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NEW INEQUALITIES FOR MAXIMUM MODULUS VALUES OF POLYNOMIAL FUNCTIONS

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Abstract

In this paper we establish new inequalities for the maximum modulus values of polynomial functions which do not have zero as a root, and which have zero as a multiple root on two hyperbolic regions. Similar inequalities for the particular case where the region is a ring are also given.

Keywords: Mathematical analysis, Hyperbolic region, Polynomials functions, Maximum modulus values, Inequality.

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1. Introduction

The Maximum Modulus Theorem [3, 4] states:

Let U be open and connected, f holomorpic on U and nonconstant.

(i) For any $z_0 \in U$, there is $z_1 \in U$ such that $|f(z_0)| < |f(z_1)|$. In other words, $|f|$ cannot attain its maximum on U .

(ii) Suppose U is bounded and f is continuous on $\mathrm{cl}\, U$. Then $|f|$ attain its maximum on the boundary ∂U.

(iii) Let U and f be as (ii). Then

$\mathrm{Sup}\{|f(z)| : z \in \mathrm{cl}\, U\} = \mathrm{Sup}\{|f(z)| : z \in \partial U\}$.

Let $f, g : C \to C$ be complex-valued polynomial functions of degrees $m \geq 1,\ n \geq 1$, respectively, of a complex variable z, and $M_f = \max\limits_{|z|=R} |f(z)|$, $M_g = \max\limits_{|z|=R} |g(z)|$ and $M_{f.g} = \max\limits_{|z|=R} |f(z).g(z)|$ $(R > 1)$. It is shown in [1] that, if $z = 0$ is not a root of the given polynomials,

$M_{f.g} \geq \delta_1 \cdot M_f \cdot M_g$i with $\delta_1 = \frac{1}{2^m} \cdot \frac{1}{2^n}$,

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if $z = 0$ is a k-multiple root of f and an r-multiple root of g, then

$M_{f.g} \geq \delta \cdot M_f \cdot M_g$ with $\delta = \frac{1}{2^{m-k}} \cdot \frac{1}{2^{n-r}}$.

Now let $M_f = \max\limits_{|z|=1} |f(z)|$, $M_g = \max\limits_{|z|=1} |g(z)|$ and $M_{f.g} = \max\limits_{|z|=1} |f(z).g(z)|$. It is shown in [5] that,

$M_{f.g} \geq \nu \cdot M_f \cdot M_g$ with $\nu = \frac{1}{2^m} \cdot \frac{1}{2^n}$,

and in [6] that

$M_{f.g} \geq \nu \cdot M_f \cdot M_g$ with $\nu = \sin^m \frac{\pi}{8m} \cdot \sin^n \frac{\pi}{8n}$.

If f and g accept $z = 0$ as k-multiple and r-multiple roots, respectively, in [2] the following inequality is obtained

$M_{f.g} \geq \delta \cdot M_f \cdot M_g$ with $\delta = \frac{1}{2^{m-k}} \cdot \frac{1}{2^{n-r}}$.

Let $f_1, f_2, \ldots, f_n : C \to C$ be n complex-valued polynomials of degrees $d_1, d_2, \ldots, d_n$, respectively, of a complex variable z. In [7] the following inequality is obtained

$$M_{f_1} \cdot M_{f_2} \cdots M_{f_n} \geq M_{f_1 \cdot f_2 \cdots f_n} \geq k \cdot M_{f_1} \cdot M_{f_2} \cdots M_{f_n},$$

where $k = (\sin \frac{2}{n} \frac{\pi}{8d_1})^{d_1} \cdot (\sin \frac{2}{n} \frac{\pi}{8d_2})^{d_2} \cdots (\sin \frac{2}{n} \frac{\pi}{8d_n})^{d_n}$.

Throughout our paper, $z = x + iy \in C$ is a complex variable. Given $a, b, c \in \mathbb{R}^+$, the hyperbolic regions B_i $(i = 1, 2)$ are as follows:

$$B_1 = \{z : \Re(z^2) \geq a^2,\ -b \leq \Re(z) \leq b,\ b > a\},$$
$$B_2 = \{z : \Re(z^2) \leq a^2,\ \Im(iz^2) \leq b^2,\ -c \leq \Im(z) \leq c,\ c \geq b\}.$$

We denote by ∂B_i, $(i = 1, 2)$, the boundaries of B_i $(i = 1, 2)$. We use the usual topology on the field of complex numbers.

