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A PREY PREDATOR MODEL WITH FUZZY INITIAL VALUES

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Abstract

In this paper we consider a prey-predator model with fuzzy initial values. We use the concept of generalized differentiability and obtain graphical solutions for the problem under consideration.

Keywords: Fuzzy number, Fuzzy derivative, Fuzzy differential equations (FDE), Fuzzy initial values, Fuzzy prey-predator model.

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1. Introduction

Differential equations are indispensable for modeling real world phenomena. Unfortunately every time uncertainty can intervene with real world problems. The uncertainty can arise from deficient data, measurement errors or when determining initial conditions. Fuzzy set theory is a powerful tool to overcome these problems. Initially, the derivative for fuzzy valued mappings was developed by Puri and Ralescu [8], and generalized and extended the concept of Hukuhara differentiability for set-valued mappings to the class of fuzzy mappings. Subsequently, using the Hukuhara derivative, Kaleva [5] started to develop a theory for FDE. But it soon appeared that the Hukuhara derivative has a shortcoming which fuzzifies the the solution as time goes on. To overcome this situation in [1, 2] the concept of strongly generalized derivative was introduced and in [6] this concept studied for higher order fuzzy differential equations. This concept allows us to overcome the above mentioned shortcoming.

We first recall some basic concepts used in the present paper.

A fuzzy set A in a universe set X is a mapping $A(x) : X \to [0, 1]$. We think of A as assigning to each element $x \in X$ a degree of membership, $0 \leq A(x) \leq 1$. Let us denote by $\mathcal{F}$ the class of fuzzy subsets of the real axis, $A(x) : X \to [0, 1]$ satisfying the following properties:

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i) A is a convex fuzzy set, i.e. $A(r\lambda + (1-\lambda)s) \geq \min[A(r), A(s)]$, $\lambda \in [0,1]$ and $r, s \in X$
ii) A is normal, i.e. $\exists x_0 \in X$ with $A(x_0) = 1$;
iii) A is upper semicontinuous, i.e. $A(x_0) \geq \lim_{x \to x_o^{\pm}} A(x)$;
iv) $[A]^0 = \overline{\sup p(A)} = \overline{\{x \in R \mid A(x) \geq 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

Then $\mathcal{F}$ is called the space of *fuzzy numbers*.

If A is a fuzzy set, we define $[A]^\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$, the *α-level (cut) of* A, with $0 < \alpha \leq 1$. For $u, v \in \mathcal{F}$ and $\lambda \in R$ the sum $u \oplus v$ and the product $\lambda \odot u$ are defined by $[u \oplus v]^\alpha = [u]^\alpha + [v]^\alpha$, $\lambda \odot u = \lambda [u]^\alpha \, \forall \alpha \in [0,1]$. Additionally, $u \oplus v = v \oplus u$, $\lambda \odot u = u \odot \lambda$. Also if $u \in \mathcal{F}$ the α-cut of u, denoted by $[u]^\alpha = [\underline{u^\alpha}, \overline{u^a}]$, $\forall \alpha \in [0,1]$.

Let $D : \mathcal{F} \times \mathcal{F} \to R_+ \cup \{0\}$, $D(u,v) = \sup_{\alpha \in [0,1]} \max\{|\underline{u^\alpha} - \underline{v^\alpha}|, |\overline{u^a} - \overline{v^a}|\}$ be the *Hausdorff distance between fuzzy numbers*, where $[u]^\alpha = [\underline{u^\alpha}, \overline{u^a}]$ and $[v]^\alpha = [\underline{v^\alpha}, \overline{v^a}]$. The following properties are well-known [4, 9].

$$D(u \oplus w, v \oplus w) = D(u,v), \ \forall u, v, w \in \mathcal{F},$$
$$D(k \odot u, k \odot v) = |k| \, D(u,v), \ \forall k \in R, \ u, v \in \mathcal{F},$$
$$D(u \oplus v, w \oplus e) \leq D(u,w) + D(v,e), \ \forall u, v, w, e \in \mathcal{F},$$

and $(\mathcal{F}, D)$ is a complete metric space.

