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POLYGONAL DESIGNS WITH BLOCK SIZE 3 AND SINGLE DISTANCE

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Abstract

Polygonal designs, a class of partially balanced incomplete block designs with regular polygons, are useful in survey sampling in terms of balanced sampling plans excluding contiguous units (BSECs) when neighboring units in a population provide similar information. In this paper, the method of cyclic shifts is used to construct cyclic polygonal designs (in terms of BSECs) with block size $k = 3$ for single distance, and a complete solution for $v \in \{9, 10, \dots, 100\}$ treatments is compiled.

Keywords: Cyclic BSEC, Cyclic polygonal design, Cyclic shifts, Distance between the units, Single distance, PBIBD.

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1. Introduction

The use of balanced sampling plans is essential in situations where the units in a population (or in an experimental region) are found physically close (as neighbors) to each other. Studies in ecological and environmental sciences are often conducted to investigate the abundance and diversities of species, where a balanced sampling plan serves the purpose of generating samples from the population by avoiding the selection of neighboring (contiguous or adjacent) units which essentially provide redundant information. In other words, these neighboring units are deliberately prevented (or excluded) from being selected under the situation that they provide little new information to the sampling effort. These plans attempt to provide ways of sampling the units from geographical region when a spatial pattern in the response is expected (see Christman [2] and See *et al.* [12]).

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Now we consider an example which highlight the usefulness of cyclic polygonal designs (CPDs) (or balanced sampling plans) in practical situations.

1.1. Example. A study was conducted to investigate the species abundance, diversity and richness, of certain insects in a forest. For the selected plots, a net was placed under the trees, the trees were fogged with the insecticide and the insects of the species of interest that landed on the net were counted. This is an expensive and time-consuming procedure that can be applied to only a relatively small number of plots with small areas. In this study, it is expected that counts from neighboring plots would be very similar, and that "fogging" one plot could alter the responses in neighboring plots. There were certain regions in the forest where a low insect count was expected from all trees due to recent fires, and other regions, near a creek, where a relatively high count was expected. Therefore, a sampling plan that avoids the simultaneous selection of neighboring plots within a region was utilized (See and Song [10]).

Let a population consists of circular ordered units labeled as $0, 1, \dots, v-1$. Then i and $i+1$ are said to be contiguous for all i such that $0 \leq i \leq v-2$, as are $v-1$ and 0. Let $\delta_{(i,j)}$ be the distance between the sampling units (or design points) i and j in this circular population such that $\delta_{(i,j)} = \min(|i-j|, v-|i-j|)$ and the maximum distance between any pair of units cannot exceed $[v/2]$. For simplicity, denote the distance between sampling units (or design points) i and j by $\delta_{(i,j)} = 1, 2, \dots, [v/2]$. The notation cyclic BSEC(v, k, λ) is used to denote a cyclic polygonal design CPD$(v, k, \lambda; 1)$. Similarly, the notation cyclic BSA$(v, k, \lambda; \alpha)$ is used to denote a cyclic polygonal design CPD$(v, k, \lambda; \alpha)$. A simple cyclic polygonal design with minimal distance $\alpha = 1$ is denoted by CPD$(v, k, \lambda; 1)$. According to [4, 11, 15], a CPD$(v, k, \lambda; \alpha)$ is actually a PBIBD$(v, b, r, k, \lambda; \alpha)$ for which association relations between the treatments are defined through the distance.

Hedayat, Rao and Stufken [5, 6] introduced balanced sampling plans excluding contiguous units (BSECs) in which the contiguous units do not appear together in a sample, whereas all other pairs of units appear equally often. The first - and second - order inclusion probabilities are $\pi_i = \frac{n}{N}$; $i = 1, 2, \dots, N$ and $\pi_{ij} = \frac{n(n-1)}{N(N-3)}; \forall\, i \neq j = 1, 2, \dots, N$, $|i-j| > 1$, and $\pi_{ij} = 0$ when $|i-j| \leq 1$, respectively. Hedayat *et al.* [5] showed that the variance of the Horvitz-Thompson estimator of the population mean $\mu = \sum Y_i$ is given by $\{1 - \frac{(n-1)(1-\rho_1)}{N-3}\}\frac{\sigma^2}{n}$, where σ^2 denotes the population variance and ρ_1 is the first order serial correlation between the units. Hedayat *et al.* [5] proved that their sampling plan is more efficient than the simple random sampling without replacement provided that $\rho_1 > -\frac{1}{N-1}$.

Ultimately, the objective of BSEC plans is to provide more representative samples and to provide more efficient estimators of the population mean, when neighboring (or contiguous) units are expected to provide similar responses. Simply, in a sampling plan the entire sample of units is selected such that no two neighboring units are included.

Stufken [13] generalized the concept of BSECs to balanced sampling plans excluding adjacent units (BSAs) where all those adjacent pairs of units are excluded whose distance is less than or equal to α (or m). The first - and second - order inclusion probabilities are $\pi_i = \frac{n}{N}$; $i = 1, 2, \dots, N$ and $\pi_{ij} = \frac{n(n-1)}{N(N-2\alpha-1)}$; $\forall\, i \neq j = 1, 2, \dots, N$, $|i-j| > \alpha$, and $\pi_{ij} = 0$ when $|i-j| \leq \alpha$, respectively. Two units i and j are called adjacent when their distance is less than a specified number α whose choice depends on the surveyor or experimenter (see Mandal *et al.* [9]). It is obvious that BSEC(v, k, λ) is equivalent to BSA$(v, k, \lambda; 1)$ or BSA(1) ([13, 16]).

A polygonal design (PD) is a PBIBD with 2-class associate scheme (i.e. with λ_1 and λ_2) such that the two units which are the i^{th} associate of each other occur together in

λ_i (distinct) blocks. Thus a PBIBD$(v, b, r, k; \lambda_1 = 0, \lambda_2 = \lambda)$, satisfying the necessary condition of PD having the first of the λ's equal to zero. We will add one more parameter α in PBIBD if represented in terms of PD as PBIBD$(v, b, r, k, \lambda_1, \lambda_2; \alpha)$, where α denotes the distance between the units.

Some definitions of polygonal designs which exist in the literature are presented as follows.

1.2. Definition. (Stufken *et al.* [15]) A polygonal design PD$(v, k, \lambda; \alpha)$ for v treatments, k blocks $(k < v)$ and minimum distance $\alpha+1$ is a binary block design in which treatments i and j appear together in precisely λ blocks, say, if $\alpha < |i-j| < v-\alpha$, while they do not appear together in any block otherwise. For $\alpha = 0$, this class of designs consists of BIBDs, while the designs are PBIBDs for $\alpha > 0$.

1.3. Definition. (Stufken and Wright [14]) A block design for v treatments in b blocks of size k is called a polygonal design with minimum distance 1 (a simple polygonal design) if (i) for any $i \in \{0, 1, \ldots, v-1\}$, treatments i and $i+1$ do not appear together in a block, and (ii) any two treatments appear together in precisely λ blocks for some positive integer λ.

1.4. Definition. (Mandal *et al.* [8]) A polygonal design with minimum distance α is an incomplete block design in which v treatments are arranged in b blocks of size k if (i) every treatment $\{0, 1, \ldots, v-1\}$ appears in r blocks, and (ii) every pair of treatments which differs more than the distance α appears in exactly λ blocks, whereas other pairs do not appear at all in any block and every block contains k distinct treatments.

A polygonal design with parameter v, k, b, r, λ and α must satisfy the following conditions:

(i) $bk = vr$,
(ii) $\lambda[v - (2\alpha + 1)] = r(k-1)$.

Thus, in a PD the necessary divisibility condition is $\lambda = \frac{r(k-1)}{v-(2\alpha+1)}$.

For block size $k = 3$ and single distance $\alpha = 1$, the existence and construction of CPDs (in terms of cyclic BSECs) has only been considered by See *et al.* [11], Colbourn and Ling [3], Wei[10], Zhang and Chang [17] and Mandal *et al.* [8], but for some λ and limited v only. Bhattacharyya *et al.* [1] used a λ-triple fold system and resolved the existence of PD$(6\alpha + 3, 3, 1; \alpha)$.

Stufken and Wright [14] suggested a further extension to the existing work and also the deduction of simpler (or additional) techniques for the construction of polygonal designs. Stufken and Wright ([14], p.180) pointed out that

> $\cdots$ it would of course be interesting to know, for any combination of v and k, all possible values of b for which a polygonal design exists, and in particular for smallest value.

and also in Stufken and Wright ([14], p.183) that

> $\cdots$ additional techniques are needed to learn more about designs for other combinations of v, b, and k. There is especially a need for simpler construction methods that apply for arbitrary values of the block size k.