2. Maximum modulus values of polynomial functions not admitting $z = 0$ as a root on the hyperbolic regions B_i, $(i = 1, 2)$

2.1. Theorem. *Let* $M_f = \max\limits_{z \in \partial B_i} |f(z)|$, $M_g = \max\limits_{z \in \partial B_i} |g(z)|$ *and* $M_{f.g} = \max\limits_{z \in \partial B_i} |f(z) \cdot g(z)|$ *be the maximum modulus values of the polynomials* $f(z) = \prod_{i=1}^m (z - \alpha_i)$, $(\alpha_i \neq 0)$ *and* $g(z) = \prod_{j=1}^n (z - \beta_j)$, $(\beta_j \neq 0)$, *where* $\min(|\alpha_i|, |\beta_j|) = r$, $(1 \leq i \leq m, 1 \leq j \leq n)$, *on the hyperbolic regions* B_i $(i = 1, 2)$.

(i) *Let* $a \neq r$ *be on the hyperbolic region* B_1. *Then:*

$$M_{f.g} \geq \delta \cdot M_f \cdot M_g \text{ with } \delta = \left(\frac{|a - r|}{\sqrt{2b^2 - a^2}}\right)^{m+n} \cdot \frac{1}{2^m} \cdot \frac{1}{2^n}. \tag{2.1}$$

(ii) *Let* $\min(a, b) \neq r$ *be on the hyperbolic region* B_2. *Then:*

$$M_{f.g} \geq \theta \cdot M_f \cdot M_g \text{ with } \theta = \left(\frac{|\min(a, b) - r|}{\sqrt{2c^2 + a^2}}\right)^{m+n} \cdot \frac{1}{2^m} \cdot \frac{1}{2^n}. \tag{2.2}$$

Proof. (i)

$$\text{For all } z_1, z_2 \in \partial B_1,\ \max|z_1 - z_2| = 2\sqrt{2b^2 - a^2}. \tag{2.3}$$

For $z, \alpha_i \in \overline{B_1} = B_1 \bigcup \partial B_1$, by (2.3) we have

$$|f(z)| = \prod_{i=1}^m |z - \alpha_i| \leq 2^m \cdot \left(\sqrt{2b^2 - a^2}\right)^m$$

and

$$|g(z)| = \prod_{j=1}^{n} |z - \beta_j| \le 2^n \cdot \left(\sqrt{2b^2 - a^2}\right)^n .$$

Hence, from the expressions above we write

$$M_f = \max_{z \in \partial B_1} |f(z)| \le 2^m \cdot \left(\sqrt{2b^2 - a^2}\right)^m$$

and

$$M_g = \max_{z \in \partial B_1} |g(z)| \le 2^n \cdot \left(\sqrt{2b^2 - a^2}\right)^n .$$

So, from these identities we obtain

$$M_f \cdot M_g \le 2^{m+n} \left(\sqrt{2b^2 - a^2}\right)^{m+n} . \tag{2.4}$$

Also, for all $z, \alpha_i, \beta_j \in C$, consider the inequalities $|z - \alpha_i| \ge ||z| - |\alpha_i||$ and $|z - \beta_j| \ge ||z| - |\beta_j||$. So we have

$$|f(z)| = \prod_{i=1}^{m} |z - \alpha_i| \ge \prod_{i=1}^{m} ||z| - |\alpha_i||$$

and

$$|g(z)| = \prod_{j=1}^{n} |z - \beta_j| \ge \prod_{j=1}^{n} ||z| - |\beta_j|| .$$

It follows that

$$|f(z).g(z)| = \prod_{i=1}^{m} |z - \alpha_i| \cdot \prod_{j=1}^{n} |z - \beta_j| \ge \prod_{i=1}^{m} ||z| - |\alpha_i|| \cdot \prod_{j=1}^{n} ||z| - |\beta_j|| .$$