2. The fuzzy derivative

2.1. Definition. (H-Difference) Let be $u, v \in \mathcal{F}$. If there exists $w \in \mathcal{F}$ such that $u = v \oplus w$, then w is called the *H-difference of* u *and* v and is denoted by $u \ominus v$.

2.2. Definition. (Hukuhara Derivative)[8] Consider a fuzzy mapping $F : (a,b) \to \mathcal{F}$ and $t_0 \in (a,b)$. We say that F is *differentiable at* $t_0 \in (a,b)$ if there exists an element $F'(t_0) \in \mathcal{F}$ such that for all $h > 0$ sufficiently small $\exists F(t_0 + h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0 - h)$ and the limits (in the metric D)

$$\lim_{h \to 0^+} \frac{F(t_0 + h) \ominus F(t_0)}{h} = \lim_{h \to 0^-} \frac{F(t_0) \ominus F(t_0 - h)}{h}$$

exist and are equal to $F'(t_0)$.

Note that this definition of the derivative is very restrictive; for instance in [1,2] the authors showed that if $F(t) = c.g(t)$ where c is a fuzzy number and $g : [a,b] \to R^+$ is a function with $g'(t) < 0$, then F is not differentiable. To avoid this difficulty, the authors of [1, 2] introduce a more general definition of the derivative for fuzzy mappings.

2.3. Definition. (Generalized Fuzzy Derivative)[1, 2] Let $F : (a,b) \to \mathcal{F}$ and $t_0 \in (a,b)$. We say that F is *strongly generalized differentiable at* t_0 if there exists an element $F'(t_0) \in \mathcal{F}$ such that

i. for $h > 0$ sufficiently small $\exists F(t_0 + h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0 - h)$, and the limits satisfy

$$\lim_{h \to 0} \frac{F(t_0 + h) \ominus F(t_0)}{h} = \lim_{h \to 0} \frac{F(t_0) \ominus F(t_0 - h)}{h} = F'(t_0),$$

or

ii. for $h > 0$ sufficiently small $\exists F(t_0) \ominus F(t_0 + h)$, $F(t_0 - h) \ominus F(t_0)$ and the limits satisfy

$$\lim_{h \to 0} \frac{F(t_0) \ominus F(t_0 + h)}{(-h)} = \lim_{h \to 0} \frac{F(t_0 - h) \ominus F(t_0)}{(-h)} = F'(t_0),$$

or

iii. for $h > 0$ sufficiently small $\exists F(t_0+h) \ominus F(t_0)$, $F(t_0-h) \ominus F(t_0)$, and the limits satisfy

$$\lim_{h\to 0} \frac{F(t_0+h) \ominus F(t_0)}{h} = \lim_{h\to 0} \frac{F(t_0-h) \ominus F(t_0)}{(-h)} = F'(t_0),$$

or

iv. for $h > 0$ sufficiently small $\exists F(t_0) \ominus F(t_0+h)$, $F(t_0) \ominus F(t_0-h)$ and the limits satisfy

$$\lim_{h\to 0} \frac{F(t_0) \ominus F(t_0+h)}{(-h)} = \lim_{h\to 0} \frac{F(t_0) \ominus F(t_0-h)}{h} = F'(t_0).$$

In [3], a definition equivalent to Definition 2.3. is given. We will use this in our work.

2.4. Definition. Let $F : (a,b) \to \mathcal{F}$ and $t_0 \in (a,b)$.

(1) for $h > 0$ sufficiently small $\exists F(t_0+h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0-h)$ and

$$\lim_{h\to 0^+} \frac{F(t_0+h) \ominus F(t_0)}{h} = \lim_{h\to 0^+} \frac{F(t_0) \ominus F(t_0-h)}{h} = F'(t_0)$$

or

(2) for $h > 0$ sufficiently small $\exists F(t_0+h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0-h)$ and

$$\lim_{h\to 0^-} \frac{F(t_0+h) \ominus F(t_0)}{h} = \lim_{h\to 0^-} \frac{F(t_0) \ominus F(t_0-h)}{h} = F'(t_0).$$