In the literature, different combinatorial tools have been used to show the existence and construction of PDs such as the association schemes, d-class association schemes, partial triple system PTS(v, λ), λ-fold system, Fisher's inequality, Langford sequence, symmetrically repeated differences, linear programming, linear algebraic and design theoretic, etc.

Taking into consideration the limitations in the existing literature and the suggestions of Stufken and Wright [14] mentioned above, a simple method of construction, the method of cyclic shifts, is used to extend the existence and construction of CPDs for block size $k = 3$ and single distance $\alpha = 1$. The interesting feature, in addition to the simplicity, of the proposed method is that the properties of a CPD from the sets of shifts (used in a CPD) can easily be obtained without constructing the actual blocks of a CPD. The pattern of off-diagonal zero elements (in bold form) from the main-diagonal in a concurrence matrix (or first row of the concurrence matrix) is helpful in the identification of the distance α (or m) in a CPD. Further, the off-diagonal elements in a concurrence matrix can easily be obtained from the sets of shifts or from concurrence set(s) of shifts. For more detail see Iqbal *et al.* [7].

In this paper, we used the method of cyclic shifts (which the authors have described in Iqbal *et al.* [7]) for the construction of CPD$(v; 3; \lambda; 1)$'s. In Section 2, some algorithms are given to search CPD$(v, 3, \lambda; 1)$'s with $\lambda = 1, 2, 3, 4, 6, 12$. In Section 3, the sets of shifts are given for the construction of CPD$(v, 3, \lambda; 1)$'s with $\lambda = 1, 2, 3, 4, 6, 12$, and a complete solution (in terms of cyclic BSEC) is given for $v \in \{9, 10, \ldots, 100\}$ treatments. The concluding remarks are given in Section 4.

2. Algorithms for the construction of CPD$(v, 3, \lambda; 1)$'s

Let $a \,\%\, b$ denote the residue (nonnegative least remainder) r of a modulo b; that is, $a \,\%\, b = r$ if and only if a is congruent r modulo b.

Now, suppose that $S_j = \{q_{1j}, q_{2j}\}$ is the set of shifts, where $\alpha \le q_{1j}, q_{2j} \le v - \alpha$, and q_{1j} and q_{2j} may be repeated any number of times. Then

$$S_j^\star = \{q_{1j}, q_{2j}, (q_{1j} + q_{2j}) \,\%\, v, v - q_{1j}, v - q_{2j}, v - (q_{1j} + q_{2j}) \,\%\, v\}.$$

Here $v - q_j$ is the complement of q_j.

A design will be a CPD$(v, 3, \lambda; 1)$ if

(i) $S_j^\star$ consists of $2, \ldots, v - 2$ an equal number of times, say λ;

(ii) $(q_{1j} + q_{2j}) \,\%\, v \ne 0$.

Some algorithms for the construction of CPD$(v, 3, \lambda; 1)$'s with $\lambda = 1, 2, 3, 4, 6, 12$ are presented with the conditions

(i) $2 \le q_{1j},\, q_{2j} \le v - 2$;

(ii) $(q_{1j} + q_{2j}) \,\%\, v \ne 0,\ 1, v - 1$.

2.1. Algorithm A CPD$(v, 3, \lambda; 1)$ with $\lambda = 1$ can be constructed, if $v = 3i$; $i\,(> 1)$ is odd, from the following $(i - 1)/2$ sets of shifts

$$S_j = [q_{1j}, q_{2j}];\ \text{j=1,2,...,(i-1)/2},$$

such that $2, 3, \ldots, v-2$ appear once among q_{1j}, q_{2j}, $(q_{1j}+q_{2j}) \,\%\, v$ and their complements.

2.2. Algorithm A CPD$(v, 3, \lambda; 1)$ with $\lambda = 2$ can be constructed, if $v = 12i$; $i\,(\ge 1)$ is an integer, from the following $(4i - 1)$ sets of shifts

$$S_j = [q_{1j}, q_{2j}];\ j = 1, 2, \ldots, (4i - 1)$$

such that $2, 3 \ldots, (v - 2)/2, (v + 2)/2, \ldots, v - 2$ appear twice but $(v/2)$ appears once among q_{1j}, q_{2j}, $(q_{1j} + q_{2j}) \,\%\, v$ and their complements.

2.3A. Algorithm A CPD$(v, 3, \lambda; 1)$ with $\lambda = 3$ can be constructed, if $v = 6i+1$; $i\,(> 1)$ is an integer, from the following $(3i - 1)$ sets of shifts

$$S_j = [q_{1j}, q_{2j}];\ j = 1, 2, \ldots, (3i - 1)$$

such that $2, 3, \ldots, v-2$ appear thrice among q_{1j}, q_{2j}, $(q_{1j}+q_{2j})\,\%\,v$ and their complements.

2.3B Algorithm A CPD$(v,3,\lambda;1)$ with $\lambda = 3$ can be constructed, if $v = 6i+5$; $i\,(\geq 1)$ is an integer, from the following $(3i+1)$ sets of shifts

$$S_j = [q_{1j}, q_{2j}];\ j = 1, 2, \ldots, (3i+1)$$

such that $2, 3, \ldots, v-2$ appear thrice among q_{1j}, q_{2j}, $(q_{1j}+q_{2j})\,\%\,v$ and their complements.

2.4. Algorithm A CPD$(v,3,\lambda;1)$ with $\lambda = 4$ can be constructed, if $v = 12i+6$; $i \geq 1$ is an integer, from the following $(8i+2)$ sets of shifts

$$S_j = [q_{1j}, q_{2j}];\ j = 1, 2, \ldots, (8i+2)$$

such that $2, 3 \ldots, (v-2)/2, (v+2)/2, \ldots, v-2$ appear four times but $(v/2)$ appears twice among q_{1j}, q_{2j}, $(q_{1j}+q_{2j})\,\%\,v$ and their complements.

2.5A. Algorithm A CPD$(v,3,\lambda;1)$ with $\lambda = 6$ can be constructed, if $v = 12i+4$; $i\,(\geq 1)$ is an integer, from the following $(12i+1)$ sets of shifts

$$S_j = [q_{1j}, q_{2j}];\ j = 1, 2, \ldots, (12i+1)$$

such that $2, 3 \ldots, (v-2)/2, (v+2)/2, \ldots, v-2$ appear six times but $(v/2)$ appears three times among q_{1j}, q_{2j}, $(q_{1j}+q_{2j})\,\%\,v$ and their complements.

2.5B. Algorithm A CPD$(v,3,\lambda;1)$ with $\lambda = 6$ can be constructed, if $v = 12i+8$; $i\,(\geq 1)$ is an integer, from the following $(12i+5)$ sets of shifts

$$S_j = [q_{1j}, q_{2j}];\ j = 1, 2, \ldots, (12i+5)$$

such that $2, 3 \ldots, (v-2)/2, (v+2)/2, \ldots, v-2$ appear six times but $(v/2)$ appears three times among q_{1j}, q_{2j}, $(q_{1j}+q_{2j})\,\%\,v$ and their complements.

2.6A Algorithm A CPD$(v,3,\lambda;1)$ with $\lambda = 12$ can be constructed, if $v = 12i-2$; $i\,(\geq 1)$ is an integer, from the following $(24i-10)$ sets of shifts

$$S_j = [q_{1j}, q_{2j}];\ j = 1, 2, \ldots, (24i-10)$$

such that $2, 3 \ldots, (v-2)/2, (v+2)/2, \ldots, v-2$ appear twelve times but $(v/2)$ appears six times among q_{1j}, q_{2j}, $(q_{1j}+q_{2j})\,\%\,v$ and their complements.

2.6B. Algorithm A CPD$(v,3,\lambda;1)$ for $\lambda = 12$ can be constructed, if $v = 12i+2$; $i\,(\geq 1)$ is an integer, from the following $(24i-2)$ sets of shifts

$$S_j = [q_{1j}, q_{2j}];\ j = 1, 2, \ldots, (24i-2)$$

such that $2, 3 \ldots, (v-2)/2, (v+2)/2, \ldots, v-2$ appear twelve times but $(v/2)$ appears six times among q_{1j}, q_{2j}, $(q_{1j}+q_{2j})\,\%\,v$ and their complements.