By the maximum modulus principle and the hypothesis,

$$\max_{z \in \partial B_1} |f(z) \cdot g(z)| \ge \prod_{i=1}^{m} ||z| - r| \cdot \prod_{j=1}^{n} ||z| - r| , \ z \in \partial B_1.$$

and hence

$$\max_{z \in \partial B_1} |f(z) \cdot g(z)| \ge \prod_{i=1}^{m} |a - r| \cdot \prod_{j=1}^{n} |a - r| ,$$

so we set

$$M_{f.g} = \max_{z \in \partial B_1} |f(z).g(z)| \ge |a - r|^{m+n} . \tag{2.5}$$

For $a \ne r$, (2.4) and (2.5) allow us to write

$$\frac{M_f \cdot M_g}{M_{f.g}} \le \frac{2^m \cdot 2^n \cdot \left(\sqrt{2b^2 - a^2}\right)^{m+n}}{(|a - r|)^{m+n}}. \tag{2.6}$$

Rearranging (2.6) gives us (2.1).

(ii) For all $z_1, z_2 \in \partial B_2$, $\max |z_1 - z_2| = 2\sqrt{2c^2 + a^2}$ and

$$\max_{z \in \partial B_2} \left(\prod_{i=1}^{m} |z - \alpha_i| \cdot \prod_{j=1}^{n} |z - \beta_j| \right) \ge |\min(a, b) - r|^{m+n}.$$

Now a proof similar to that given for (i) leads us to the desired conclusion. □

2.2. Corollary. *Let*

$$f_1(z) = \prod_{i_1=1}^{m_1} (z - \alpha_{i_1}),\ (\alpha_{i_1} \neq 0),$$

$$f_2(z) = \prod_{i_2=1}^{m_2} (z - \alpha_{i_2}),\ (\alpha_{i_2} \neq 0),$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$f_n(z) = \prod_{i_n=1}^{m_n} (z - \alpha_{i_n}),\ (\alpha_{i_n} \neq 0),$$

be n, $(n \geq 3)$, *polynomials on the hyperbolic regions* B_i, $(i = 1, 2)$, *and satisfying* $\min(|\alpha_{i1}|, |\alpha_{i2}|, \ldots, |\alpha_{in}|) = r$, $(1 \leq i_1 \leq m_1, 1 \leq i_2 \leq m_2, \ldots, 1 \leq i_n \leq m_n)$.

(i) *let* $a \neq r$ *be on the hyperbolic region* B_1. *Then*

$$M_{f_1 \cdot f_2 \cdots f_n} \geq \delta_1 \cdot M_{f_1} \cdot M_{f_2} \cdots M_{f_n}, \tag{2.7}$$
$$\text{where } \delta_1 = \left(\frac{|a - r|}{\sqrt{2b^2 - a^2}} \right)^{m_1 + m_2 + \cdots + m_n} \cdot \frac{1}{2^{m_1}} \cdot \frac{1}{2^{m_2}} \cdots \frac{1}{2^{m_n}}.$$

(ii) *Let* $\min(a, b) \neq r$ *be on the hyperbolic region* B_2. *Then*

$$M_{f_1 \cdot f_2 \cdots f_n} \geq \theta_1 \cdot M_{f_1} \cdot M_{f_2} \cdots M_{f_n}, \tag{2.8}$$
$$\text{where } \theta_1 = \left(\frac{|\min(a, b) - r|}{\sqrt{2c^2 + a^2}} \right)^{m_1 + m_2 + \cdots + m_n} \cdot \frac{1}{2^{m_1}} \cdot \frac{1}{2^{m_2}} \cdots \frac{1}{2^{m_n}}.$$

3. Maximum modulus values of polynomial functions having $z = 0$ as a root on the hyperbolic regions B_i, $(i = 1, 2)$

In this section we will give some relations concerning maximum modulus values of polynomials which admit $z = 0$ as a multiple root on hyperbolic regions B_i, $(i = 1, 2)$

3.1. Theorem. *Let*

$$f(z) = z^k \prod_{i=1}^{m-k} (z - \alpha_i),\ (\alpha_i \neq 0, k \leq m)$$

and

$$g(z) = z^p \prod_{j=1}^{n-p} (z - \beta_j),\ (\beta_j \neq 0, p \leq n)$$

be polynomials on the hyperbolic regions B_i, $(i = 1, 2)$, *satisfying*

$$\min(|\alpha_i|, |\beta_j|) = r,\ (1 \leq i \leq m - k, 1 \leq j \leq n - p).$$

