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FUZZY SOFT TOPOLOGY

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Abstract

In the present paper we introduce the topological structure of fuzzy soft sets and fuzzy soft continuity of fuzzy soft mappings. We show that a fuzzy soft topological space gives a parametrized family of fuzzy topological spaces. Furthermore, with the help of an example it is shown that the constant mapping is not continuous in general. Then the notions of fuzzy soft closure and interior are introduced and their basic properties are investigated. Finally, the initial fuzzy soft topology and some properties of projection mappings are studied.

Keywords: Fuzzy soft sets, Fuzzy soft topology, Fuzzy soft continuity, Fuzzy soft closure and interior, Fuzzy soft product topology.

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1. Introduction

The notion of a fuzzy set was introduced by Zadeh [21] in his classical paper of 1965. Three years later, Chang [4] gave the definition of fuzzy topology, which is a family of fuzzy sets satisfying the three classical axioms. Since Chang applied fuzzy set theory into topology many topological notions were introduced in a fuzzy setting. In 1976, Lowen [9] introduced a more natural definition of fuzzy topology which was different from Chang's definition.

In 1999, the Russian researcher Molodtsov [14] introduced the concept of a soft set, and started to develop the basics of the corresponding theory as a new approach for modeling uncertainties. He pointed out several directions for the applications of soft sets, such as game theory, Riemann integration, theory of measurement, smoothness of functions and so on. At present, works on soft set theory and its applications are progressing rapidly in various fields. Maji *et al.* [11, 12] presented some new definitions on soft sets and discussed in detail the application of soft set theory in decision making problems. Chen *et al.* [5] studied the parametrization reduction of soft sets. Maji *et al.* [10] combined fuzzy sets and soft sets and introduced the concept of fuzzy soft sets. To continue the

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investigation on fuzzy soft sets, Ahmad and Kharal [1] presented some more properties of fuzzy soft sets and introduced the notion of a mapping on fuzzy soft sets.

The theoretical structures of soft set and fuzzy soft set theories have been studied increasingly in recent years. Aktaş and Cağman [2] defined the notion of soft groups and derived some properties. Feng *et al.* [7] initiated the study of soft semirings by using the soft set theory and investigated several related properties. By using the t-norm, the concept of fuzzy soft group was introduced by Aygünoğlu and Aygün [3]. In 2010, Nazmul and Samanta [15] defined soft topological groups, normal soft topological groups and homomorphisms. Furthermore, Shabir and Naz [18] introduced the concept of soft topological space and studied neighborhoods and separation axioms. As a different approach to soft topology, B. Pazar Varol *et al.* [16] interpreted categories related to categories of topological spaces as special categories of soft sets. In 2011, Tanay *et al.* [19] gave the topological structure of fuzzy soft sets.

In the present study we consider the topological structure of fuzzy soft set theory. Firstly, as a preliminaries, we give some basic definitions and results in fuzzy soft set theory. After giving these preliminaries, we define fuzzy soft topology in Chang's sense (It is similar to Tanay and Kandemir's definition). It is shown that a fuzzy soft topological space gives a parametrized family of fuzzy topological spaces. Furthermore, we also define fuzzy soft topology in Lowen's sense and call it enriched fuzzy soft topology. We introduce fuzzy soft continuity of fuzzy soft mappings and with the help of an example it is shown that a constant mapping is not continuous in general. However, it is continuous between enriched fuzzy soft topological spaces. Then we study the fuzzy soft closure and fuzzy soft interior operators. Finally, we introduce the initial fuzzy soft topology and study its topological properties.

2. Fuzzy soft set theory

In this section, we give new definitions and various results on fuzzy soft set theory. Throughout this paper, let X be a nonempty set refereed to as the universe, E the set of all parameters for the universe X and $A \subseteq E$.

2.1. Definition. [21] A fuzzy set f on X is a mapping $f : X \to I$. The value $f(x)$ represents the degree of membership of $x \in X$ in the fuzzy set f, for $x \in X$.

Let I^X denotes the family of all fuzzy sets on X. If $f, g \in I^X$ then some basic set operation for fuzzy sets are given by Zadeh [21] as follows:

(1) $f \leqslant g \iff f(x) \leqslant g(x)$, for all $x \in X$.
(2) $f = g \iff f(x) = g(x)$, for all $x \in X$.
(3) $h = f \vee g \iff h(x) = f(x) \vee g(x)$, for all $x \in X$.
(4) $k = f \wedge g \iff k(x) = f(x) \wedge g(x)$, for all $x \in X$.
(5) $t = f^c \iff t(x) = 1 - f(x)$, for all $x \in X$.

2.2. Definition. [14] A pair (F, A) is called a *soft set over* X if F is a mapping defined by $F : A \longrightarrow 2^X$, where 2^X is the power set of X.

In other words, a soft set is a parameterized family of subsets of the set X. Each set $F(e)$, $e \in A$, from this family may be considered as the set of e-elements of the soft set (F, A).

2.3. Definition. [10] A pair (f, A) is called a *fuzzy soft set over* X, where $f : A \longrightarrow I^X$ is a function.

That is, for each $a \in A$, $f(a) = f_a : X \longrightarrow I$ is a fuzzy set on X.

According to [13], a soft set (F, A) can be extended to a *soft set type* (F, E), where $F(e) \neq \emptyset$ if $e \in A \subseteq E$ and $F(e) = \emptyset$ if $e \in E - A$. Thus,

2.4. Definition. [13] A soft set F_A on the universe X is a mapping from the parameter set E to 2^X, i.e., $F_A : E \to 2^X$, where $F_A(e) \neq \emptyset$ if $e \in A \subseteq E$ and $F_A(e) = \emptyset$ if $e \notin A$.

The subscript A in the notation F_A indicates where the image of F_A is non-empty.

A soft set can be defined by the set of ordered pairs

$$F_A = \{(e, F_A(e)) : e \in E, F_A(e) \in 2^X\}$$

The value $F_A(e)$ is a set called the e-element of the soft set for all $e \in E$. [6]

Analogously to the above ideas we can offer the definition of "fuzzy soft set" as follows:

2.5. Definition. A fuzzy soft set f_A on the universe X is a mapping from the parameter set E to I^X, i.e., $f_A : E \to I^X$, where $f_A(e) \neq 0_X$ if $e \in A \subseteq E$ and $f_A(e) = 0_X$ if $e \notin A$, where 0_X is empty fuzzy set on X.

From now on, we will use $\mathcal{F}(X, E)$ instead of the family of all fuzzy soft sets over X.

Obviously, a classical soft set F_A over a universe X can be seen as a fuzzy soft set by using the characteristic function of the set $F_A(e)$:

$$f_A(e)(a) = \chi_{F_A(e)}(a) = \begin{cases} 1, & \text{if } a \in F_A(e); \\ 0, & \text{otherwise.} \end{cases}$$

2.6. Definition. Let $f_A, g_B \in \mathcal{F}(X, E)$. Then f_A is called a *fuzzy soft subset of* g_B if $f_A(e) \leq g_B(e)$, for each $e \in E$, and we write $f_A \sqsubseteq g_B$. Also f_a is called a *fuzzy soft superset* of g_B if g_B is a fuzzy soft subset of f_A, and we write $f_X \sqsupseteq g_B$.