3. Detection of the properties in CPD$(v,3,\lambda;1)$'s

In this section, it is shown how CPD$(v,3,\lambda;1)$'s (in terms of BSECs) for some λ can be constructed, and their concurrence matrix (or first row of the concurrence matrix) derived from the set(s) of shifts or from concurrence set(s) of shifts. To demonstrate this, some examples of CPD$(v,3,\lambda;1)$'s, along with their concurrence matrices (or first row of the concurrence matrices), for $\lambda = 1, 2, 3, 4, 6, 12$ and for $v \in \{9, 10, 11, 12, 13, 14\}$ are given below.

3.1. Example. The set of shifts for CPD$(9,3,\lambda;1)$ with $\lambda = 1$ and $r = 3$ is $[2,3]$.

The corresponding CPD$(9,3,1;1)$ and the concurrence matrix are

0	1	2	3	4	5	6	7	8
2	3	4	5	6	7	8	0	1
5	6	7	8	0	1	2	3	4

and

$$\mathbf{NN}' = \begin{pmatrix} 3 & \mathbf{0} & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \mathbf{0} & 3 & \mathbf{0} & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \mathbf{0} & 3 & \mathbf{0} & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \mathbf{0} & 3 & \mathbf{0} & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \mathbf{0} & 3 & \mathbf{0} & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \mathbf{0} & 3 & \mathbf{0} & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & \mathbf{0} & 3 & \mathbf{0} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & \mathbf{0} & 3 & \mathbf{0} \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & \mathbf{0} & 3 \end{pmatrix}$$

From the above design and the corresponding concurrence matrix, we can see that the treatments 0 and 1, 1 and 2, …, 7 and 8, and 8 and 0 do not appear together within blocks. Therefore, we can say that the above design is a CPD of distance $\alpha = 1$. The concurrence set of shifts for the above design is 2, 3 and 5. In this set, treatment 1 (and its complement 8) does not appear and in other shifts, treatments 2, 3 and 5 (and their complements treatments 7, 6 and 4) appear once. In the above concurrence matrix, the elements are 0 and 1. 0 for treatment 1 and 8, and 1 for the rest of the treatments.

3.2. Example. The sets of shifts for CPD$(10, 3, \lambda; 1)$ with $\lambda = 12$ and $r = 42$ are $[2, 2] + [2, 3](6) + [2, 4](4) + [3, 3](3)$.

The corresponding CPD$(10, 3, 12; 1)$ and the first row of concurrence matrix are

1×{0	1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	0	1
4	5	6	7	8	9	0	1	2	3}
6×{0	1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	0	1
5	6	7	8	9	0	1	2	3	4}
4×{0	1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	0	1
6	7	8	9	0	1	2	3	4	5}
3×{0	1	2	3	4	5	6	7	8	9
3	4	5	6	7	8	9	0	1	2
6	7	8	9	0	1	2	3	4	5}

and

$$(42 \quad \mathbf{0} \quad 12 \quad 12 \quad 12 \quad 12 \quad 12 \quad 12 \quad 12 \quad 0)'.$$

3.3. Example. The sets of shifts for CPD$(11, 3, \lambda; 1)$ with $\lambda = 3$ and $r = 12$ are $[2, 3](2) + [2, 4] + [4, 4]$.

The corresponding CPD$(11, 3, 3; 1)$ and the first row of the concurrence vector are

2×{0	1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	0	1
5	6	7	8	9	10	0	1	2	3	4}
1×{0	1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	0	1
6	7	8	9	10	0	1	2	3	4	5}
1×{0	1	2	3	4	5	6	7	8	9	10
4	5	6	7	8	9	10	0	1	2	3
8	9	10	0	1	2	3	4	5	6	7}

and

$$(12 \quad \mathbf{0} \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 0)'.$$

3.4. Example. The sets of shifts for $CPD(12, 3, \lambda; 1)$ with $\lambda = 2$ and $r = 9$ are $[2, 3] + [2, 4] + [3, 4]$.

The corresponding $CPD(12, 3, 2; 1)$ and the first row of the concurrence matrix are

0	1	2	3	4	5	6	7	8	9	10	11
2	3	4	5	6	7	8	9	10	11	0	1
5	6	7	8	9	10	11	0	1	2	3	4
0	1	2	3	4	5	6	7	8	9	10	11
2	3	4	5	6	7	8	9	10	11	0	1
6	7	8	9	10	11	0	1	2	3	4	3
0	1	2	3	4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10	11	0	1	2
7	8	9	10	11	0	1	2	3	4	3	4

and

$$(9 \quad \mathbf{0} \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 0)'.$$

3.5. Example. The sets of shifts for $CPD(13, 3, \lambda; 1)$ with $\lambda = 3$ and $r = 15$ are $[2, 3] + [2, 4](2) + [3, 4] + [5, 5]$.

The first row of concurrence matrix for $CPD(13, 3, 3; 1)$ is

$$(15 \quad \mathbf{0} \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 0)'.$$

3.6. Example. The sets of shifts for $CPD(14, 3, \lambda; 1)$ with $\lambda = 12$ and $r = 66$ are $[2, 3](4) + [2, 4](5) + [2, 5](3) + [3, 4](3) + [3, 5](5) + [4, 4](2)$.

The first row of the concurrence matrix for $CPD(14, 3, 12; 1)$ is

$$(66 \quad \mathbf{0} \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 0)'.$$

Similarly, the remaining $CPD(v, 3, \lambda; 1)$'s with some λ can be constructed and their concurrence matrices (or the first row of the concurrence matrices) can also be developed.

4. Construction of $CPD(v, 3, \lambda; 1)$'s with $\lambda = 1, 2, 3, 4, 6, 12$

In this section, the $CPD(v, 3, \lambda; 1)$'s with $\lambda = 1, 2, 3, 4, 6, 12$ are constructed, and a complete solution for treatments $v \in \{9, 10, \ldots, 100\}$ are given. These CPDs exist under the necessary condition $\lambda = \frac{\beta k(k-1)}{v-(2\alpha+1)}$, where $\beta = \frac{\lambda(v-3)}{6}$ denotes the number of shifts required for a CPD with $\alpha = 1$.

The sets of shifts have been searched through the Algorithms given in Section 2 to construct the CPD$(v, 3, \lambda; 1)$'s with $\lambda = 1, 2, 3, 4, 6, 12$ for treatments $v \in \{9, 10, \ldots, 100\}$.

The following Lemmas give the complete constructions.

4.1. Lemma. *There exists a CPD*$(v, 3, 1; 1)$ *if and only if* $v \equiv 3 \,\%\, 6$ *and* $v \geq 9$.

Proof. Using Algorithm 2.1, the sets of shifts are searched to construct the CPD$(v, 3, 1; 1)$'s for $v \in \{9, 15, 21, 27, 33, 39, 45, 51, 57, 63, 69, 75, 81, 87, 93, 99\}$, which satisfy the necessary condition $\lambda = \left[\frac{r(k-1)}{v-3}\right]$.

$v = 9 : [2, 3]$

$v = 15 : [2, 4] + [3, 5]$

$v = 21 : [2, 4] + [3, 8] + [5, 7]$

$v = 27 : [2, 3] + [4, 9] + [6, 10] + [7, 8]$

$v = 33 : [2, 4] + [3, 13] + [5, 9] + [7, 8] + [10, 11]$

$v = 39 : [2, 16] + [3, 8] + [4, 15] + [5, 9] + [6, 7] + [10, 12]$

$v = 45 : [2, 19] + [3, 17] + [4, 5] + [6, 7] + [8, 10] + [11, 12] + [14, 15]$

$v = 51 : [2, 16] + [3, 6] + [4, 21] + [5, 8] + [7, 15] + [10, 14] + [11, 17] + [12, 19]$

$v = 57 : [2, 21] + [3, 13] + [4, 11] + [5, 22] + [6, 12] + [7, 19] + [8, 17] + [9, 20]$
$+ [10, 14]$

$v = 63 : [2, 9] + [3, 25] + [4, 26] + [5, 24] + [6, 13] + [7, 16] + [8, 10] + [12, 15]$
$+ [14, 17] + [20, 21]$

$v = 69 : [2, 22] + [3, 23] + [4, 31] + [5, 15] + [6, 8] + [7, 25] + [9, 19] + [10, 11]$
$+ [12, 18] + [13, 27] + [16, 17]$

$v = 75 : [2, 21] + [3, 10] + [4, 26] + [5, 6] + [7, 27] + [8, 24] + [9, 22] + [12, 28]$
$+ [14, 15] + [16, 20] + [17, 25] + [18, 19]$

$v = 81 : [2, 3] + [4, 23] + [6, 32] + [7, 8] + [9, 13] + [10, 30] + [11, 25] + [12, 17]$
$+ [14, 33] + [16, 21] + [18, 28] + [19, 20] + [24, 26]$