2.5. Theorem. [3, 5] *Let $F : T \to \mathcal{F}$ be a function and set $[F(t)]^\alpha = [f_\alpha(t), g_\alpha(t)]$ for each $\alpha \in [0,1]$. Then*

1. *If F is differentiable in the first form (1), then $f_\alpha(t)$ and $g_\alpha(t)$ are differentiable functions and $[F'(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)]$.*
2. *If F is differentiable in the second form (2), then $f_\alpha(t)$ and $g_\alpha(t)$ are differentiable functions and $[F'(t)]^\alpha = [g'_\alpha(t), f'_\alpha(t)]$.* □

This theorem is fundamental in solving fuzzy differential equations.

3. Solving fuzzy differential equations with fuzzy initial values

Consider the equation

(3.1) $\quad x'(t) = F(t, x(t)),\ x(0) = x_0,$

where $F : [0,\alpha] \times \mathcal{F} \to \mathcal{F}$ and x_0 is a fuzzy number, $[x(t)]^\alpha = [u_\alpha(t), v_\alpha(t)]$, $[x_0]^\alpha = [u^0_\alpha, v^0_\alpha]$ and $[F(t,x(t))]^\alpha = [f_\alpha(t, u_\alpha(t), v_\alpha(t)), g_\alpha(t, u_\alpha(t), v_\alpha(t))]$.

Then, we have the following alternatives for solving the initial value problem (3.1):

1- If we consider $x'(t)$ by using the derivative in the first form (1), then from Theorem 2.5 $[x'(t)]^\alpha = [u'_\alpha(t), v'_\alpha(t)]$. So we get the following equalities:

$$u'_\alpha(t) = f_\alpha(t, u_\alpha(t), v_\alpha(t)),\ u_\alpha(0) = u^0_\alpha,$$
$$v'_\alpha(t) = g_\alpha(t, u_\alpha(t), v_\alpha(t)),\ v_\alpha(0) = v^0_\alpha.$$

By solving the above system for u_α and v_α, we get the fuzzy solution $[x(t)]^\alpha = [u_\alpha(t), v_\alpha(t)]$. Finally we ensure that $[x(t)]^\alpha = [u_\alpha(t), v_\alpha(t)]$ and $[x'(t)]^\alpha = [u'_\alpha(t), v'_\alpha(t)]$ are valid level sets.

2- If we consider $x'(t)$ by using the derivative in the second form (2), then from Theorem 2.5 $[x'(t)]^\alpha = [u'_\alpha(t), v'_\alpha(t)]$. So we get the following:

$$u'_\alpha(t) = g_\alpha(t, u_\alpha(t), v_\alpha(t)),\ u_\alpha(0) = u^0_\alpha,$$
$$v'_\alpha(t) = f_\alpha(t, u_\alpha(t), v_\alpha(t)),\ v_\alpha(0) = v^0_\alpha.$$