2.7. Definition. Let $f_A, g_B \in \mathcal{F}(X, E)$. Then f_A and g_B are said to be *equal*, denoted by $f_A = g_B$, if $f_A \sqsubseteq g_B$ and $g_B \sqsubseteq f_A$.

2.8. Definition. Let $f_A, g_B \in \mathcal{F}(X, E)$. The *union of* f_A *and* g_B, denoted by $f_A \sqcup g_B$, is the fuzzy soft set $h_{A\cup B}$ defined by $h_{A\cup B}(e) = f_A(e) \vee g_B(e)$, $\forall\, e \in E$.

That is, $h_{A\cup B} = f_A \sqcup g_B$.

2.9. Definition. Let $f_A, g_B \in \mathcal{F}(X, E)$. The *intersection of* f_A *and* g_B, denoted by $f_A \sqcap g_B$, is the fuzzy soft set $h_{A\cap B}$ defined by $h_{A\cap B}(e) = f_A(e) \wedge g_B(e)$, $\forall\, e \in E$.

That is, $h_{A\cap B} = f_A \sqcap g_B$.

2.10. Proposition. *Let* $f_A, g_B \in \mathcal{F}(X, E)$. *Then*

$$f_A \sqsubseteq g_B \text{ iff } f_A = f_A \sqcap g_B \text{ or } g_B = f_A \sqcup g_B.$$

Proof. Straightforward. □

2.11. Definition. Let $f_A \in \mathcal{F}(X, E)$. Then the *complement of* f_A, denoted by f_A^c, is the fuzzy soft set defined by $f_A^c(e) = 1_X - f_A(e), \forall\, e \in E$.

Let us call f_A^c the *fuzzy soft complement function* of f_A. Clearly $(f_A^c)^c = f_A$.

Let $f_E \in \mathcal{F}(X, E)$. The fuzzy soft set f_E is called the *null fuzzy soft set*, denoted by $\widetilde{0}_E$, if $f_E(e) = 0_X, \forall\, e \in E$.

2.12. Definition. Let $f_E \in \mathcal{F}(X, E)$. The fuzzy soft set f_E is called the *universal fuzzy soft set*, denoted by $\widetilde{1}_E$, if $f_E(e) = 1_X, \forall\, e \in E$.

Clearly $(\widetilde{1}_E)^c = \widetilde{0}_E$ and $(\widetilde{0}_E)^c = \widetilde{1}_E$.

2.13. Definition. Let $f_A \in \mathcal{F}(X, E)$. The fuzzy soft set f_A is called the *A-universal fuzzy soft set*, denoted by $\widetilde{1}_A$, if $f_A(e) = 1_X, \forall\, e \in A$ and $f_A(e) = 0_X, \forall\, e \in E \backslash A$.

We denote by α_X the constant fuzzy set on X, i.e. $\alpha_X(x) = \alpha$ for all $x \in X$ and $\alpha \in I$.

2.14. Definition. Let $f_E \in \mathcal{F}(X,E)$. The fuzzy soft set f_E is called the *α-universal fuzzy soft set*, denoted by $\widetilde{\alpha}_E$, if $f_E(e) = \alpha_X$ for each $e \in E$. Clearly,

$$(\widetilde{\alpha}_E)^c = (\widetilde{1-\alpha})_E.$$

2.15. Definition. Let $f_A \in \mathcal{F}(X,E)$. The fuzzy soft set f_A is called the *α-A- universal fuzzy soft set*, denoted by $\widetilde{\alpha}_A$, if $f_A(e) = \alpha_X,\ \forall e \in A$ and $f_A(e) = 0_X,\ \forall e \in E\backslash A$.

2.16. Remark. The complement of an α-A- universal fuzzy soft set $\widetilde{\alpha}_A$ is not an α-A- universal fuzzy soft set. Indeed,

$$\widetilde{\alpha}_A = \begin{cases} f_A(e) = \alpha_X, & \text{if } e \in A; \\ f_A(e) = 0_X, & \text{otherwise.} \end{cases} \implies (\widetilde{\alpha}_A)^c = \begin{cases} f_A^c(e) = 1_X - \alpha_X, & \text{if } e \in A; \\ f_A^c(e) = 1_X, & \text{otherwise.} \end{cases}$$

2.17. Theorem. *Let J be an index set and $f_A, g_B, h_C, (f_A)_i, (g_B)_i \in \mathcal{F}(X,E)\ \forall i \in J$, then*

(1) $f_A \sqcap f_A = f_A,\ f_A \sqcup f_A = f_A.$
(2) $f_A \sqcap g_B = g_B \sqcap f_A,\ f_A \sqcup g_B = g_B \sqcup f_A.$
(3) $f_A \sqcup (g_B \sqcup h_C) = (f_A \sqcup g_B) \sqcup h_C,\ f_A \sqcap (g_B \sqcap h_C) = (f_A \sqcap f_B) \sqcap h_C.$
(4) $f_A \sqcap \left(\bigsqcup_{i\in J} (g_B)_i\right) = \bigsqcup_{i\in J} (f_A \sqcap (g_B)_i)\ \ f_A \sqcup \left(\bigsqcap_{i\in J} (g_B)_i\right) = \bigsqcap_{i\in J} (f_A \sqcup (g_B)_i).$
(5) $\widetilde{0}_E \sqsubseteq f_A \sqsubseteq \widetilde{1}_A \sqsubseteq \widetilde{1}_E.$
(6) $\left(\bigsqcap_{i\in J} (f_A)_i\right)^c = \bigsqcup_{i\in J} (f_A)_i^c,\ \left(\bigsqcup_{i\in J} (f_A)_i\right)^c = \bigsqcap_{i\in J} (f_A)_i^c$
(7) *If $f_A \sqsubseteq f_B$, then $(f_B)^c \sqsubseteq (f_A)^c$.*
(8) $f_A \sqcap g_B \sqsubseteq f_A, g_B$ *and* $f_A, g_B \sqsubseteq f_A \sqcup g_B$

Proof. We give here the proof of (3), (4) and (6). The others can be proved in a similar way.

(3) For each $e \in E$, according to the Definition 2.8 and since $f_A(e),\ g_B(e),\ h_C(e) \in I^X$,

$$\begin{aligned} (f_A \sqcup (g_B \sqcup h_C))(e) &= f_A(e) \vee (g_B \sqcup h_C)(e) \\ &= f_A(e) \vee (g_B(e) \vee h_C(e)) \\ &= (f_A(e) \vee g_B(e)) \vee h_C(e) \\ &= (f_A \sqcup g_B)(e) \vee h_C(e) \\ &= ((f_A \sqcup g_B) \sqcup h_C)(e). \end{aligned}$$

Hence we obtain $f_A \sqcup (g_B \sqcup h_C) = (f_A \sqcup g_B) \sqcup h_C$.