$v = 87 : [2, 42] + [3, 16] + [4, 20] + [5, 29] + [6, 11] + [7, 31] + [8, 27] + [9, 28]$
$+ [10, 22] + [12, 18] + [13, 33] + [14, 26] + [15, 21] + [23, 25]$

$v = 93 : [2, 43] + [3, 44] + [4, 34] + [5, 36] + [6, 20] + [7, 14] + [8, 25] + [9, 28]$
$+ [10, 13] + [11, 29] + [12, 15] + [16, 19] + [17, 22] + [18, 24] + [30, 31]$

$v = 99 : [2, 48] + [3, 41] + [4, 35] + [5, 14] + [6, 21] + [7, 24] + [8, 15] + [9, 16] + [10, 32]$
$+ [11, 31] + [12, 26] + [13, 20] + [17, 37] + [18, 28] + [22, 39] + [29, 30]$ □

4.2. Lemma. *There exists a CPD*$(v, 3, 2; 1)$ *if and only if* $v \equiv 0 \,\%\, 12$ *and* $v \geq 12$.

Proof. Using Algorithm 2.2, the sets of shifts are searched to construct CPD$(v, 3, 2; 1)$'s for treatments $v \in$ {12,24,36,48,60,72,84,96}, which satisfy the necessary condition $\lambda = \left[\frac{r(k-1)}{v-3}\right]$.

$v = 12 : [2, 3] + [2, 4] + [3, 4]$

$v = 24 : [2, 3] + [2, 9] + [3, 7] + [4, 6] + [4, 8] + [5, 6] + [7, 8]$

$v = 36 : [2, 3] + [2, 14] + [3, 12] + [4, 7] + [4, 14] + [5, 10] + [6, 7] + [6, 10]$
$+ [8, 9] + [8, 9] + [11, 12]$

$$v = 48 : [2,6] + [2,11] + [3,17] + [3,20] + [4,8] + [4,9] + [5,14] + [5,16] + [6,18] + [7,15] + [7,18] + [9,10] + [10,11] + [12,14] + [15,16]$$

$$v = 60 : [2,6] + [2,6] + [3,12] + [3,21] + [4,18] + [4,22] + [5,20] + [5,23] + [7,10] + [7,24] + [9,10] + [9,25] + [11,12] + [11,17] + [13,14] + [13,16] + [14,16] + [15,18] + [19,20]$$

$$v = 72 : [2,26] + [2,31] + [3,24](2) + [4,7] + [4,28] + [5,12] + [5,20] + [6,9] + [6,12] + [7,23] + [8,17] + [8,23] + [9,20] + [10,22] + [10,26] + [11,22] + [13,16] + [13,21] + [14,16] + [14,21] + [15,19] + [18,19]$$

$$v = 84 : [2,29] + [2,33] + [3,16] + [3,34] + [4,5] + [4,38] + [5,23] + [6,21] + [6,30] + [7,18] + [7,33] + [8,16] + [8,26] + [9,28] + [10,15] + [10,22] + [11,29] + [11,30] + [12,19] + [12,24] + [13,14] + [13,26] + [14,18] + [15,20] + [17,21] + [17,22] + [20,23]$$

$$v = 96 : [2,29] + [2,33] + [3,12] + [3,28] + [4,37] + [4,42] + [5,11] + [5,29] + [6,33] + [6,36] + [7,34] + [7,39] + [8,22] + [8,32] + [9,28] + [9,38] + [10,11] + [10,38] + [12,23] + [13,14] + [13,27] + [14,16] + [15,17] + [17,19] + [18,26](2) + [19,24] + [20,23] + [20,25] + [21,24] + [22,25]$$

□

4.3. Lemma. *There exists a CPD$(v,3,3;1)$ if and only if $v \equiv 1 \,\%\, 6$ and $v \geq 13$.*

Proof. Using Algorithm 2.3A, the sets of shifts are searched to construct CPD$(v,3,3;1)$'s for treatments $v \in \{13, 19, 25, 31, 37, 43, 49, 55, 61, 67, 73, 79, 85, 91, 97\}$, which satisfy the necessary condition $\lambda = \left[\frac{r(k-1)}{v-3}\right]$.

$$v = 13 : [2,3] + [2,4](2) + [3,4] + [5,5]$$

$$v = 19 : [2,5] + [2,6](2) + [3,4] + [3,5] + [3,7] + [4,5] + [4,6]$$

$$v = 25 : [2,8] + [2,9](2) + [3,4] + [3,6] + [3,7] + [4,6] + [4,8] + [5,6] + [5,7] + [5,8]$$

$$v = 31 : [2,10] + [2,11] + [2,12] + [3,7] + [3,11] + [3,12] + [4,5] + [4,6] + [4,7] + [5,8] + [5,9] + [6,7] + [6,9] + [8,8]$$

$$v = 37 : [2,12] + [2,13] + [2,14] + [3,10] + [3,11] + [3,15] + [4,5] + [4,6] + [4,12] + [5,7] + [5,13] + [6,11](2) + [7,8] + [7,9] + [8,9] + [8,10]$$

$$v = 43 : [2,4] + [2,10] + [2,15] + [3,14] + [3,16](2) + [4,8] + [4,16] + [5,13](2) + [5,14] + [6,7] + [6,15] + [7,8] + [7,11] + [8,9] + [9,11] + [9,14] + [10,11] + [10,12]$$

$$v = 49 : [2,12] + [2,20](2) + [3,9] + [3,16] + [3,18] + [4,11] + [4,14] + [4,17] + [5,6] + [5,8] + [5,10] + [6,16] + [6,17] + [7,10] + [7,12] + [7,18] + [8,13] + [8,15] + [9,14] + [9,16] + [10,19] + [11,13]$$

$$v = 55 : [2,8] + [2,19](2) + [3,14] + [3,19] + [3,21] + [4,12] + [4,20](2) + [5,17](2) + [5,23] + [6,7](2) + [6,9] + [7,18] + [8,12] + [8,18] + [9,14] + [9,16] + [10,16] + [10,18] + [11,12] + [11,14] + [11,15] + [13,15]$$

$$v = 61 : [2,12] + [2,26](2) + [3,17] + [3,19](2) + [4,14] + [4,25] + [4,26] + [5,7](2) + [5,23] + [6,15] + [6,17](2) + [7,14] + [8,16](2) + [8,21] + [9,13] + [9,18](2) + [10,15](2) + [10,20] + [11,13] + [11,16] + [11,20] + [13,19]$$

$$v = 67 : [2,15](2) + [2,21] + [3,27](3) + [4,19] + [4,22] + [4,25] + [5,10] + [5,13] + [5,28] + [6,12] + [6,13] + [6,19] + [7,21] + [7,22](2) + [8,10] + [8,16](2) + [9,23] + [9,26](2) + [10,14] + [11,14] + [11,20](2) + [12,16] + [12,21] + [13,20] + [14,17]$$

$$v = 73 : [2,15] + [2,24] + [2,25] + [3,19](2) + [3,21] + [4,29](3) + [5,17] + [5,26] + [5,27] + [6,13] + [6,28] + [6,30] + [7,8](2) + [7,16] + [8,12] + [9,18] + [9,23](2) + [10,18] + [10,25] + [10,28] + [11,20](2) + [11,25] + [12,14] + [12,24] + [13,21](2) + [14,16](2) + [17,18]$$

$$v = 79 : [2,17] + [2,28](2) + [3,6](2) + [3,27] + [4,32](3) + [5,21](2) + [5,23] + [6,12] + [7,15] + [7,17](2) + [8,23] + [8,29](2) + [9,25] + [10,23] + [10,24] + [10,25] + [11,27](2) + [11,29] + [12,19](2) + [13,20](2) + [13,26] + [14,21] + [14,25] + [15,20] + [15,2] + [16,18](2) + [16,22]$$

$$v = 85 : [2,14] + [2,16] + [2,32] + [3,24] + [3,35](2) + [4,6](2) + [4,29] + [5,31](2) + [5,36] + [6,32] + [7,18] + [7,23](2) + [8,15] + [8,20] + [8,25] + [9,19](2) + [9,25] + [10,19] + [11,26](2) + [11,31] + [12,20] + [12,29] + [12,30] + [13,21] + [13,27](2) + [14,21] + [14,26] + [15,18] + [15,24] + [16,21] + [17,22](2) + [17,24] + [20,22]$$

$$v = 91 : [2,32](2) + [2,36] + [3,25](2) + [3,37] + [4,7](2) + [4,33] + [5,17] + [5,25] + [5,39] + [6,17] + [6,35] + [6,39] + [7,26] + [8,16] + [8,35](2) + [9,19] + [9,22](2) + [10,16] + [10,30](2) + [11,34] + [12,29](2) + [12,33] + [13,14] + [13,18] + [13,24] + [14,24] + [14,29] + [15,17] + [15,21](2) + [16,35] + [18,20] + [18,21] + [19,23](2) + [20,27](2)$$

$$v = 97 : [2,36] + [2,39](2) + [3,18](2) + [3,30] + [4,26] + [4,38] + [4,43] + [5,22] + [5,24](2) + [6,17](2) + [6,39] + [7,11] + [7,19] + [7,33] + [8,20] + [8,37](2) + [9,25](2) + [9,32] + [10,22](2) + [10,34] + [11,35](2) + [12,25] + [12,31](2) + [13,35] + [13,36](2) + [14,15] + [14,19] + [14,24] + [15,27](2) + [16,28](2) + [16,31] + [17,23] + [19,21] + [20,26] + [20,30]$$ □