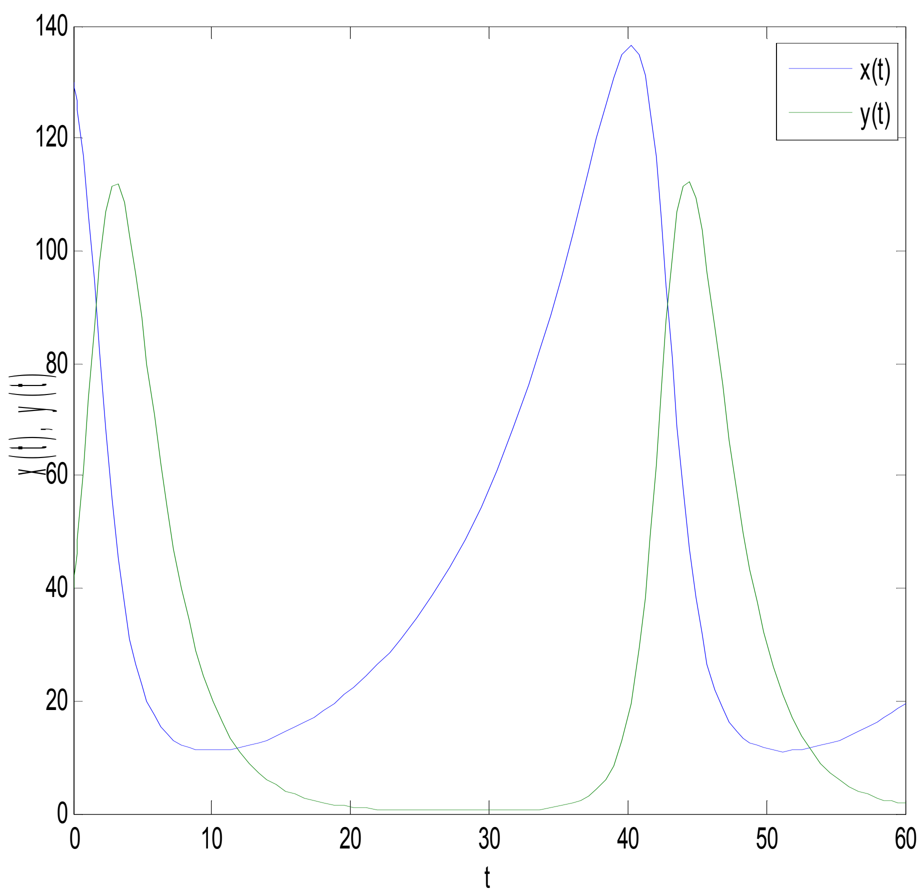
Solving the above system for u_α and v_α, we get the fuzzy solution $[x(t)]^\alpha = [u_\alpha(t), v_\alpha(t)]$. Finally we ensure that $[x(t)]^\alpha = [u_\alpha(t), v_\alpha(t)]$ and $[x'(t)]^\alpha = [u'_\alpha(t), v'_\alpha(t)]$ are valid level sets.

Now we consider the following prey-predator model with fuzzy initial values. Before giving a solution of the fuzzy problem we give a crisp solution.

$$
\begin{aligned}
&\frac{dx}{dt} = 0.1x - 0.005xy,\\
&\frac{dy}{dt} = -0.4y + 0.008xy,\\
&x(0) = 130 \in \mathcal{F},\ y(0) = 40 \in \mathcal{F},
\end{aligned}
\tag{3.2}
$$

where $x(t)$ and $y(t)$ are the number of preys and predators at time t, respectively. Crisp solutions for the problem (3.2) are given in Figure 1.

Figure 1. Crisp solution for Problem (3.2)



Let the initial values be fuzzy, e.g. $x(0) = \widetilde{130}$, $y(0) = \widetilde{40}$ and let their α-level sets be as follows.

$$\begin{aligned} x(0) &= \left[\widetilde{130}\right]^{\alpha} = [100 + 30\alpha, 160 - 30\alpha]\,, \\ y(0) &= \left[\widetilde{40}\right] = [20 + 20\alpha, 60 - 20\alpha]\,. \end{aligned}$$

Let the α-level sets of $x(t, \alpha)$ be $[x(t, \alpha)]^{\alpha} = [u(t, \alpha), v(t, \alpha)]$, and for simplicity denote them as $[u, v]$, similarly $[y(t, \alpha)]^{\alpha} = [r(t, \alpha), s(t, \alpha)] = [r, s]$.

According to Definition 2.4, $x(t, \alpha)$ may be differentiable in the first form (1) or in the second form (2), similarly $y(t, \alpha)$ may be differentiable in the first form (1) or in the second form (2). So we have four different initial value problems. We give two of them explicitly, the other cases can be obtained similarly. But we give graphical solutions of all cases.

