Y \xi, Z) - S(Y, \nabla_Z \xi) + \nabla_Y S(\xi, Z) - S(\nabla_Y \xi, Z) \\ &\qquad - S(\xi, \nabla_Y Z) + \nabla_Z S(\xi, Y) - S(\nabla_Z \xi, Y) - S(\xi, \nabla_Z Y) = 0. \end{aligned} \tag{4.3}$$

After some calculations, we have

$$\begin{aligned} &- 2S(\nabla_Y \xi, Z) - 2S(Y, \nabla_Z \xi) + (n-1)\nabla_Y \eta(Z) \\ &\qquad + (n-1)\nabla_Z \eta(Y) - S(\xi, \nabla_Y Z) - S(\xi, \nabla_Z Y) = 0. \end{aligned} \tag{4.4}$$

Using (2.5) in (4.4), we obtain

$$2S(\phi Y, Z) + 2S(\phi Z, Y) + (n-1)(\nabla_Y \eta)(Z) + (n-1)(\nabla_Z \eta)(Y) = 0, \tag{4.5}$$

which implies that

$$S(\phi Y, \phi Z) + S(\phi^2 Z, Y) = 0. \tag{4.6}$$

Now using (2.3) in (4.6), we have

$$S(\phi Y, \phi Z) = S(Y,Z) - (n-1)\eta(Y)\eta(Z). \tag{4.7}$$

Thus, if a K-contact manifold satisfies the cyclic parallel Ricci tensor then the Ricci tensor satisfies (4.7).

It may be noted that in a Sasakian manifold the relation (4.7) holds without assuming a cyclic parallel Ricci tensor.

A contact manifold is said to be η-Einstein if

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{4.8}$$

where a, b are smooth functions. On the other hand, it is known that [2], in a K-contact η-Einstein manifold a, b are constants. Thus from (4.8), we obtain

$$\begin{aligned} (\nabla_X S)(Y,Z) &= b[(\nabla_X \eta)(Y)\eta(Z) + (\nabla_X \eta)(Z)\eta(Y)], \\ (\nabla_Z S)(X,Y) &= b[(\nabla_Z \eta)(X)\eta(Y) + (\nabla_Z \eta)(Y)\eta(X)], \\ (\nabla_Y S)(X,Z) &= b[(\nabla_Y \eta)(X)\eta(Z) + (\nabla_Y \eta)(Z)\eta(X)]. \end{aligned} \tag{4.9}$$

Therefore using (4.9), we have

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(X,Z) + (\nabla_Z S)(X,Y)$$
$$= b[\{(\nabla_Z \eta)(X) + (\nabla_X \eta)(Z)\}\eta(Y) + \{(\nabla_X \eta)(Y)(\nabla_Y \eta)(X)\}\eta(Z) + \{(\nabla_Z \eta)(Y) + (\nabla_Y \eta)(Z)\}\eta(X)].$$

Since ξ is a Killing vector field in a K-contact manifold, we get

$$(\nabla_X \eta)(Y) + (\nabla_Y \eta)(X) = 0, \text{ for all } X, Y.$$

Hence we can state the following:

4.1. Theorem. *A K-contact η-Einstein manifold satisfies the cyclic parallel Ricci tensor.* □

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