(i) *Let* $a \neq r$ *be on the hyperbolic region* B_1. *Then:*

$$M_{f.g} \geq \varepsilon_2 \cdot M_f \cdot M_g, \tag{3.1}$$
$$\text{where } \varepsilon_2 = \left(\frac{a}{\sqrt{2b^2 - a^2}} \right)^{k+p} \cdot \left(\frac{|a - r|}{\sqrt{2b^2 - a^2}} \right)^{m+n-k-p} \cdot \frac{1}{2^{m-k}} \cdot \frac{1}{2^{n-p}}.$$

(ii) *Let* $\min(a,b) \neq r$ *be on the hyperbolic region* B_2*. Then:*

$$M_{f.g} \geq \varepsilon_3 \cdot M_f \cdot M_g, \tag{3.2}$$
$$\text{where } \varepsilon_3 = \left(\frac{\min(a,b)}{\sqrt{2c^2+a^2}}\right)^{k+p} \cdot \left(\frac{|\min(a,b)-r|}{\sqrt{2c^2+a^2}}\right)^{m+n-k-p} \cdot \frac{1}{2^{m-k}} \frac{1}{2^{n-p}}.$$

Proof. (i) For all $z \in \partial B_1$, $|z| \leq \sqrt{2b^2-a^2}$. Hence, we can write

$$|z|^k \leq \sqrt{2b^2-a^2}^{\,k} \text{ and } |z|^p \leq \sqrt{2b^2-a^2}^{\,p}. \tag{3.3}$$

Also, from (2.3) we have

$$\prod_{i=1}^{m-k} |z-\alpha_i| \leq 2^{m-k} \cdot \left(\sqrt{2b^2-a^2}\right)^{m-k} \tag{3.4}$$

and

$$\prod_{j=1}^{n-p} |z-\beta_j| \leq 2^{n-p} \cdot \left(\sqrt{2b^2-a^2}\right)^{n-p}. \tag{3.5}$$

Then by (3.3), (3.4) and (3.5) we set

$$M_f = \max_{z\in\partial B_1} |f(z)| \leq 2^{m-k} \cdot \left(\sqrt{2b^2-a^2}\right)^{m}, \text{ and}$$
$$M_g = \max_{z\in\partial B_1} |g(z)| \leq 2^{n-p} \cdot \left(\sqrt{2b^2-a^2}\right)^{n}. \tag{3.6}$$

It follows from (3.6) that

$$M_f \cdot M_g \leq 2^{m-k} \cdot 2^{n-p} \left(\sqrt{2b^2-a^2}\right)^{m+n}. \tag{3.7}$$

Moreover, consider

$$\prod_{i=1}^{m-k} |z-\alpha_i| \geq \prod_{i=1}^{m-k} ||z|-|\alpha_i|| \text{ and } \prod_{j=1}^{n-p} |z-\beta_j| \geq \prod_{j=1}^{n-p} ||z|-|\beta_j||. \tag{3.8}$$

From (3.8) we can write

$$|f(z)\cdot g(z)| \geq |z|^{p+k} \prod_{i=1}^{m-k} \big||z|-|\alpha_i|\big| \cdot \prod_{j=1}^{n-p} \big||z|-|\beta_j|\big|.$$

Passing to the maximum

$$\max_{z\in B_1} |f(z)\cdot g(z)| \geq |z|^{p+k} \prod_{i=1}^{m-k} \big||z|-|\alpha_i|\big| \cdot \prod_{j=1}^{n-p} \big||z|-|\beta_j|\big|, \; z \in \partial B_1,$$

hence

$$\max_{z\in B_1} |f(z)\cdot g(z)| \geq |z|^{p+k} \prod_{i=1}^{m-k} \big||z|-r\big| \cdot \prod_{j=1}^{n-p} \big||z|-r\big|, \; z \in \partial B_1,$$

and

$$\max_{z\in B_1} |f(z)\cdot g(z)| \geq |z|^{p+k} \prod_{i=1}^{m-k} \big|a-r\big| \cdot \prod_{j=1}^{n-p} \big|a-r\big|.$$

Thus we find that

$$M_{f.g} = \max_{z\in\partial B_1} |f(z)\cdot g(z)| \geq a^{k+p} \cdot |a-r|^{m+n-k-p}. \tag{3.9}$$

By (3.7) and (3.9) we write

$$\frac{M_f \cdot M_g}{M_{f.g}} \leq \frac{2^{m-k} \cdot 2^{n-p} \cdot \left(\sqrt{2b^2 - a^2}\right)^{m+n}}{(|a-r|)^{m+n-k-p} \cdot a^{k+p}}. \tag{3.10}$$

Now, (3.10) yields (3.1).