If $x(t, \alpha)$ and $y(t, \alpha)$ are (1) differentiable according to Definition 2.4, Problem 3.2 becomes

$$\begin{aligned} [u, v] &= 0.1\,[u, v] - 0.005\,[u, v]\,.\,[r, s]\,, \\ [r, s] &= -0.4\,[r, s] + 0.008\,[u, v]\,[r, s]\,. \end{aligned}$$

Hence for $\alpha = 0$ the following initial value problem derives from 3.2:

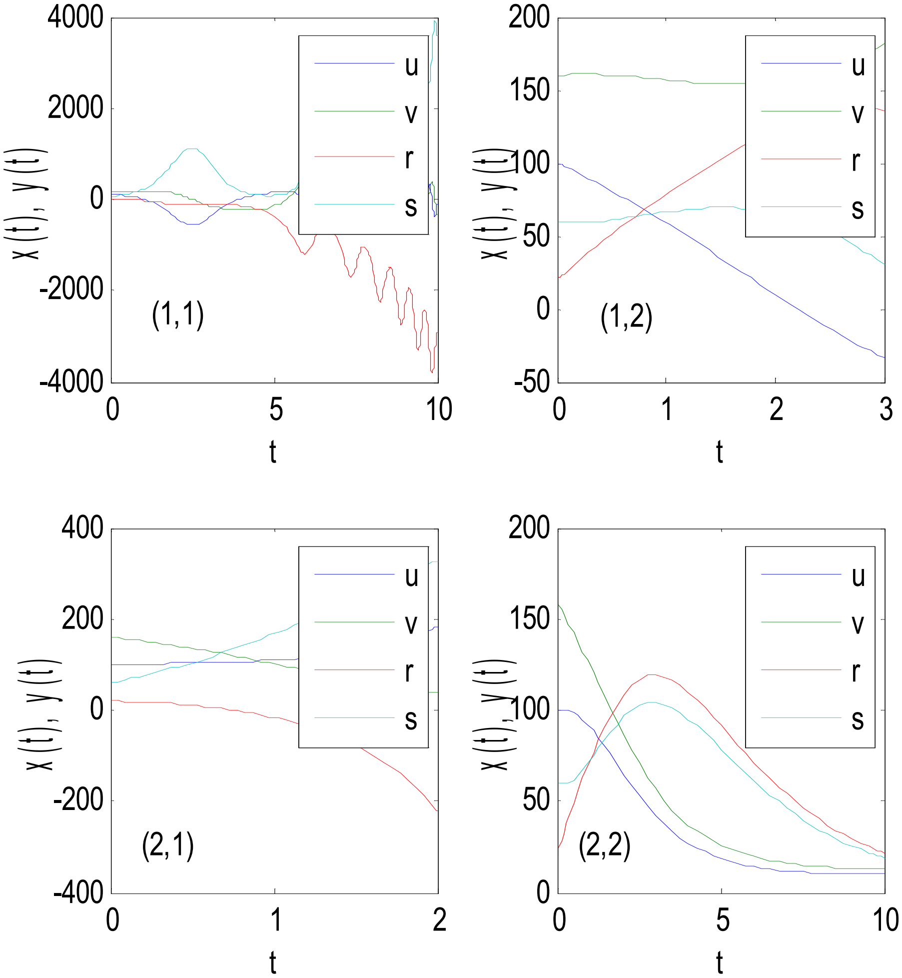
$$\begin{aligned} u' &= 0.1u - 0.005vs, \\ v' &= 0.1v - 0.005ur, \\ r' &= 0.008ur - 0.4s, \\ s' &= 0.008vs - 0.4r, \\ u(0) &= 100,\ v(0) = 160,\ r(0) = 20,\ s(0) = 60. \end{aligned}$$

Now if $x(t, \alpha)$ and $y(t, \alpha)$ are (2) differentiable according to Definition 2.4, Problem 3.2 becomes

$$\begin{aligned} v' &= 0.1u - 0.005vs, \\ u' &= 0.1v - 0.005ur, \\ s' &= 0.008ur - 0.4s, \\ r' &= 0.008vs - 0.4r, \\ u(0) &= 100,\ v(0) = 160,\ r(0) = 20,\ s(0) = 60. \end{aligned}$$

for $\alpha = 0$ the graphical solutions of all cases are given in Figure 2.