The proof of $f_A \sqcap (g_B \sqcap h_C) = (f_A \sqcap g_B) \sqcap h_C$ can be made similarly.

(4) According to the Definitions2.8 and 2.9, for $e \in E$,

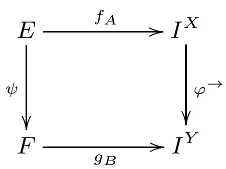
$$\begin{aligned} \left(f_A \sqcap \left(\bigsqcup_{i\in J} (g_B)_i\right)\right)(e) &= f_A(e) \wedge \left(\bigsqcup_{i\in J} (g_B)_i\right)(e) \\ &= f_A(e) \wedge \left(\bigvee_{i\in J} (g_B)_i(e)\right) \\ &= \bigvee_{i\in J} (f_A(e) \wedge (g_B)_i(e)) \\ &= \bigvee_{i\in J} (f_A \sqcap (g_B)_i)(e) \\ &= \left(\bigsqcup_{i\in J} (f_A \sqcap (g_B)_i)\right)(e). \end{aligned}$$

(6) For $e \in E$,

$$\left(\prod_{i\in J}(f_A)_i\right)^c(e) = 1_X - \left(\prod_{i\in J}(f_A)_i\right)(e)$$
$$= 1_X - \left(\bigwedge_{i\in J}(f_A)_i(e)\right)$$
$$= \bigvee_{i\in J}\left((f_A)_i^c(e)\right)$$
$$= \bigsqcup_{i\in J}\left((f_A)_i^c\right)(e).$$ □

To consider fuzzy soft sets as a category we have to define morphisms between two fuzzy soft sets.

2.18. Definition. [3] Let $\mathcal{F}(X,E)$ and $\mathcal{F}(Y,K)$ be the families of all fuzzy soft sets over X and Y, respectively. Let $\varphi : X \to Y$ and $\psi : E \to K$ be two functions. Then the pair (φ,ψ) is called a *fuzzy soft mapping from X to Y*, and denoted by $(\varphi,\psi) : \mathcal{F}(X,E) \to \mathcal{F}(Y,K)$.



In the diagram, $f_A \in \mathcal{F}(X,E)$, $g_B \in \mathcal{F}(Y,K)$ and $\varphi^{\to} : I^X \to I^Y$ is the forward powerset operator (see e.g. [17]), that is $\varphi^{\to}(h) := \varphi(h)$ for all $h \in I^X$.

Since componentwise composition of two fuzzy soft functions (φ,ψ) from X to Y and (φ',ψ') from Y to Z is obviously a fuzzy soft function $(\varphi'\circ\varphi, \psi'\circ\psi)$ from X to Z, where $\psi : E \to K$ and $\psi' : K \to G$ and the pair of identities (id_X, id_E) from X to X is the identical morphism.

(1) Let $f_A \in \mathcal{F}(X,E)$. Then the *image* of f_A under the fuzzy soft mapping (φ,ψ) is the fuzzy soft set over Y defined by $(\varphi,\psi)(f_A)$, where $\forall\, k \in \psi(E)$, $\forall\, y \in Y$,

$$\varphi(f_A)(k)(y) = \begin{cases} \bigvee_{\varphi(x)=y}\bigvee_{\psi(e)=k} f_A(e)(x), & \text{if } x \in \varphi^{-1}(y); \\ 0_X, & \text{otherwise.} \end{cases}$$

(2) Let $g_B \in \mathcal{F}(Y,K)$. Then the *pre-image* of g_B under the fuzzy soft mapping (φ,ψ) is the fuzzy soft set over X defined by $(\varphi,\psi)^{-1}(g_B)$, where $\forall\, e \in \psi^{-1}(K)$, $\forall\, x \in X$,

$$\varphi^{-1}(g_B)(e)(x) = g_B(\psi(e))(\varphi(x)).$$

If φ and ψ is injective then the fuzzy soft mapping (φ,ψ) is said to be injective. If φ and ψ is surjective then the fuzzy soft mapping (φ,ψ) is said to be surjective.

The fuzzy soft mapping (φ,ψ) is called constant, if φ and ψ are constant.

2.19. Theorem. [8] *Let X and Y crisp sets $f_A, (f_A)_i \in \mathcal{F}(X,E)$ and $g_B, (g_B)_i \in \mathcal{F}(Y,K)$ $\forall\, i \in J$, where J is an index set.*

(1) *If $(f_A)_1 \sqsubseteq (f_A)_2$, then $(\varphi,\psi)(f_A)_1 \sqsubseteq (\varphi,\psi)(f_A)_2$.*
(2) *If $(g_B)_1 \sqsubseteq (g_B)_2$, then $(\varphi,\psi)^{-1}((g_B)_1) \sqsubseteq (\varphi,\psi)^{-1}((g_B)_2)$.*
(3) *$f_A \sqsubseteq (\varphi,\psi)^{-1}((\varphi,\psi)(f_A))$, the equality holds if (φ,ψ) is injective.*
(4) *$(\varphi,\psi)\big((\varphi,\psi)^{-1}(g_B)\big) \sqsubseteq g_B$, the equality holds if (φ,ψ) is surjective.*

(5) $(\varphi,\psi)\left(\bigsqcup_{i\in J}(f_A)_i\right) = \bigsqcup_{i\in J}(\varphi,\psi)(f_A)_i$.
(6) $(\varphi,\psi)\left(\prod_{i\in J}(f_A)_i\right) \sqsubseteq \prod_{i\in J}(\varphi,\psi)(f_A)_i$, *the equality holds if* (φ,ψ) *is injective.*
(7) $(\varphi,\psi)^{-1}\left(\bigsqcup_{i\in J}(g_B)_i\right) = \bigsqcup_{i\in J}(\varphi,\psi)^{-1}(g_B)_i$.
(8) $(\varphi,\psi)^{-1}\left(\prod_{i\in J}(g_B)_i\right) = \prod_{i\in J}(\varphi,\psi)^{-1}(g_B)_i$.
(9) $(\varphi,\psi)^{-1}((g_B)^c) = \left((\varphi,\psi)^{-1}(g_B)\right)^c$.
(10) $((\varphi,\psi)(f_A))^c \sqsubseteq (\varphi,\psi)((f_A)^c)$.
(11) $(\varphi,\psi)^{-1}\left(\widetilde{1}_K\right) = \widetilde{1}_E,\ (\varphi,\psi)^{-1}\left(\widetilde{0}_K\right) = \widetilde{0}_E$.
(12) $(\varphi,\psi)\left(\widetilde{1}_E\right) = \widetilde{1}_K$ *if* (φ,ψ) *is surjective.*
(13) $(\varphi,\psi)\left(\widetilde{0}_E\right) = \widetilde{0}_K$.