(ii) For all $z \in \partial B_2$ we have $|z| \leq \sqrt{2c^2 + a^2}$. Hence $|z|^k \leq \sqrt{2c^2 + a^2}^k$ and $|z|^p \leq \sqrt{2c^2 + a^2}^p$. Also, for all $z_1, z_2 \in \partial B_2$, consider $\max|z_1 - z_2| = 2\sqrt{2c^2 + a^2}$ Then $|f(z)| \leq 2^{m-k} \cdot \left(\sqrt{2c^2 + a^2}\right)^m$ and $|g(z)| \leq 2^{n-p} \cdot \left(\sqrt{2c^2 + a^2}\right)^n$. On the other hand, we have

$$\max_{z \in \partial B_2} |f(z) \cdot g(z)| \geq (\min(a,b))^{k+p} |\min(a,b) - r|^{m+n-k-p}.$$

Now a similar argument to that used in the proof of (i) gives the result. □

3.2. Corollary. *Let*

$$f_1(z) = z^{k_1} \prod_{i_1=1}^{m_1-k_1} (z - \alpha_{i_1}), (\alpha_{i_1} \neq 0,\ k_1 \leq m_1),$$

$$f_2(z) = z^{k_2} \prod_{i_2=1}^{m_2-k_2} (z - \alpha_{i_2}), (\alpha_{i_2} \neq 0,\ k_2 \leq m_2),$$

$$\cdots\cdots\cdots\cdots$$

$$f_n(z) = z^{k_n} \prod_{i_n=1}^{m_n-k_n} (z - \alpha_{i_n}), (\alpha_{i_n} \neq 0, k_n \leq m_n)$$

be n, $(n \geq 3)$, polynomials on the hyperbolic regions B_i, $(i = 1, 2)$, and let

$$\min\left(|\alpha_{i_1}|, |\alpha_{i2}|, \ldots, |\alpha_{in}|\right) = r,\ (1 \leq i_1 \leq m_1,\ 1 \leq i_2 \leq m_2, \ldots, 1 \leq i_n \leq m_n).$$

(i) *Let $a \neq r$ be on the hyperbolic region B_1. Then:*

$$\begin{aligned} &M_{f_1 \cdot f_2 \cdots f_n} \geq \delta_2 \cdot M_{f_1} \cdot M_{f_2} \cdots M_{f_n}, \\ &\textit{where } \delta_2 = \left(\frac{a}{\sqrt{2b^2 - a^2}}\right)^{k_1 + k_2 + \cdots + k_n} \\ &\qquad \cdot \left(\frac{|a-r|}{\sqrt{2b^2 - a^2}}\right)^{m_1 + m_2 + \cdots + m_n - k_1 - k_2 - \cdots - k_n} \\ &\qquad \cdot \frac{1}{2^{m-k_1}} \cdot \frac{1}{2^{m_2-k_2}} \cdots \frac{1}{2^{m_n-k_n}}. \end{aligned} \tag{3.11}$$

(ii) *Let $\min(a,b) \neq r$ be on the hyperbolic region B_2. Then:*

$$\begin{aligned} &M_{f_1 \cdot f_2 \cdots f_n} \geq \theta_2 \cdot M_{f_1} \cdot M_{f_2} \cdots M_{f_n}, \\ &\textit{where } \theta_2 = \left(\frac{\min(a,b)}{\sqrt{2c^2 + a^2}}\right)^{k_1 + k_2 + \cdots + k_n} \\ &\qquad \cdot \left(\frac{|\min(a,b) - r|}{\sqrt{2c^2 + a^2}}\right)^{m_1 + m_2 + \cdots + m_n - k_1 - k_2 - \cdots - k_n} \\ &\qquad \cdot \frac{1}{2^{m_1-k_1}} \cdot \frac{1}{2^{m_2-k_2}} \cdots \frac{1}{2^{m_n-k_n}}. \end{aligned} \tag{3.12}$$