Figure 2. Fuzzy solutions of Problem (3.2) for $\alpha = 0$



In Figure 2, (1,1) means that $x(t, \alpha)$ and $y(t, \alpha)$ are (1) differentiable according to Definition 2.4, (2,1) means $x(t, \alpha)$ is (2) differentiable and $y(t, \alpha)$ is (1) differentiable, and so on. Now if we analyze Figure 2. we observe that when $x(t, \alpha)$ and $y(t, \alpha)$ are (2) differentiable the graphical solution is biologically meaningful, furthermore the graphical solution is coherent with the crisp solution. On the contrary, when $x(t, \alpha)$ and $y(t, \alpha)$ are differentiable as (1,1), (1,2), (2,1) the graphical solutions are incompatible with biological facts.

So we focus on the situation when $x(t, \alpha)$ and $y(t, \alpha)$ are (2) differentiable. We give the crisp graphical solution and fuzzy graphical solution when $x(t, \alpha)$ and $y(t, \alpha)$ are (2) differentiable on the same graph for $\alpha = 0$ and $\alpha \in [0, 1]$. The crisp solution and fuzzy solution for $\alpha = 0$ are given in Figure 3.

Figure 3. Crisp solution and fuzzy solution for $\alpha = 0$

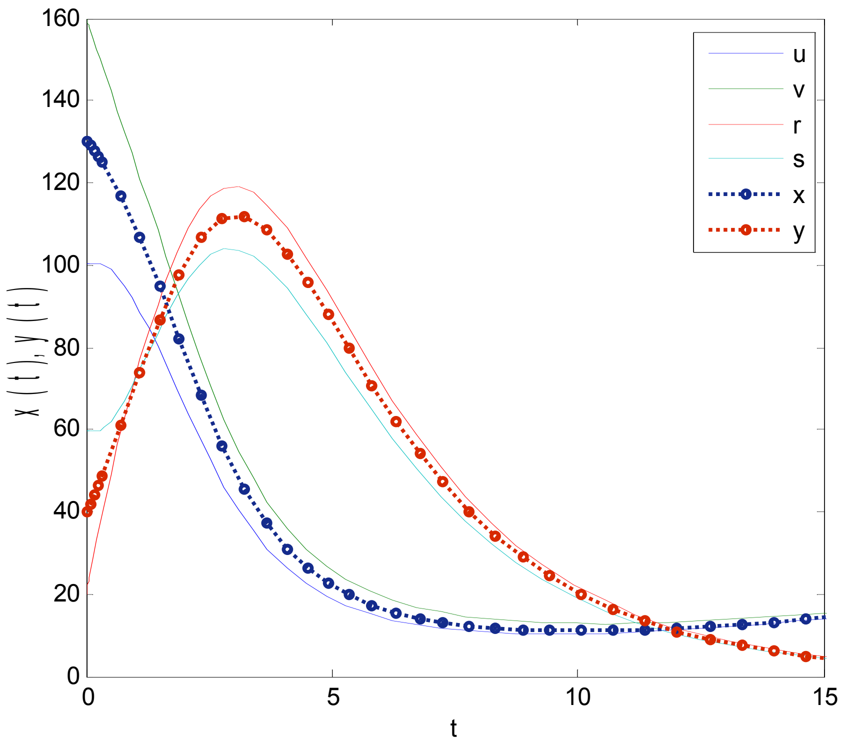


Figure 4. $x(t)$ for $\alpha \in [0, 1]$

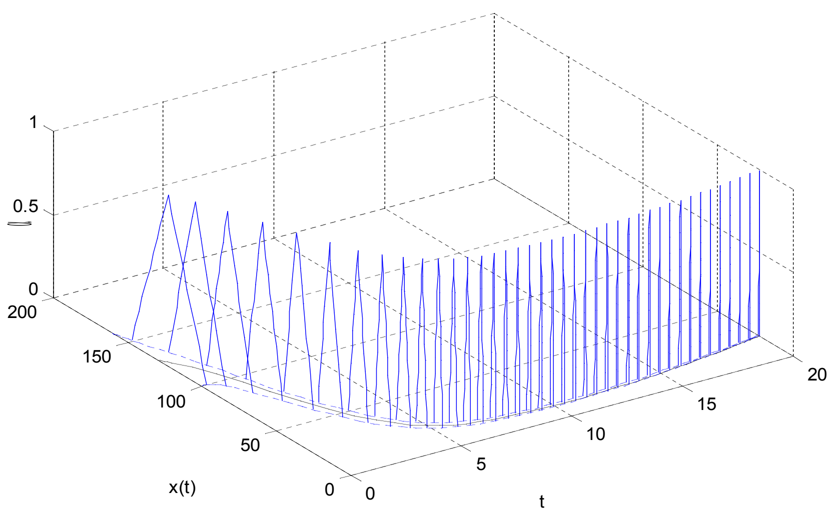
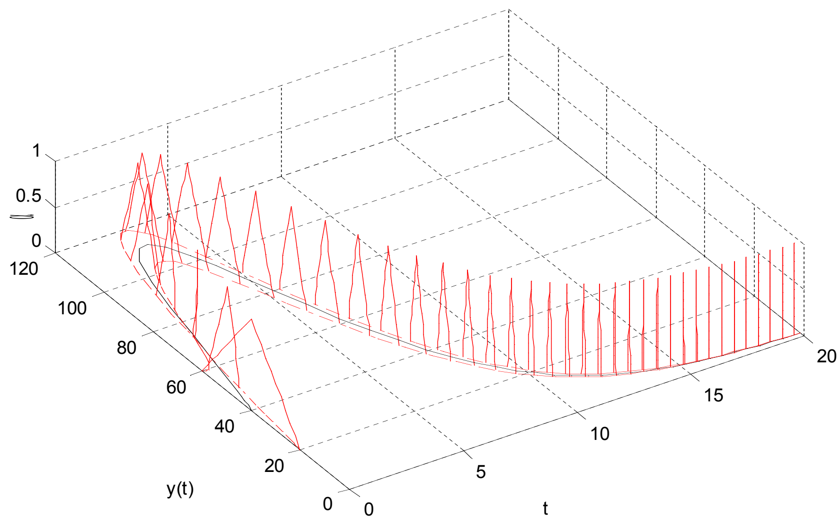


Figure 5. $y(t)$ for $\alpha \in [0,1]$



3.1. Remark. In Figures 4 and 5, if we set $\alpha = 0$ we see the crisp solution confined by the left and right branches of the dependent variables $x(t),\ y(t)$. For example in Figure 4, $x(t)$ is confined by u and v for $\alpha = 0$. Additionally, if we set $\alpha = 1$ the projection of the peaks of the triangles coincides with the crisp solution.