2.20. Definition. (Construction of the product) Let $f_A \in \mathcal{F}(X,E)$ and $g_B \in \mathcal{F}(Y,K)$. The *fuzzy product* $f_A \times g_B$ is defined by $(f\times g)_{A\times B}$ where

$$(f\times g)_{A\times B}(e,k) = f_A(e)\times g_B(k) \in I^X \times I^Y \subseteq I^{X\times Y},\ \forall (e,k)\in A\times B,$$

and for all $(x,y)\in X\times Y,\ (f_A(e)\times g_B(k))\,(x,y) = f_A(e)(x)\wedge g_B(k)(y)$.

According to this definition the fuzzy soft set $f_A\times g_B$ is a fuzzy soft set over $X\times Y$ and its parameter universe is $E\times K$.

One can easily see that the pairs of projections $p_X : X\times Y\to X$, $q_E : E\times K\to E$ and $p_Y : X\times Y\to Y$, $q_K : E\times K\to K$ determine morphisms respectively (p_X,q_E) from $X\times Y$ to X and (p_Y,q_K) from $X\times Y$ to Y, where

$$(p_X,q_E)(f_A\times g_B) = p_X(f\times g)_{q_E(A\times B)} = f_A$$

and

$$(p_Y,q_K)(f_A\times g_B) = p_Y(f\times g)_{q_K(A\times B)} = g_B.$$

3. Fuzzy soft topological spaces

3.1. Fuzzy soft topological spaces. In this section, we give the definition of fuzzy soft topological space and study some basic structures.

3.1. Definition. [19] A fuzzy soft topological space is a pair $(X,\mathcal{T})$ where X is a nonempty set and $\mathcal{T}$ a family of fuzzy soft sets over X satisfying the following properties:

(1) $\widetilde{0}_E, \widetilde{1}_E \in \mathcal{T}$,
(2) If $f_A,\ g_B\in\mathcal{T}$, then $f_A\sqcap g_B\in\mathcal{T}$,
(3) If $(f_A)_i\in\mathcal{T}, \forall i\in J$, then $\bigsqcup_{i\in J}(f_A)_i\in\mathcal{T}$

$\mathcal{T}$ is called a *topology* of fuzzy soft sets on X. Every member of $\mathcal{T}$ is called *fuzzy soft open.* g_B is called *fuzzy soft closed* in $(X,\mathcal{T})$ if $(g_B)^c\in\mathcal{T}$.

3.2. Examples. $\mathcal{T}^0 = \{\widetilde{0}_E, \widetilde{1}_E\}$ is a fuzzy soft topology on X.

$\mathcal{T}^1 = \mathcal{F}(X,E)$ is a fuzzy soft topology on X.

The intersection of any family of fuzzy soft topologies on X is also a fuzzy soft topology on X.

A fuzzy soft topology $\mathcal{T}_1$ is called *weaker* (or *coarser*) than a fuzzy soft topology $\mathcal{T}_2$ if and only if $\mathcal{T}_1\subset\mathcal{T}_2$. In that case $\mathcal{T}_2$ is said to be *stronger* (or *finer*) than $\mathcal{T}_1$.

A fuzzy soft topology is called *enriched* if it satisfies

(1)′ $\widetilde{\alpha}_E\in\mathcal{T}, \forall\alpha\in I$.

3.3. Example. Let $(X, \mathcal{T})$ be a fuzzy soft topological spaces, where $\mathcal{T} = \{(f_A)_\lambda \mid \lambda \in \Delta\}$. Then we can also construct fuzzy topologies from each parameter in the following way:

Let $\tau_{e_i} = \{(f_A)_\lambda(e_i) \mid (f_A)_\lambda \in \mathcal{T}\}, \forall e_i \in E$.

Indeed τ_{e_i} is a fuzzy topology on X.

(1) Since $\widetilde{0}_E, \widetilde{1}_E \in \mathcal{T}$, it follows that $\exists \lambda_1, \lambda_2 \in \Delta$ such that $(f_A)_{\lambda_1}(e_i) = 1_X$ and $(f_A)_{\lambda_2}(e_i) = 0_X$. So, $0_X, 1_X \in \tau_{e_i}$.

(2) For $(f_A)_{\lambda_1}(e_i), (f_A)_{\lambda_2}(e_i) \in \tau_{e_i}$, we know that $(f_A)_{\lambda_1}, (f_A)_{\lambda_2} \in \mathcal{T}$. Hence, $(f_A)_{\lambda_1} \sqcap (f_A)_{\lambda_2} \in \mathcal{T}$. Therefore

$$(f_A)_{\lambda_1}(e_i) \wedge (f_A)_{\lambda_2}(e_i) = \left((f_A)_{\lambda_1} \sqcap (f_A)_{\lambda_2}\right)(e_i) \in \tau_{e_i}.$$

(3) Similarly, if $\forall (f_A)_\lambda(e_i) \in \tau_{e_i}$, then $\bigvee_\lambda (f_A)_\lambda(e_i) \in \tau_{e_i}$.

Hence for each $e_i \in E$, τ_{e_i} is a fuzzy topology on X, which is called the "e_i-parameter topology" of the fuzzy soft topology $\mathcal{T}$.

3.4. Theorem. *Let $(X, \mathcal{T})$ be a fuzzy soft topological space and let $\mathcal{T}'$ denote the collection of all fuzzy soft closed sets. Then:*

1. $\widetilde{0}_E, \widetilde{1}_E \in \mathcal{T}'$.
2. *If $f_A, g_B \in \mathcal{T}'$, then $f_A \sqcup g_B \in \mathcal{T}'$.*
3. *If $(f_A)_i \in \mathcal{T}'$, $\forall i \in J$, then $\prod_{i \in J} (f_A)_i \in \mathcal{T}'$.*

Proof. Straightforward. □

3.5. Definition. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two fuzzy soft topological spaces.

1. A fuzzy soft mapping $(\varphi, \psi) : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is called *fuzzy soft continuous* if $(\varphi, \psi)^{-1}(g_B) \in \mathcal{T}_1,\ \forall g_B \in \mathcal{T}_2$.
2. A fuzzy soft mapping $(\varphi, \psi) : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is called *fuzzy soft open* if $(\varphi, \psi)(f_A) \in \mathcal{T}_2,\ \forall f_A \in \mathcal{T}_1$.

If $(\varphi, \psi) : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ and $(\varphi', \psi') : (Y, \mathcal{T}_2) \to (Z, \mathcal{T}_3)$ are fuzzy soft continuous then clearly $(\varphi', \psi') \circ (\varphi, \psi)$ is also fuzzy soft continuous because for a fuzzy soft set f_A on Z,

$$\begin{aligned}
\left((\varphi', \psi') \circ (\varphi, \psi)\right)^{-1}(f_A)(e) &= \left(\varphi' \circ \varphi, \psi' \circ \psi\right)^{-1}(f_A)(e) \\
&= (\varphi' \circ \varphi)^{-1}\left(f_A(\psi'(\psi(e)))\right) \\
&= \varphi^{-1}\left((\varphi')^{-1}(f_A(\psi'(\psi(e))))\right) \\
&= (\varphi, \psi)^{-1}\left((\varphi', \psi')^{-1}(f_A)\right)(e).
\end{aligned}$$

Hence we have $\left((\varphi', \psi') \circ (\varphi, \psi)\right)^{-1}(f_A) = (\varphi, \psi)^{-1}\left((\varphi', \psi')^{-1}(f_A)\right)$.