4.4. Lemma. *There exists a CPD*$(v, 3, 3; 1)$ *if and only if* $v \equiv 5 \,\%\, 6$ *and* $v \geq 11$.

Proof. Using Algorithm 2.3B, the sets of shifts are searched to construct CPD$(v, 3, 3; 1)$'s for treatments $v \in \{11, 17, 23, 29, 35, 41, 47, 53, 59, 65, 71, 77, 83, 89, 95\}$, which satisfy the necessary condition $\lambda = \left[\frac{r(k-1)}{v-3}\right]$.

$$v = 11 : [2,3](2) + [2,4] + [4,4]$$

$$v = 17 : [2,4] + [2,6](2) + [3,4](2) + [3,5] + [5,5]$$

$v = 23 : [2, 6] + [2, 7] + [2, 8] + [3, 5] + [3, 7] + [3, 9] + [4, 5] + [4, 6] + [4, 7] + [5, 6]$

$v = 29 : [2, 6] + [2, 7] + [2, 10] + [3, 10](2) + [3, 12] + [4, 7] + [4, 8] + [4, 9] + [5, 6](2) + [5, 9] + [7, 8]$

$v = 35 : [2, 6] + [2, 13](2) + [3, 7] + [3, 11](2) + [4, 5] + [4, 12](2) + [5, 7] + [5, 9] + [6, 10] + [6, 11] + [7, 13] + [8, 9] + [8, 10]$

$v = 41 : [2, 6] + [2, 15] + [2, 17] + [3, 11] + [3, 12] + [3, 13] + [4, 12] + [4, 14] + [4, 17] + [5, 6] + [5, 13](2) + [6, 8] + [7, 8] + [7, 9] + [7, 12] + [9, 10] + [9, 11] + [10, 10]$

$v = 47 : [2, 11] + [2, 17] + [2, 19] + [3, 6] + [3, 7] + [3, 17] + [4, 9] + [4, 14] + [4, 15] + [5, 12] + [5, 16] + [5, 20] + [6, 9] + [6, 16] + [7, 11] + [7, 15] + [8, 10] + [8, 12] + [8, 16] + [10, 14] + [11, 13] + [12, 14]$

$v = 53 : [2, 22](2) + [2, 24] + [3, 11] + [3, 14] + [3, 15] + [4, 14] + [4, 18] + [4, 19] + [5, 10](2) + [5, 16] + [6, 7] + [6, 19](2) + [7, 16](2) + [8, 12](2) + [8, 13] + [9, 11] + [9, 17](2) + [10, 11] + [12, 13]$

$v = 59 : [2, 19] + [2, 22] + [2, 28] + [3, 13] + [3, 16] + [3, 23] + [4, 14] + [4, 22] + [4, 23] + [5, 6] + [5, 7] + [5, 12] + [6, 18] + [6, 19] + [7, 13] + [7, 14] + [8, 15] + [8, 17](2) + [9, 12] + [9, 15] + [9, 22] + [10, 18] + [10, 20](2) + [11, 15] + [11, 16] + [13, 14]$

$v = 65 : [2, 20](2) + [2, 27] + [3, 15] + [3, 23] + [3, 28] + [4, 13] + [4, 17](2) + [5, 23](2) + [5, 25] + [6, 9] + [6, 27](2) + [7, 14] + [7, 18](2) + [8, 8] + [8, 12] + [9, 10] + [9, 15] + [10, 24](2) + [11, 19](2) + [11, 22] + [12, 14](2) + [13, 16](2)$

$v = 71 : [2, 21](2) + [2, 27] + [3, 29](2) + [3, 31] + [4, 12] + [4, 26](2) + [5, 14] + [5, 17](2) + [6, 7] + [6, 15] + [6, 24] + [7, 18] + [7, 28] + [8, 10] + [8, 17] + [8, 18] + [9, 14] + [9, 19](2) + [10, 27](2) + [11, 14] + [11, 22] + [11, 24] + [12, 20] + [12, 24] + [13, 20](2) + [15, 16](2)$

$v = 77 : [2, 6] + [2, 17] + [2, 28] + [3, 6] + [3, 17] + [3, 32] + [4, 19] + [4, 20] + [4, 25] + [5, 7] + [5, 28](2) + [6, 27] + [7, 27](2) + [8, 18](2) + [9, 22](2) + [10, 11] + [10, 21] + [10, 30] + [11, 24](2) + [12, 20] + [12, 26] + [13, 16](2) + [13, 23] + [14, 23] + [14, 25](2) + [15, 21] + [15, 22] + [15, 30] + [16, 18] + [17, 19]$

$v = 83 : [2, 14](2) + [2, 28] + [3, 31](3) + [4, 16] + [4, 32] + [4, 37] + [5, 11] + [5, 19] + [5, 30] + [6, 27] + [6, 29] + [6, 33] + [7, 8](2) + [7, 14] + [8, 17] + [9, 10] + [9, 27](2) + [10, 22](2) + [11, 28] + [11, 30] + [12, 23] + [12, 26] + [12, 29] + [13, 15] + [13, 16] + [13, 20] + [14, 23](2) + [17, 21](2) + [18, 22] + [18, 25](2) + [19, 20]$

$v = 89 : [2, 17] + [2, 31](2) + [3, 35](3) + [4, 18] + [4, 23](2) + [5, 7] + [5, 23] + [5, 36] + [6, 14](2) + [6, 21] + [7, 30] + [7, 37] + [8, 17] + [8, 29] + [8, 33] + [9, 17] + [9, 19](2) + [10, 22](2) + [10, 32] + [11, 25](2) + [11, 31] + [12, 34](2) + [13, 26] + [13, 29] + [13, 30] + [14, 20]$

$$+ [15,26] + [15,29] + [15,30] + [16,24](3) + [18,21](2)$$

$$v = 95 : [2,21] + [2,26](2) + [3,11] + [3,27](2) + [4,5] + [4,21](2) + [5,38](2)$$
$$+ [6,34] + [6,35](2) + [7,22](2) + [7,36] + [8,20] + [8,25] + [8,30] + [9,10]$$
$$+ [9,37] + [10,34] + [10,35] + [11,31](2) + [12,20] + [12,24](2) + [13,16]$$
$$+ [13,32](2) + [14,19](2) + [15,26] + [15,31] + [15,34]$$
$$+ [16,23](2) + [17,27](2) + [18,24] + [18,30](2) + [20,27]$$

□

4.5. Lemma. *There exists a CPD*$(v,3,4;1)$ *if and only if* $v \equiv 6 \,\%\, 12$ *and* $v \geq 18$.

Proof. Using Algorithm 2.4, the sets of shifts are searched to construct CPD$(,3,4;1)$'s for treatments $v \in \{18,30,42,54,66,78,90\}$ which satisfy the necessary condition $\lambda = \left[\frac{r(k-1)}{v-3}\right]$.