4. Conclusions

When vagueness occurs in problems which are analyzed, fuzzy differential equations arise naturally. Expressing vagueness with fuzzy sets is a more realistic approach than the classical one, so using fuzzy differential equations is unavoidable.

As we know from fuzzy theory if we have the α-cut $[u]^\alpha = [\underline{u^\alpha}, \overline{u^a}]$ of a fuzzy number, then the inequality $\underline{u^\alpha} \le \overline{u^a}$ must be satisfied. But we can observe from the graphs of the solution curves this rule does not always hold. We propose two alternatives for this shortcoming. The first is using crisp solutions instead of fuzzy solutions in intervals where the rule does not hold, and for the second we can take $\underline{u^{*\alpha}} = \min\left(\underline{u^\alpha}, \overline{u^a}\right)$ and $\overline{u^a} = \max\left(\underline{u^\alpha}, \overline{u^a}\right)$. Then $[u^*]^\alpha = \left[\underline{u^{*\alpha}}, \overline{u^{*a}}\right]$ is the α-cut of the new situations.

As we see in this work, the uniqueness of the solution of a fuzzy initial value problem is lost when we use the strongly generalized derivative concept. This situation is looked on as a disadvantage, but actually it is not [2, 7] because researchers can choose the best solution which better reflects the behavior of the system under consideration, from multiple solutions. Here a question may arise. Which solution is the best? The answer for this may come after a precise analysis of the physical properties of the system which is under study.

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