□

4. A special case and the maximum modulus values of polynomial functions on a ring region

Let us denote by $B_2(R)$ the region B_2 which is formed by taking $a = b = c = R$, $(R \geq 1)$. Then

$$\min(a, b) = R, \ \sqrt{2c^2 + a^2} = \sqrt{3}R.$$

We obtain the followings identities:

From (2.1),

$$\theta = \left(\frac{|R-r|}{\sqrt{3}R}\right)^{m+n} \cdot \frac{1}{2^m} \cdot \frac{1}{2^n}; \tag{4.1}$$

From (2.8),

$$\theta_1 = \left(\frac{|R-r|}{\sqrt{3}R}\right)^{m_1+m_2+\cdots+m_n} \cdot \frac{1}{2^{m_1}} \cdot \frac{1}{2^{m_2}} \cdots \frac{1}{2^{m_n}}; \tag{4.2}$$

From (3.2),

$$\varepsilon_3 = \left(\frac{1}{\sqrt{3}}\right)^{k+p} \cdot \left(\frac{|R-r|}{\sqrt{3}R}\right)^{m+n-k-p} \cdot \frac{1}{2^{m-k}} \cdot \frac{1}{2^{n-p}}; \tag{4.3}$$

From (3.12),

$$\gamma_2 = \left(\frac{1}{\sqrt{3}}\right)^{k_1+k_2+\cdots+k_n} \cdot \left(\frac{|R-r|}{\sqrt{3}R}\right)^{m_1+m_2+\cdots+m_n-k_1-k_2-\cdots-k_n} \cdot \frac{1}{2^{m_1-k_1}} \cdot \frac{1}{2^{m_2-k_2}} \cdots \frac{1}{2^{m_n-k_n}}. \tag{4.4}$$

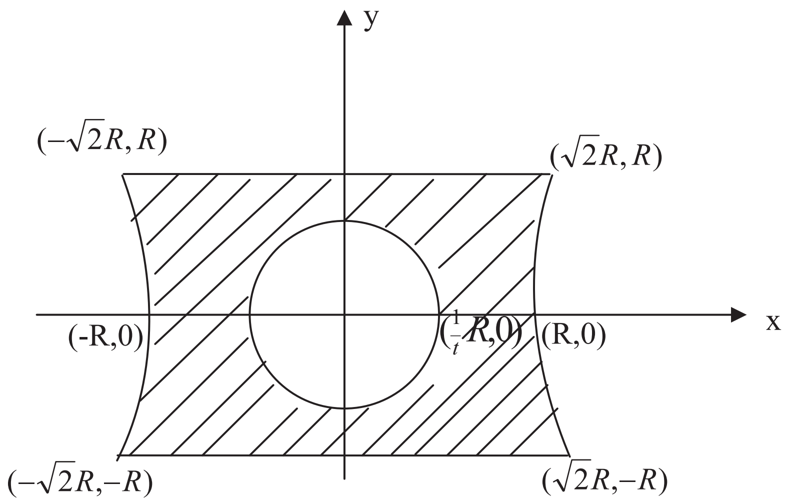
Now consider the circular region $D(R) = \{z \in C : |z| \leq R\}$, where $R \geq 1$ and R is finite. Then $D(R) \subset B_2(R)$. Now consider $C(t, r) = \{z \in C : |z| \geq \frac{1}{t}R, (1 < t < \infty)\}$ and the region $H = B_2(R) \bigcap C(t, R)$. The region H is the ring in Figure 1. Also, take $r = \frac{1}{t}R$, $(1 < t < \infty)$ in (4.1) and (4.2). In this case let $M_f = \max_{z \in \partial H} |f(z)|$, $M_g = \max_{z \in \partial H} |g(z)|$ and $M_{f.g} = \max_{z \in \partial H} |f(z) \cdot g(z)|$. We have, respectively,

$$M_{f.g} \geq \gamma \cdot M_f \cdot M_g, \quad \text{where } \gamma = \left(\frac{t-1}{\sqrt{3}t}\right)^{m+n} \cdot \frac{1}{2^m} \cdot \frac{1}{2^n}, \tag{4.5}$$

and

$$M_{f_1.f_2\cdots f_n} \geq \gamma_1 \cdot M_{f_1} \cdot M_{f_2} \cdots M_{f_n}, \quad \text{where } \gamma_1 = \left(\frac{t-1}{\sqrt{3}t}\right)^{m_1+m_2+\cdots+m_n} \cdot \frac{1}{2^{m_1}} \cdot \frac{1}{2^{m_2}} \cdots \frac{1}{2^{m_n}}. \tag{4.6}$$