$$v = 18 : [2,5](2) + [2,6](2) + [3,4](2) + [3,5] + [3,6] + [4,5] + [4,6]$$

$$v = 30 : [2,5] + [2,6] + [2,9] + [2,10] + [3,8] + [3,10] + [3,11] + [3,12]$$
$$+ [4,5] + [4,8] + [4,10] + [4,11] + [5,7] + [5,9] + [6,7](2) + [6,10] + [8,9]$$

$$v = 42 : [2,3] + [2,8] + [2,11] + [2,16] + [3,12] + [3,13] + [3,17] + [4,6] + [4,11]$$
$$+ [4,14](2) + [5,12] + [5,15] + [5,16] + [6,11] + [6,13] + [6,14] + [7,8]$$
$$+ [7,9] + [7,12] + [7,14] + [8,9] + [8,12] + [9,10](2) + [11,13]$$

$$v = 54 : [2,14] + [2,17] + [2,21] + [2,22] + [3,4] + [3,11] + [3,13] + [3,18] + [4,9]$$
$$+ [4,20] + [4,23] + [5,15] + [5,17](2) + [5,19] + [6,7] + [6,15] + [6,19]$$
$$+ [6,20] + [7,15] + [7,16] + [8,10] + [8,17] + [8,18](2) + [9,10] + [9,11]$$
$$+ [9,12] + [10,14] + [10,16] + [11,12] + [11,14] + [12,13] + [12,15]$$

$$v = 66 : [2,3] + [2,16] + [2,18] + [2,22] + [3,27](2) + [3,29] + [4,12] + [4,17] + [4,23]$$
$$+ [4,29] + [5,8] + [5,19] + [5,23] + [6,11] + [6,15] + [6,24] + [6,25]$$
$$+ [7,19](3) + [7,21] + [8,15] + [8,22] + [8,25] + [9,11] + [9,12] + [9,13]$$
$$+ [9,23] + [10,15] + [10,17] + [10,18] + [10,22] + [11,13] + [11,20]$$
$$+ [12,14] + [12,16] + [13,16] + [14,15] + [14,17] + [14,18] + [20,21]$$

$$v = 78 : [2,19] + [2,21] + [2,34] + [2,35] + [3,22] + [3,28] + [3,31](2) + [4,15]$$
$$+ [4,20] + [4,29] + [4,31] + [5,7] + [5,14] + [5,16] + [5,32] + [6,17]$$
$$+ [6,24] + [6,27] + [6,30] + [7,15] + [7,17] + [7,32] + [8,16] + [8,20]$$
$$+ [8,22] + [8,30] + [9,17] + [9,18] + [9,27](2) + [10,18] + [10,23]$$
$$+ [10,25] + [10,28] + [11,14](2) + [11,18] + [11,23] + [12,20]$$
$$+ [12,26](2) + [13,16] + [13,20] + [13,22] + [13,26] + [14,15]$$
$$+ [15,17] + [16,21] + [18,19]$$

$$v = 90 : [2,11] + [2,14] + [2,24] + [2,41] + [3,15] + [3,21] + [3,32] + [3,36] + [4,13]$$
$$+ [4,28](2) + [4,35] + [5,14] + [5,21] + [5,25] + [5,29] + [6,34](2)$$
$$+ [6,36] + [6,37] + [7,27] + [7,31](2) + [7,36] + [8,17] + [8,25] + [8,29]$$
$$+ [8,37] + [9,15] + [9,22] + [9,33] + [9,36] + [10,21] + [10,29](2)$$
$$+ [10,30] + [11,16] + [11,19] + [11,21] + [12,14] + [12,26]$$
$$+ [12,28] + [12,30] + [13,20] + [13,22] + [14,23] + [15,18]$$

$$+ [15,23] + [16,19] + [16,25] + [17,27](2) + [18,23] + [18,28] + [19,22] + [20,22] + [20,23] + [20,24] \qquad \square$$

4.6. Lemma. *There exists a CPD$(v,3,6;1)$ if and only if $v \equiv 4 \,\%\, 12$ and $v \geq 16$.*

Proof. By using Algorithm 2.5A, the sets of shifts are searched to construct CPD$(v,3,6;1)$'s for treatments $v \in \{16,\ 28,\ 40,\ 52,\ 64,\ 76,\ 88,\ 100\}$ which satisfy the necessary condition $\lambda = \left[\frac{r(k-1)}{v-3}\right]$.

$$v = 16 : [2,3] + [2,4] + [2,5](2) + [2,6] + [2,7] + [3,3] + [3,4](2) + [3,5] + [4,4] + [4,6] + [5,5]$$

$$v = 28 : [2,6](2) + [2,7] + [2,8] + [2,10] + [2,11] + [3,6] + [3,9](3) + [3,10](2) + [4,7](3) + [4,8] + [4,10](2) + [5,6](2) + [5,7] + [5,8](2) + [5,9] + [6,7]$$

$$v = 40 : [2,8] + [2,9] + [2,12] + [2,15] + [2,18](2) + [3,4] + [3,12] + [3,13](2) + [3,14](2) + [4,9] + [4,12] + [4,15](3) + [5,9] + [5,10] + [5,11] + [5,12](3) + [6,7](2) + [6,8](2) + [6,10] + [6,13] + [7,9] + [7,11](2) + [8,10](2) + [8,11] + [9,10] + [9,11]$$

$$v = 52 : [2,10](3) + [2,20](3) + [3,8](3) + [3,20](3) + [4,13](2) + [4,14](3) + [4,17] + [5,7](2) + [5,8] + [5,16](3) + [6,6] + [6,7] + [6,19](3) + [7,22](3) + [8,13](3) + [10,14](3) + [11,15](3) + [16,17](3)$$

$$v = 64 : [2,14](3) + [2,25](3) + [3,12](3) + [3,26](3) + [4,18](3) + [4,23](3) + [5,10](3) + [6,11] + [6,14] + [6,22](3) + [6,24] + [7,7] +]7,10] + [7,19](3) + [8,16](3) + [8,17](3) + [9,12](3) + [9,21](3) + [10,24](2) + [11,18](3) + [11,20](2) + [13,19](3) + [13,20](3) + [14,17]$$

$$v = 76 : [2,13](3) + [2,16](3) + [3,14](3) + [3,21](3) + [4,29](3) + [4,34](3) + [5,20](3) + [5,26](3) + [6,6] + [6,9](2) + [6,23](2) + [7,26](3) + [7,27](3) + [8,19](3) + [8,22](3) + [9,22](2) + [9,23](2) + [10,18](2) + [10,29] + [10,30](3) + [11,17](3) + [11,24](3) + [12,19] + [12,20](2) + [12,23](2) + [13,19](2) + [13,28] + [14,25](3) + [15,21] + [16,20] + [16,21](2) + [18,22]$$

$$v = 88 : [2,18](3) + [2,29](3) + [3,15](3) + [3,33](3) + [4,23](3) + [4,30] + [4,37](2) + [15,17](3) + [15,19](3) + [6,17](3) + [6,19](3) + [7,25](3) + [7,28](3) + [8,32](3) + [8,35](3) + [9,12](3) + [9,29](3) + [10,20](2) + [10,31](3) + [10,37] + [11,28](3) + [11,33](3) + [12,34](3) + [13,26](3) + [13,27](3) + [14,22](3) + [14,24](3) + [15,30](3) + [16,21](3) + [16,26](3) + [20,34]$$

$$v = 100 : [2,20] + [2,22](2) + [2,43](3) + [3,34](3) + [3,41](3) + [4,30](3) + [4,42](3) + [5,20](3) + [5,31] + [5,33](2) + [6,16](2) + [6,27] + [6,38](3) + [7,14](3) + [7,14](3) + [7,21](3) + [8,24](3) + [8,25](3) + [9,13](2) + [9,41] + [10,13](2) + [10,39](3) + [10,40] + [11,19](3) + [11,37](3) + [12,23] + [12,29](2) + [12,35](2) + [12,38] + [13,29](3) + [14,26](3) + [15,31](3) + [15,36](3) + [16,23](3) + [16,23](3) + [16,32]$$

$$+ [17, 18](3) + [17, 26](3) + [17, 26](3) + [18, 27](3)$$
$$+ [19, 28](3) + [20, 32](2) + [24, 29]$$ □

4.7. Lemma. *There exists a CPD$(v, 3, 6; 1)$ if and only if $v \equiv 8 \% 12$ and $v \geq 20$.*

Proof. Using Algorithm 2.5B, the sets of shifts are searched to construct CPD$(v, 3, 6; 1)$'s for treatments $v \in \{20,\ 32,\ 44,\ 56,\ 68,\ 80,\ 92\}$ which satisfy the necessary condition $\lambda = \left[\frac{r(k-1)}{v-3}\right]$.