3.6. Example. The constant mapping $(\varphi, \psi) : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2),\ \varphi(x) = y_0,\ \psi(e) = k_0\ \forall x \in X,\ \forall e \in E$, is not continuous in general.

Let $X = Y = \{x_1, x_2, x_3\}$, $E = \{e_1, e_2, e_3\}$ and $(\varphi, \psi) : (X, \mathcal{T}^0) \to (Y, \mathcal{T}^1)$ a constant mapping, where $\varphi(x) = x_1,\ \forall x \in X$ and $\psi(e) = e_1,\ \forall e \in E$. If we take

$$f_A(e_1) = \{\frac{x_1}{0.2}, \frac{x_2}{0.5}, \frac{x_3}{0}\},\ f_A(e_2) = \{\frac{x_1}{0.6}, \frac{x_2}{0}, \frac{x_3}{0}\}\ f_A(e_3) = \{\frac{x_1}{0}, \frac{x_2}{0}, \frac{x_3}{0}\}, \text{where } A = \{e_1, e_2\},$$

then

$$(\varphi, \psi)^{-1}(f_A)(e_1)(x_1) = f_A(\psi(e_1))(\varphi(x_1)) = f_A(e_1)(x_1) = 0.2$$

and similarly,

$$(\varphi, \psi)^{-1}(f_A)(e_1)(x_2) = (\varphi, \psi)^{-1}(f_A)(e_1)(x_3) = 0.2,$$

$$(\varphi,\psi)^{-1}(f_A)(e_2)(x_1) = f_A(\psi(e_2))(\varphi(x_1)) = f_A(e_1)(x_1) = 0.2$$

and similarly,

$$(\varphi,\psi)^{-1}(f_A)(e_2)(x_2) = (\varphi,\psi)^{-1}(f_A)(e_2)(x_3) = 0.2,$$
$$(\varphi,\psi)^{-1}(f_A)(e_3)(x_1) = f_A(\psi(e_3))(\varphi(x_1)) = f_A(e_1)(x_1) = 0.2,$$

and similarly,

$$(\varphi,\psi)^{-1}(f_A)(e_3)(x_2) = (\varphi,\psi)^{-1}(f_A)(e_3)(x_3) = 0.2.$$

Hence $(\varphi,\psi)^{-1}(f_A) \notin \mathcal{T}^0$, while $f_A \in \mathcal{T}^1$.

3.7. Theorem. *Let* $(X,\mathcal{T}_1)$ *and* $(Y,\mathcal{T}_2)$ *be two enriched fuzzy soft topological spaces and* $(\varphi,\psi) : (X,\mathcal{T}_1) \to (Y,\mathcal{T}_2)$ *a constant mapping, where* $\varphi(x) = y_0,\ \psi(e) = k_0\ \forall x \in X,\ \forall e \in E$. *Then* (φ,ψ) *is fuzzy soft continuous.*

Proof. Let $g_B \in \mathcal{T}_2$. Then

$$(\varphi,\psi)^{-1}(g_B)(e)(x) = g_B(\psi(e))(\varphi(x))$$
$$= \begin{cases} \alpha_X, & \text{if } \psi(e) = k_0 \in B \text{ and } g_B(\psi(e))(y_0) \neq 0; \\ 0_X, & \text{otherwise.} \end{cases}$$

Hence, if $k_0 \in B$, $(\varphi,\psi)^{-1}(g_B) = \widetilde{\alpha}_E \in \mathcal{T}_1$ and if $k_0 \notin B$, $\widetilde{0}_E \in \mathcal{T}_1$ where $\alpha = g_B(\psi(e))(y_0)$. □

3.8. Definition. Let $(X,\mathcal{T})$ be a fuzzy soft topological space and $f_A \in \mathcal{F}(X,E)$. The *fuzzy soft closure* of f_A, denoted by $\overline{f_A}$, is the intersection of all fuzzy soft closed supersets of f_A.

Clearly, $\overline{f_A}$ is the smallest fuzzy soft closed set over X which contains f_A, and $\overline{f_A}$ is closed.

3.9. Theorem. *Let* $(X,\mathcal{T})$ *be a fuzzy soft topological space and* $f_A, g_B \in \mathcal{F}(X,E)$. *Then,*

(1) $\overline{\widetilde{0}_E} = \widetilde{0}_E$ *and* $\overline{\widetilde{1}_E} = \widetilde{1}_E$.
(2) $f_A \sqsubseteq \overline{f_A}$.
(3) $\overline{\overline{f_A}} = \overline{f_A}$.
(4) *If* $f_A \sqsubseteq g_B$, *then* $\overline{f_A} \sqsubseteq \overline{g_B}$.
(5) f_A *is a fuzzy soft closed set if and only if* $f_A = \overline{f_A}$.
(6) $\overline{f_A \sqcup g_B} = \overline{f_A} \sqcup \overline{g_B}$.

Proof. (1) to (4): Clear from the definition of closure.

(5) Let f_A be a fuzzy soft closed set. By (2) we have $f_A \sqsubseteq \overline{f_A}$. Since $\overline{f_A}$ is the smallest fuzzy soft closed set over X which contains f_A, then $\overline{f_A} \sqsubseteq f_A$. Hence, $f_A = \overline{f_A}$.

Conversely, assume that $f_A = \overline{f_A}$. Since $\overline{f_A}$ is a fuzzy soft closed set, then f_A is closed.

(6) By (4), $\overline{f_A}, \overline{g_B} \sqsubseteq \overline{f_A \sqcup g_B}$. So, $\overline{f_A} \sqcup \overline{g_B} \sqsubseteq \overline{f_A \sqcup g_B}$.

Conversely, by (2), $f_A \sqcup g_B \sqsubseteq \overline{f_A} \sqcup \overline{g_B}$. Since $\overline{f_A}$ and $\overline{g_B}$ are fuzzy soft closed sets and $\overline{f_A \sqcup g_B}$ is the smallest closed set which contains $f_A \sqcup g_B$, then $\overline{f_A \sqcup g_B} \sqsubseteq \overline{f_A} \sqcup \overline{g_B}$. Hence, we obtain the equality. □

3.10. Definition. Let $(X,\mathcal{T})$ be a fuzzy soft topological space and $f_A \in \mathcal{F}(X,E)$. The fuzzy soft interior of f_A denoted by f_A^o is the union of all fuzzy soft open subsets of f_A.

Clearly, f_A^o is the largest fuzzy soft open set contained in f_A and f_A^o is open.