Figure 1 (Region *H*)



5. Conclusion and results

Formulas (2.1), (2.1), (2.7), (2.8),(3.1) and (3.2) gives inequalities about maximum modulus values of polynomials on hyperbolic regions B_1, B_2. We can also say that on the region $B_2(R)$ (a special hyperbolic region) the formulas (4.1), (4.2), (4.3) and (4.4) hold. On the ring region H the formulas (4.5) and (4.6) are considered. If the formulas given in a disc with radius $r = \frac{1}{t} \cdot R$ $(1 < t < \infty)$ are taken into consideration, formulas (4.5) and (4.6) hold on the region $B_2(R)$ outside the disc (or on the region H).

On the other hand, the formulas given in [1, 2, 5] hold in a disc with center $(0, 0)$ (for example, a disc of radius $\sqrt{3}R$) which contains the region $B_2(R)$. Yet there exist polynomial functions for which the formulas (4.1),(4.2), (4.3) and (4.4) hold in the hyperbolic region $B_2(R)$ which is in a disc of radius $\geq \sqrt{3} \cdot R$.

5.1. Example. Let $t = \sqrt{3}$. Then $r = \frac{1}{\sqrt{3}}R$. Let the center of the disc be $(0, 0)$ and its radius $\sqrt{3}R$. [There exist polynomials $f, g, f_1, f_2, \ldots, f_n$ suitable for the hyperbolic region $B_2(R)$]. On this disc the following coefficients are given for the formulas in [1, 2, 5]:

$$\begin{aligned} \delta_1 = \nu &= \frac{1}{2^m} \cdot \frac{1}{2^n}, \\ \delta &= \frac{1}{2^{m-k}} \cdot \frac{1}{2^{n-r}}, \\ \varepsilon &= \frac{1}{2^{m_1}} \cdot \frac{1}{2^{m_2}} \cdots \frac{1}{2^{m_n}}, \text{ and} \\ \varepsilon_2 &= \frac{1}{2^{m_1-r_1}} \cdot \frac{1}{2^{m_2-r_2}} \cdots \frac{1}{2^{m_n-r_n}}. \end{aligned}$$

Yet on the hyperbolic region $B_2(R)$, which is completely inside the disc, the coefficients of the same formulas will be

$$\theta = \left(\frac{\sqrt{3}-1}{3}\right)^{m+n} \cdot \frac{1}{2^m} \cdot \frac{1}{2^n},$$

$$\varepsilon_3 = \left(\frac{1}{\sqrt{3}}\right)^{k+p} \cdot \left(\frac{\sqrt{3}-1}{3}\right)^{m+n-k-p} \cdot \frac{1}{2^{m-k}} \cdot \frac{1}{2^{n-p}},$$

$$\theta_1 = \left(\frac{\sqrt{3}-1}{3}\right)^{m_1+m_2+\cdots+m_n} \cdot \frac{1}{2^{m_1}} \cdot \frac{1}{2^{m_2}} \cdots \frac{1}{2^{m_n}}, \text{ and}$$

$$\gamma_2 = \left(\frac{1}{\sqrt{3}}\right)^{r_1+r_2+\cdots+r_n} \cdot \left(\frac{\sqrt{3}-1}{3}\right)^{m_1+m_2+\cdots+m_n-r_1-r_2-\cdots-r_n} \cdot \frac{1}{2^{m_1-r_1}} \cdots \frac{1}{2^{m_2-r_2}} \cdots \frac{1}{2^{m_n-r_n}}.$$

The corresponding coefficients are seen to be smaller.

Naturally, similar inequalities can be obtained on an hyperbolic region which is bounded completely by hyperboles.

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