$v = 20 : [2, 5](2) + [2, 6](2) + [2, 7] + [2, 8] + [3, 4] + [3, 5] + [3, 6](3) + [3, 7]$
$+ [4, 4] + [4, 5](2) + [4, 6] + [5, 7]$

$v = 32 : [2, 7] + [2, 8] + [2, 9] + [2, 10] + [2, 12] + [2, 14] + [3, 6] + [3, 9] + [3, 10](2)$
$+ [3, 12] + [3, 13] + [4, 7](2) + [4, 10] + [4, 11](2) + [4, 12] + [5, 6] + [5, 7]$
$+ [5, 8](2) + [5, 9] + [5, 10] + [6, 7] + [6, 8](2) + [6, 9] + [7, 8]$

$v = 44 : [2, 8] + [2, 12] + [2, 15] + [2, 18] + [2, 19](2) + [3, 6] + [3, 10] + [3, 11](2)$
$+ [3, 15] + [3, 16] + [4, 9](2) + [4, 12] + [4, 14](2) + [4, 16] + [5, 8] + [5, 11]$
$+ [5, 15] + [5, 16](2) + [5, 17] + [6, 9] + [6, 11] + [6, 12] + [6, 13] + [6, 15]$
$+ [7, 10](3) + [7, 12] + [7, 13] + [7, 15] + [8, 10] + [8, 11]$
$+ [8, 12] + [8, 14] + [9, 11] + [9, 12]$

$v = 56 : [2, 19] + [2, 20](5) + [3, 3](2) + [3, 16] + [3, 22] + [4, 71](6) + [5, 16](5)$
$+ [5, 20] + [6, 19](4) + [8, 18](6) + [9, 15](6) + [10, 13](6) + [12, 17](6)$
$+ [14, 14](3)$

$v = 68 : [2, 28](6) + [3, 4](5) + [3, 18] + [4, 17] + [5, 18](5) + [5, 24] + [6, 25](6)$
$+ [7, 22] + [8, 13] + [8, 14](5) + [9, 10] + [9, 17](5) + [10, 13] + [10, 19](4)$
$+ [11, 16](6) + [12, 14] + [12, 24](5) + [13, 19] + [13, 21](3) + [15, 20](6)$

$v = 80 : [2, 21](6) + [3, 27] + [3, 33](4) + [3, 37] + [4, 27](3) + [4, 32] + [4, 33] + [4, 36]$
$+ [5, 12](6) + [6, 12](2) + [6, 19] + [6, 33] + [6, 34] + [6, 37] + [7, 18](5)$
$+ [7, 31] + [8, 20] + [8, 22](5) + [9, 19](5) + [9, 22] + [10, 16] + [10, 27](2)$
$+ [10, 32](3) + [11, 24](6) + [13, 16](5) + [13, 18]$
$+ [14, 15] + [14, 20](5) + [15, 26](5)$

$v = 92 : [2, 8] + [2, 32](5) + [3, 39](6) + [4, 15](5) + [4, 36] + [5, 13] + [5, 14] + [5, 18](2)$
$+ [5, 27] + [5, 35] + [6, 25](6) + [7, 23] + [7, 28](5) + [8, 10](2) + [8, 16]$
$+ [8, 36] + [8, 38] + [9, 9] + [9, 24](3) + [9, 27] + [10, 23](3) + [11, 13]$
$+ [11, 26](5) + [12, 22] + [12, 26] + [12, 36](4) + [13, 27](4) + [14, 29](5)$
$+ [15, 29] + [16, 22](4) + [16, 27] + [17, 28]$
$+ [17, 30](5) + [20, 21](6) + [22, 24]$

□

4.8. Lemma. *There exists a CPD$(v, 3, 12; 1)$ if and only if $v \equiv 10\%12)$ and $v \geq 10$.*

Proof. By using Algorithm 2.6A, the sets of shifts are searched to construct CPD$(v, 3, 12; 1)$'s for treatments $v \in \{10,\ 22,\ 34,\ 46,\ 58,\ 70,\ 82,\ 94\}$ which satisfy the necessary condition $\lambda = \left[\frac{r(k-1)}{v-3}\right]$.

$v = 10 : [2, 2] + [2, 3](6) + [2, 4](4) + [3, 3](3)$

$$v=22:[2,3]+[2,5](2)+[2,6](3)+[2,7](2)+[2,8](4)+[3,5](2)+[3,6](5)$$
$$+[3,7](2)+[3,8](2)+[4,5](4)+[4,6](4)+[4,7](4)+[5,7](2)+[5,8]$$
$$v=34:[2,7](2)+[2,8]+[2,9](2)+[2,12](3)+[2,13](2)+[2,15](2)+[3,3]$$
$$+[3,8](2)+[3,9](4)+[3,13](4)+[4,7](2)+[4,8](2)+[4,10](6)$$
$$+[4,13](2)+[5,7](2)+[5,8](2)+[5,10](4)+[5,11](4)+[6,7](2)$$
$$+[6,8](3)+[6,9](2)+[6,10]+[6,11](2)+[6,12]+[7,8](2)+[7,9](2)$$
$$v=46:[2,10](2)+[2,11]+[2,12]+[2,18]+[2,20](5)+[2,21](2)+[3,5]+[3,11](7)$$
$$+[3,14](2)+[3,20](2)+[4,13](7)+[4,15](2)+[4,16](2)+[4,17]$$
$$+[5,16](9)+[5,17](2)+[6,8](2)+[6,12](9)+[6,16]+[7,8](7)+[7,11](2)$$
$$+[7,13](2)+[7,15]+[8,15](2)+[9,10](10)+[9,13](2)+[11,11]$$
$$v=58:[2,9](6)+[2,19](6)+[3,4](6)+[3,15]+[3,25](5)+[4,18](5)+[4,23]$$
$$+[5,11](6)+[5,12](6)+[6,15](2)+[6,16](6)+[6,23](2)+[6,25](2)$$
$$+[7,19](6)+[8,18](6)+[8,21](4)+[8,23](2)+[9,15](6)+[10,13](6)$$
$$+[10,20](6)+[12,13](5)+[12,15]+[13,22]$$
$$+[14,17](6)+[14,20](6)+[15,15]$$
$$v=70:[2,12](6)+[2,14](6)+[3,3](2)+[3,20](7)+[3,21]+[4,27](11)+[4,29]$$
$$+[5,8](6)+[5,17](5)+[5,26]+[6,24](10)+[7,15]+[7,18](6)+[7,20]$$
$$+[7,26](4)+[8,15](5)+[8,29]+[9,19](10)+[9,20](2)+[10,25](6)$$
$$+[10,26](6)+[11,13]+[11,21](11)+[12,17](6)+[13,13]+[13,17]$$
$$+[13,28](2)+[15,22](6)+[16,18](6)+[19,19]+[20,20]$$
$$v=82:[2,19]+[2,24](5)+[2,25](6)+[3,8](2)+[3,17](4)+[3,23](6)+[4,22]$$
$$+[4,30](6)+[4,32](5)+[5,33](6)+[5,35](6)+[6,16](6)+[6,19](6)$$
$$+[7,29](6)+[7,31](6)+[8,9](8)+[8,21](2)+[9,11](2)+[9,24]+[9,32]$$
$$+[10,11](2)+[10,18]+[10,19](4)+[10,22](5)+[11,20](6)+[12,27](6)$$
$$+[12,30](6)+[13,19]+[13,24](6)+[13,28](5)+[14,21](6)$$
$$+[14,23](6)+[15,18](5)+[15,21]+[15,28](6)+[16,18](6)$$
$$v=94:[2,19](6)+[2,36](6)+[3,22](6)+[3,32](6)+[4,9]+[4,11]+[4,22](4)$$
$$+[4,31](6)+[5,17]+[5,31](5)+[5,41](6)+[6,12](5)+[6,21](6)+[6,24]$$
$$+[7,8](6)+[7,33](6)+[8,29](6)+[9,9]+[9,11]+[9,17](2)+[9,32](6)$$
$$+[10,33](6)+[10,34](6)+[11,11]+[11,17]+[17,29](5)+[11,31]$$
$$+[11,34]+[12,18](6)+[12,24]+[13,24](6)+[13,25](6)+[14,24]$$
$$+[14,26]+[14,28](5)+[14,29]+[14,34]+[15,30](5)+[16,23](6)$$
$$+[16,28](6)+[17,17]+[17,26](5)+[17,29]+[19,23](6)$$
$$+[20,27](6)+[20,29](5)+[24,24]+[24,25]$$

□

4.9. Lemma. *There exists a CPD*$(v,3,12;1)$ *if and only if* $v\equiv 2\,\%\,12$ *and* $v\geq 14$.

Proof. Using Algorithm 2.6B, the sets of shifts are searched to construct CPD$(v,3,12;1)$'s for treatments $v\in\{14,\ 26,\ 38,\ 50,\ 62,\ 74,\ 86,\ 98\}$ which satisfy the necessary condition

$\lambda = \left[\frac{r(k-1)}{v-3}\right]$.