3.11. Theorem. *Let* $(X,\mathcal{T})$ *be a fuzzy soft topological space and* $f_A, g_B \in \mathcal{F}(X,E)$. *Then,*

(1) $(\widetilde{0}_E)^o = \widetilde{0}_E$ *and* $(\widetilde{1}_E)^o = \widetilde{1}_E$.
(2) $f_A^o \sqsubseteq f_A$.
(3) $(f_A^o)^o = f_A^o$.
(4) *If* $f_A \sqsubseteq g_B$, *then* $f_A^o \sqsubseteq g_B^o$.
(5) f_A *is fuzzy soft open set if and only if* $f_A = f_A^o$.
(6) $(f_A \sqcup g_B)^o = f_A^o \sqcap g_B^o$.

Proof. Straightforward. □

3.12. Theorem. *Let* $(X, \mathcal{T})$ *be a fuzzy soft topological space and* $f_A \in \mathcal{F}(X, E)$. *Then,*

(1) $(f_A^o)^c = \overline{(f_A^c)}$.
(2) $(\overline{f_A})^c = (f_A^c)^o$.

Proof. (1) Certainly, $f_A^o \sqsubseteq f_A$ so by Theorem 2.17 (7), $f_A^c \sqsubseteq (f_A^o)^c$. Since $(f_A^o)^c$ is a fuzzy soft closed set and by Theorem 3.9 (4),

$$\overline{(f_A^c)} \sqsubseteq \overline{(f_A^o)^c} = (f_A^o)^c.$$

Conversely, by Theorem 3.9 (2) $f_A^c \sqsubseteq \overline{f_A^c}$. By Theorem 2.17 (7), $(\overline{f_A^c})^c \sqsubseteq (f_A^c)^c = f_A$. Since $\overline{f_A^c}$ is a fuzzy soft closed set, then $(\overline{f_A^c})^c$ is open. By the definition of interior, $(\overline{f_A^c})^c \sqsubseteq f_A^o$ and again using Theorem 2.17 (7) we obtain $(f_A^o)^c \sqsubseteq ((\overline{f_A^c})^c)^c = \overline{(f_A^c)}$.

(2) Similar to (1). □

3.13. Theorem. *Let* $(X, \mathcal{T}_1)$ *and* $(Y, \mathcal{T}_2)$ *be two fuzzy soft topological spaces and* $(\varphi, \psi) : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ *a fuzzy soft mapping. Then the following are equivalent:*

(1) (φ, ψ) *is continuous.*
(2) $(\varphi, \psi)^{-1}(g_B) \in \mathcal{T}_1', \forall\, g_B \in \mathcal{T}_2'$.
(3) $(\varphi, \psi)(\overline{f_A}) \sqsubseteq \overline{(\varphi, \psi)(f_A)}, \forall\, f_A \in \mathcal{F}(X, E)$.
(4) $\overline{(\varphi, \psi)^{-1}(g_B)} \sqsubseteq (\varphi, \psi)^{-1}(\overline{g_B}),\ \forall\, g_B \in \mathcal{F}(Y, K)$.
(5) $(\varphi, \psi)^{-1}(g_B^o) \sqsubseteq ((\varphi, \psi)^{-1}(g_B))^o,\ \forall\, g_B \in \mathcal{F}(Y, K)$.

Proof. (1) $\Longrightarrow$ (2) By Theorem 2.19 (10).

(2) $\Longrightarrow$ (3) Let $f_A \in \mathcal{F}(X, E)$. Then,

$$f_A \sqsubseteq (\varphi, \psi)^{-1}\left((\varphi, \psi)(f_A)\right) \sqsubseteq (\varphi, \psi)^{-1}\overline{((\varphi, \psi)(f_A))} \in \mathcal{T}_1',$$

and then $\overline{f_A} \sqsubseteq (\varphi, \psi)^{-1}\overline{((\varphi, \psi)(f_A))}$. By Theorem 2.19 (4), we get

$$(\varphi, \psi)(\overline{f_A}) \sqsubseteq (\varphi, \psi)(\varphi, \psi)^{-1}\overline{((\varphi, \psi)(f_A))} \sqsubseteq \overline{(\varphi, \psi)(f_A)}.$$

(3) $\Longrightarrow$ (4) Let $g_B \in \mathcal{F}(Y, K)$.

If we write $(\varphi, \psi)^{-1}(g_B)$ instead of $f_A$in (3), we obtain

$$(\varphi, \psi)\overline{(\varphi, \psi)^{-1}(g_B)} \sqsubseteq \overline{(\varphi, \psi)\left((\varphi, \psi)^{-1}(g_B)\right)} \sqsubseteq \overline{g_B},$$

and by Theorem 2.19 (3),

$$\overline{(\varphi, \psi)^{-1}(g_B)} \sqsubseteq (\varphi, \psi)^{-1}(\varphi, \psi)\overline{(\varphi, \psi)^{-1}(g_B)} \sqsubseteq (\varphi, \psi)^{-1}(\overline{g_B}).$$

(4) $\Longrightarrow$ (5) Let $g_B \in \mathcal{F}(Y, K)$. We have

$$\overline{(\varphi, \psi)^{-1}(g_B^c)} \sqsubseteq (\varphi, \psi)^{-1}(\overline{g_B^c}).$$

By Theorem 3.12 (1) and Theorem 2.19 (9),

$$\begin{aligned}\left(((\varphi,\psi)^{-1}(g_B))^o\right)^c &= \overline{(\varphi,\psi)^{-1}(g_B^c)}\\ &\sqsubseteq (\varphi,\psi)^{-1}(\overline{g_B^c})\\ &= (\varphi,\psi)^{-1}(g_B^o)^c\\ &= \left((\varphi,\psi)^{-1}(g_B^o)\right)^c.\end{aligned}$$

Hence, $(\varphi,\psi)^{-1}(g_B^o) \sqsubseteq \left((\varphi,\psi)^{-1}(g_B)\right)^o$.

(5) $\Longrightarrow$ (1) Let $g_B \in \mathcal{T}_2$. Then $g_B = g_B^o$. From the hypothesis,

$$(\varphi,\psi)^{-1}(g_B) = (\varphi,\psi)^{-1}(g_B^o) \sqsubseteq \left((\varphi,\psi)^{-1}(g_B)\right)^o \sqsubseteq (\varphi,\psi)^{-1}(g_B).$$

So, $(\varphi,\psi)^{-1}(g_B) = \left((\varphi,\psi)^{-1}(g_B)\right)^o \in \mathcal{T}_1$. Consequently, (φ,ψ) is fuzzy soft continuous. □

3.14. Theorem. *Let $i : \mathcal{F}(X,E) \to \mathcal{F}(X,E)$ be an operator satisfying the following:*

(i1) $i(\widetilde{1}_E) = \widetilde{1}_E$.
(i2) $i(f_A) \sqsubseteq f_A, \forall f_A \in \mathcal{F}(X,E)$.
(i3) $i(f_A \sqcap g_B) = i(f_A) \sqcap i(g_B), \forall f_A, g_B \in \mathcal{F}(X,E)$.
(i4) $i\,(i(f_A)) = i(f_A), \forall f_A \in \mathcal{F}(X,E)$.

Then we can associate a fuzzy soft topology in the following way:

$$\mathcal{T} = \{f_A \in \mathcal{F}(X,E) \mid i(f_A) = f_A\}.$$

Moreover with this fuzzy soft topology $\mathcal{T}$, $f_A^o = i(f_A)$ for every $f_A \in \mathcal{F}(X,E)$.