$$\begin{aligned}
v = 14 :\ & [2,3](4) + [2,4](5) + [2,5](3) + [3,4](3) + [3,5](5) + [4,4](2) \\
v = 26 :\ & [2,5](2) + [2,6](2) + [2,7](2) + [2,9](4) + [2,10](2) + [3,5](2) + [3,7](2) \\
& + [3,8](4) + [3,9](4) + [4,6](6) + [4,7] + [4,8](2) + [4,9](2) + [4,10] \\
& + [5,6](2) + [5,7](3) + [5,8](2) + [5,10] + [6,7](2) \\
v = 38 :\ & [2,8] + [2,9](2) + [2,11] + [2,13](6) + [2,16](2) + [3,5] + [3,6](2) + [3,7] \\
& + [3,9](2) + [3,14](4) + [3,15](2) + [4,6] + [4,7](2) + [4,9](2) + [4,12](4) \\
& + [4,14](3) + [5,9](2) + [5,10](2) + [5,11](4) + [5,12](2) + [5,13] + [6,8](3) \\
& + [6,11](2) + [6,12](2) + [6,13](2) + [7,8] + [7,9](2) \\
& + [7,10](4) + [7,12](2) + [8,10](3) + [8,11](2) \\
v = 50 :\ & [2,5] + [2,7] + [2,15](2) + [2,16](2) + [2,18](2) + [2,19](2) + [2,21](2) \\
& + [3,4](2) + [3,10] + [3,13](3) + [3,18](2) + [3,19](4) + [4,10](2) + [4,11](2) \\
& + [4,17](2) + [4,18](2) + [4,20](2) + [5,7](4) + [5,12] + [5,13](2) \\
& + [5,14](2) + [5,17](2) + [6,9](2) + [6,13](2) + [6,14](2) + [6,15](2) \\
& + [6,17](2) + [6,18](2) + [7,10] + [7,12](2) + [7,16] + [8,15](4) \\
& + [8,16](6) + [8,17](2) + [9,11](6) + [9,13] + [9,14](2) + [10,11](2) \\
& + [10,12](3) + [10,13] + [10,14](2) + [11,14](2) + [12,13](2) \\
v = 62 :\ & [2,6] + [2,14](6) + [2,26](5) + [3,19](6) + [3,23](6) + [4,17](6) + [4,27](6) \\
& + [5,15] + [5,20](5) + [5,24](6) + [6,11](6) + [6,15] + [6,18] + [6,23] \\
& + [6,27](2) + [7,8](5) + [7,13](6) + [7,21] + [8,16](5) + [8,18] \\
& + [9,12](4) + [9,16] + [9,18] + [9,19](6) + [10,13](6) + [10,15] \\
& + [10,22](6) + [11,14](5) + [11,18] + [12,15](2) \\
& + [12,18](6) + [13,14] + [15,18](2) \\
v = 74 :\ & [2,13] + [2,16](5) + [2,23](5) + [2,31] + [3,16](5) + [3,24] + [3,28](6) \\
& + [4,19](6) + [4,25](6) + [5,15] + [5,28](6) + [5,32](5) + [6,11](5) \\
& + [6,15](5) + [6,25] + [6,30] + [7,8] + [7,13](5) + [7,17] + [7,27](5) \\
& + [8,21](5) + [8,24](6) + [9,12] + [9,26](6) + [9,30](5) + [10,10] \\
& + [10,11] + [10,13] + [10,17](6) + [10,24](2) + [11,13] + [11,15] \\
& + [11,22](5) + [12,18](6) + [12,26](5) + [13,18] + [13,19] + [13,22] \\
& + [13,24] + [14,15] + [14,20](5) + [14,22](6) + [15,16](2) \\
v = 86 :\ & [2,17](5) + [2,22] + [2,32](6) + [3,3](2) + [3,10] + [3,27] + [3,28](6) \\
& + [4,19](6) + [4,24](5) + [4,36] + [5,29](6) + [5,37](6) + [6,10](4) \\
& + [6,20](6) + [7,18](6) + [7,36](6) + [8,8](2) + [8,9] + [8,11] \\
& + [8,30](6) + [9,22](5) + [9,32] + [9,36](5) + [10,21] + [10,29](6) \\
& + [11,21](5) + [11,26](6) + [12,21](6) + [12,23](6) + [13,27](11) \\
& + [14,25](6) + [14,28] + [14,30](5) + [15,18](6) \\
& + [15,20](6) + [16,22](6) + [17,24](6) \\
v = 98 :\ & [2,17](5) + [2,39](6) + [2,43] + [3,18](6) + [3,39](6) + [4,22] + [4,25](6) \\
& + [4,32](5) + [5,9](2) + [5,40](4) + [5,43](6) + [6,9] + [6,22](5) + [6,26]
\end{aligned}$$

$$+ [6,37](5) + [7,19] + [7,28](6) + [7,30](4) + [7,40] + [8,16](6)$$
$$+ [8,26](6) + [9,17] + [9,22](6) + [9,27] + [9,28] + [10,27](2)$$
$$+ [10,31](4) + [10,34](6) + [11,30] + [11,31] + [11,38](6) + [11,40](4)$$
$$+ [12,15] + [12,23](6) + [12,30](5) + [13,20](6) + [13,27]$$
$$+ [13,32](5) + [14,26](2) + [14,27] + [14,31] + [14,36](6)$$
$$+ [15,15] + [15,23](6) + [15,30] + [15,32] + [16,17](6)$$
$$+ [18,29](6) + [19,27](6) + [20,24](6) + [21,25](6)$$

□

5. Concluding remarks

Hedayat *et al.* [5] first introduced CPDs (in terms of BSECs) and gave constructions of CPD$(9,3,1;1)$, CPD$(10,3,12;1)$ and CPD$(11,3,3;1)$. See *et al.* [11] constructed CPDs (in terms of PBIBDs) for CPD$(v,3,1;1)$'s for $v \in \{9,15,21,27,33\}$, and CPD$(v,3,2;1)$'s for $v \in \{9,12,15\}$.

Colbourn and Ling [3] used PTS(v,λ) and gave a complete solution for CPD$(v,3,\lambda;1)$ with some λ for $v \in$ {9, 11, 12, 13, 14, 15, 16, 17, 18,19, 20, 21, 22, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 39, 42, 45, 51, 88}. Wei [16] used the Langford sequence to show the existence of CPDs (in terms of CBSEs) with block size $k = 3$ and $\lambda = 1, 2$. Zhang and Chang [17] gave the existence and construction of CPDs (in terms of BSAs) for $k = 3$ and $\alpha = 1$ for some v.

Mandal *et al.* [8] used symmetrically repeated differences and a linear programming technique to construct CPDs with $k = 3$ and $\alpha = 1$ for some $v \in$ {16, 18, 19, 20, 2, 22, 23, 24, 25, 26, 27, 29, 32, 33, 35}.

It can be noted from the work of Colbourn and Ling [3], Zhang and Chang [17] and Mandal *et al.* [8] that CPD$(v,3,\lambda;1)$'s are available only for some λ and limited v. In this paper, CPD$(v,3,\lambda;1)$'s (in terms of BSECs) with $\lambda = 1,2,3,4,6,12$ are constructed to provide complete solution for treatments $v \in \{9,10,\ldots,100\}$ (see Table 1).

Table 1. Proposed CPD$(v,3,\lambda;1)$'s with $\lambda = 1,2,3,4,6,12$ for $v \in \{9,10,\ldots,100\}$

λ	v (CPDs using the method of cyclic shifts)	Existence
1	{9, 15, 21, 27, 33, 39, 45, 51, 57, 63, 69, 75, 81, 87, 93, 99}	$v \equiv 3\,\%\,6$
2	{12, 24, 36, 48, 60, 72, 84, 96}	$v \equiv 0\,\%\,12$
3	{13, 19, 25, 31, 37, 43, 49, 55, 61, 67, 73, 79, 85, 91, 97}	$v \equiv 1\,\%\,6$
3	{11,17, 23, 29, 35, 41, 47, 53, 59, 65, 71, 77, 83, 89, 95}	$v \equiv 5\,\%\,6$
4	{18, 30, 42, 54, 66, 78, 90}	$v \equiv 6\,\%\,12$
6	{16, 28, 40, 52, 64, 76, 88, 100}	$v \equiv 4\,\%\,12$
6	{20, 32, 44, 56, 68, 80, 92}	$v \equiv 8\,\%\,12$
12	{10, 22, 34, 46, 58, 70, 82, 94}	$v \equiv 10\,\%\,12$
12	{14, 26, 38, 50, 62, 74, 86, 98}	$v \equiv 2\,\%\,12$

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