Proof. (T1) By (i1) $\widetilde{1}_E \in \mathcal{T}$. By (i2) $i(\widetilde{0}_E) \sqsubseteq \widetilde{0}_E$, so $i(\widetilde{0}_E) = \widetilde{0}_E$ and $\widetilde{0}_E \in \mathcal{T}$.

(T2) Let f_A and $g_B \in \mathcal{T}$. By the definition of $\mathcal{T}$, $i(f_A) = f_A$ and $i(g_B) = g_B$. By (i3), $i(f_A \sqcap g_B) = i(f_A) \sqcap i(g_B) = f_A \sqcap g_B$. So $f_A \sqcap g_B \in \mathcal{T}$.

(T3) Let $\{(f_A)_j \mid j \in J\} \subset \mathcal{T}$. By (i3), i is order preserving and by the definition of union, $(f_A)_k \sqsubseteq \sqcup_{j\in J}(f_A)_j, \forall k \in J$ and $i(f_A)_k \sqsubseteq i\,(\sqcup_{j\in J}(f_A)_j)$. By the definition of $\mathcal{T}$, $i(f_A)_k = (f_A)_k$, and so $(f_A)_k = i(f_A)_k \sqsubseteq i\,(\sqcup_{j\in J}(f_A)_j)$. Then $\sqcup_{j\in J}(f_A)_j \sqsubseteq i(\sqcup_{j\in J}(f_A)_j)$, $\forall j \in J$.

Conversely, by (i2) we have $i(\sqcup_{j\in J}(f_A)_j) \sqsubseteq \sqcup_{j\in J}(f_A)_j$. Hence, $i(\sqcup_{j\in J}(f_A)_j) = \sqcup_{j\in J}(f_A)_j$ and

$$\sqcup_{j\in J}(f_A)_j \in \mathcal{T}.$$

For the second part, let $f_A \in \mathcal{F}(X,E)$. Since $f_A^o \in \mathcal{T}$, then $i(f_A^o) = f_A^o$. By (i3), i is order preserving and $f_A^o = i(f_A^o) \sqsubseteq i(f_A)$. Conversely, by (i4) we have $i\,(i(f_A)) = i(f_A)$, then $i(f_A) \sqsubseteq \sqcup\{g_B \in \mathcal{F}(X,E) \mid i(g_B) = g_B \sqsubseteq f_A\} = f_A^o$. Thus, $f_A^o = i(f_A)$. □

The operator i is called the *fuzzy soft interior operator*.

3.15. Remark. By Theorem 3.11 (1), (2) , (3), (6) and Theorem 3.14, we see that with a fuzzy soft interior operator we can associate a fuzzy soft topology and conversely with a given fuzzy soft topology we can associate a fuzzy soft interior operator.

3.16. Theorem. *Let $c : \mathcal{F}(X,E) \to \mathcal{F}(X,E)$ be an operator satisfying the following:*

(c1) $c(\widetilde{0}_E) = \widetilde{0}_E$.
(c2) $f_A \sqsubseteq c(f_A)$, $\forall f_A \in \mathcal{F}(X,E)$.
(c3) $c(f_A \sqcup g_B) = c(f_A) \sqcup c(g_B)$, $\forall f_A, g_B \in \mathcal{F}(X,E)$.
(c4) $c\,(c(f_A)) = c(f_A)$, $\forall f_A \in \mathcal{F}(X,E)$.

Then we can associate a fuzzy soft topology in the following way:

$$\mathcal{T} = \{f_A^c \in \mathcal{F}(X,E) \mid c(f_A) = f_A\}.$$

Moreover with this fuzzy soft topology $\mathcal{T}, \overline{f_A} = c(f_A)$ *for every* $f_A \in \mathcal{F}(X,E)$.

Proof. Similar to proof of Theorem 3.14. □

The operator c is called the *fuzzy soft closure operator*.

3.17. Remark. By Theorem 3.9 (1), (2) , (3), (6) and Theorem 3.16, we see that with a fuzzy soft closure operator we can associate a fuzzy soft topology and conversely with a given fuzzy soft topology we can associate a fuzzy soft closure operator.

3.2. Initial fuzzy soft topology. In this subsection, we introduce the fuzzy soft product topology and study some properties of projection mappings.

3.18. Definition. Let $(X, \mathcal{T})$ be a fuzzy soft topological space. A subcollection $\mathcal{B}$ of $\mathcal{T}$ is called a base for $\mathcal{T}$ if every member of $\mathcal{T}$ can be expressed as a union of members of $\mathcal{B}$.

3.19. Theorem. *Let* $(\varphi, \psi) : (X, \mathcal{T}) \to (Y, \mathcal{T}^*)$ *be a fuzzy soft mapping and* $\mathcal{B}$ *a base for* $\mathcal{T}^*$. *Then* (φ, ψ) *is fuzzy soft continuous if and only if* $(\varphi, \psi)^{-1}(f_B) \in \mathcal{T},\ \forall f_B \in \mathcal{B}$.

Proof. Straightforward. □

3.20. Definition. Let $(X, \mathcal{T})$ be a fuzzy soft topological space. A subcollection $\mathcal{S}$ of $\mathcal{T}$ is called a subbase for $\mathcal{T}$ if the family of all finite intersections of members of $\mathcal{S}$ forms a base for $\mathcal{T}$.

3.21. Theorem. *Let* $\mathcal{S}$ *be a family of fuzzy soft sets over* X *such that* $\widetilde{1}_E, \widetilde{0}_E \in \mathcal{S}$. *Then* $\mathcal{S}$ *is a base for the topology* $\mathcal{T}$, *whose members are of the form* $\bigsqcup_{i \in J} \left(\prod_{k \in \Delta_i} (f_A)_{i,k} \right)$, *where* J *is arbitrary index set and for each* $i \in J$, Δ_i *is a finite index set,* $(f_A)_{i,k} \in \mathcal{S}$ *for* $i \in J$ *and* $k \in \Delta_i$.

Proof. Straightforward. □

3.22. Definition. Let $\{(\varphi, \psi)_i : \mathcal{F}(X,E) \to (Y_i, \mathcal{T}_i)\}_{i \in J}$ be a family of fuzzy soft mappings and $\{(Y_i, \mathcal{T}_i)\}_{i \in J}$ a family of fuzzy soft topological spaces. Then the topology $\mathcal{T}$ generated from the subbase $\mathcal{S} = \{(\varphi, \psi)_i^{-1}(f_A) \mid f_A \in \mathcal{T}_i, i \in J\}$ is called the *fuzzy soft topology* (or *initial fuzzy soft topology*) induced by the family of fuzzy soft mappings $\{(\varphi, \psi)_i\}_{i \in J}$ and from the family of fuzzy soft topological spaces $\{(Y_i, \mathcal{T}_i)\}_{i \in J}$.

The initial fuzzy soft topology $\mathcal{T}$ on X induced by the family $\{(\varphi, \psi)_i : \mathcal{F}(X,E) \to (Y_i, \mathcal{T}_i)\}_{i \in J}$ is the coarsest fuzzy soft topology with respect to which each $(\varphi, \psi)_i : (X, \mathcal{T}) \longrightarrow (Y_i, \mathcal{T}_i)$ is fuzzy soft continuous, $i \in J$.

3.23. Definition. Let $\{(X, \mathcal{T}_i)\}_{i \in J}$ be a family of fuzzy soft topological spaces. Then the initial fuzzy soft topology on X $(= \prod_{i \in J} X_i)$ generated by the family $\{(p_{X_i}, q_{E_i})_i\}_{i \in J}$, where $p_{X_i} : \prod_{i \in J} X_i \to X_i$ and $q_{E_i} : \prod_{i \in J} E_i \to E_i$, is called the *product fuzzy soft topology* on X.

3.24. Theorem. *Let* $\{(X_i, \mathcal{T}_i)\}_{i \in J}$ *be a family of fuzzy soft topological spaces and* $\mathcal{T}$ *the product fuzzy soft topology on* X $(= \prod_{i \in J} X_i)$. $\mathcal{T}$ *has as a base the set of finite intersections of fuzzy soft sets of the form* $(p_{X_i}, q_{E_i})_i^{-1}(f_A)_i$, *where* $(f_A)_i \in \mathcal{T}_i,\ i \in J$.

Proof. Straightforward. □

3.25. Theorem. *Let $\{(X_i, \mathfrak{T}_i)\}_{i\in J}$ be a family of fuzzy soft topological spaces and $\mathfrak{T}$ the product fuzzy soft topology on X $(= \prod_{i\in J} X_i)$. Let $(Y, \mathfrak{T}^*)$ be a fuzzy soft topological space and $(\varphi, \psi) : (Y, \mathfrak{T}^*) \to (X, \mathfrak{T})$ a fuzzy soft mapping. Then (φ, ψ) is fuzzy soft continuous if and only if $(p_{X_i}, q_{E_i}) \circ (\varphi, \psi) : (Y, \mathfrak{T}^*) \to (X_i, \mathfrak{T}_i)$ is fuzzy soft continuous, $i \in J$.*

Proof. Straightforward. □

3.26. Theorem. *Let $\{(X_i, \mathfrak{T}_i)\}_{i\in J}$, $\{(Y_i, \mathfrak{T}_i^*)\}_{i\in J}$ be two families of fuzzy soft topological spaces and $(X, \mathfrak{T})$, $(Y, \mathfrak{T}^*)$ their product fuzzy soft topological spaces, respectively. Let $(\varphi, \psi)_i : (X_i, \mathfrak{T}_i) \to (Y_i, \mathfrak{T}_i^*)$ be a fuzzy soft mapping for each $i \in J$. Then the product mapping $(\varphi, \psi) = \prod_{i\in J}(\varphi, \psi)_i : (X, \mathfrak{T}) \to (Y, \mathfrak{T}^*)$, where $x_i \to \varphi_i(x_i)$, $e_i \to \psi_i(e_i)$ is fuzzy soft continuous if $(\varphi, \psi)_i$ is fuzzy soft continuous, $\forall\, i \in J$.*

Proof. The fuzzy soft mapping (φ, ψ) can be written as $x \to \varphi_i(p_{X_i}(x))$ and $e \to \psi_i(q_{E_i}(e))$, where $x = (x_i)$, $e = (e_i)$ and (φ, ψ) is fuzzy soft continuous by Theorem 3.25. □

3.27. Theorem. *Let $\{(X_i, \mathfrak{T}_i)\}_{i=1}^n$, $\{(Y_i, \mathfrak{T}_i^*)\}_{i=1}^n$ be families of fuzzy soft topological spaces and $(X, \mathfrak{T})$, $(Y, \mathfrak{T}^*)$ be the product fuzzy soft topological spaces, respectively. Let $(\varphi, \psi)_i : (X_i, \mathfrak{T}_i) \to (Y_i, \mathfrak{T}_i^*)$ be a fuzzy soft mapping. Then the mapping $(\varphi, \psi) := \prod_{i=1}^n(\varphi, \psi)_i : (X, \mathfrak{T}) \to (Y, \mathfrak{T}^*)$, where $\varphi(x_1, \ldots, x_n) = (\varphi_1(x_1), \ldots, \varphi_n(x_n))$, $\psi(e_1, \ldots, e_n)$ $= (\psi_1(e_1), \ldots, \psi_n(e_n))$ is fuzzy soft open if $(\varphi, \psi)_i$ is fuzzy soft open, $\forall\, i = \overline{1, n}$.*

Proof. Let $f_A \in \mathfrak{T}$. For each $i = 1, 2, \ldots, n$ and $j \in \triangle$ there exist $(f_A)_{i_j} \in \mathfrak{T}_i$ such that $f_A = \sqcup_{j\in\triangle} \prod_{i=1}^n (f_{A\,i_j})$. Then:

$$\begin{aligned}
\varphi(f_A)(k)(y) &= \varphi\left(\sqcup_{j\in\triangle} \prod_{i=1}^n (f_A)_{i_j}\right)(k)(y)\\
&= \bigvee_{j\in\triangle} \bigvee_{\varphi(x)=y} \bigvee_{\psi(e)=k} \left(\sqcup_{j\in\triangle} \prod_{i=1}^n (f_{A\,i_j})\right)(e)(x)\\
&= \bigvee_{j\in\triangle} \bigvee_{\varphi_1(x_1)=y_1} \bigvee_{\psi_1(e_1)=k_1} \cdots\\
&\qquad \cdots \bigvee_{\varphi_n(x_n)=y_n} \bigvee_{\psi_n(e_n)=k_n} \Big((f_A)_{1_j}(e_1)(x_1) \wedge \cdots \wedge (f_A)_{n_j}(e_n)(x_n)\Big)\\
&= \bigvee_{j\in\triangle} \Big(\varphi_1(f_A)_1(k_1)(y_1) \wedge \cdots \wedge \varphi_n(f_A)_n(k_n)(y_n)\Big)\\
&= \left(\sqcup_{j\in\triangle} \prod_{i=1}^n \varphi_i(f_A)_i\right)(k)(y).
\end{aligned}$$

Hence, $(\varphi, \psi)(f_A) = \sqcup_{j\in\triangle} \prod_{i=1}^n \varphi_i(f_A)_i$. Since $(f_A)_{i_j} \in \mathfrak{T}_i$ and $(\varphi, \psi)_i$ is fuzzy soft open, we obtain $(\varphi, \psi)_i(f_A)_{i_j}$ is fuzzy soft open. □

4. Conclusion

In this paper, a new structure called a "fuzzy soft topology" is introduced and studied. The notions of fuzzy soft continuity, initial and product fuzzy soft topology, fuzzy soft closure (interior) are introduced and some results are investigated. To extend this work, one could study the properties of fuzzy soft topological spaces in other topological structures.

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