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ON FUNCTIONS THAT ARE JANOWSKI STARLIKE WITH RESPECT TO N-SYMMETRIC POINTS

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Abstract

In this paper sufficient conditions for a function to be Janowski starlike with respect to N-symmetric points are given in terms of the quotient of analytical representations of starlikeness and convexity with respect to N-symmetric points.

Keywords: Analytic function, Janowski starlike, N-symmetric points, Sufficient condition.

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1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of analytic functions in the unit disk $\mathcal{U} = \{z : |z| < 1\}$ normalized such that $f(0) = f'(0) - 1 = 0$.

Further, let f and g be analytic functions in the unit disc $\mathcal{U}$. Then we say that f is *subordinate* to g, and we write $f \prec g$ or $f(z) \prec g(z)$ if there exists function ω, analytic in the unit disc $\mathcal{U}$, such that $\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$ for all $z \in \mathcal{U}$. Specially, if g is univalent in $\mathcal{U}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Now, a function $f \in \mathcal{A}$ is said to be in the class of *Janowski starlike functions with respect to N-symmetric points*, denoted by $S^*_N[A, B]$, $-1 \leq B < A \leq 1$, $N = 1, 2, 3, \ldots$, if

$$\frac{zf'(z)}{f_N(z)} \prec \frac{1 + Az}{1 + Bz},$$

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where

$$f_N(z) = z + \sum_{m=2}^{\infty} a_{m\cdot N+1} z^{m\cdot N+1}.$$

Geometrically, this means that the image of $\mathcal{U}$ by $zf'(z)/f_N(z)$ is inside the open disk centered on the real axis, on the right hand side of the complex plane, with diameter end points $(1-A)/(1-B)$ and $(1+A)/(1+B)$. Thus, $S_N^*[A,B]$ is subclass of the class of close-to-convex (univalent) functions (see [2, p.314]). Special selections of A and B lead us to the following classes:

- $S_N^*(\alpha) \equiv S_N^*[1-2\alpha,-1]$, $0 \le \alpha < 1$, with analytic representation

$$\operatorname{Re} \frac{zf'(z)}{f_N(z)} > \alpha, \quad z \in \mathcal{U},$$

is the class of *functions that are starlike of order α with respect to N-symmetric points*;

- $S_N^* \equiv S_N^*[1,-1] = S_N^*(0)$, with analytic representation

$$\operatorname{Re} \frac{zf'(z)}{f_N(z)} > 0, \quad z \in \mathcal{U},$$

is the class of *functions that are starlike with respect to N-symmetric points*, first introduced by Sakaguchi in [4];

- $S_N^*[\alpha, 0]$, $0 < \alpha \le 1$ is the class defined by

$$\left| \frac{zf'(z)}{f_N(z)} - 1 \right| < \alpha, \quad z \in \mathcal{U}.$$

Bearing in mind that $f_1(z) = f(z)$, for $N = 1$ classes defined above yield the well known classes of univalent functions: $S^*[A,B] \equiv S_1^*[A,B]$ is the class of *Janowski starlike functions*, first defined in [1]; $S^*(\alpha) \equiv S_1^*(\alpha)$, $0 \le \alpha < 1$, is the class of *starlike functions of order α,* and $S^* \equiv S_1^*$ is the class of *starlike functions.* One can also note that $S_N^*[A,B] \subseteq S_N^*((1-A)/(1-B))$.

Geometrical characterization of a function $f(z) \in S_N^*$, $N \ge 2$, is: if $\varepsilon = \exp(2\pi i/N)$ and r is close to 1, $r < 1$, then the angular velocity of $f(z)$ about the point

$$M_{f,N}(z_0) = \frac{1}{\sum_{j=1}^{N-1} \varepsilon^{-j}} \cdot \sum_{j=1}^{N-1} \varepsilon^{-j} \cdot f(\varepsilon^j z_0)$$

is positive at $z = z_0$ as z traverses the circle $|z| = r$ in the positive direction.

Further, a function $f \in \mathcal{A}$ is *Janowski convex with respect to N-symmetric points* if

$$\frac{[zf'(z)]'}{f_N'(z)} \prec \frac{1+Az}{1+Bz},$$

and the corresponding class is denoted by $\mathcal{K}_N[A,B]$. In a similar way as before, special choices of N, A and B lead to some well known classes of univalent functions.

In this paper we will study the quotient of analytical representations of starlikeness and convexity with respect to N-symmetric points, i.e. we will study the expression

$$\frac{[zf'(z)]'/f_N'(z)}{zf'(z)/f_N(z)} = \frac{1 + zf''(z)/f'(z)}{zf_N'(z)/f_N(z)}$$

and obtain necessary conditions that will embed $f(z)$ in the class $S_N^*[A,B]$.

The classical case when $N = 1$ is studied in the following papers: $N = a = b = 1$ in [3], [5], [8]; $N = b = 1$ and a real in [6], [7]; and the most general case when $N = 1$ and a, b real in [9]. Therefore we will consider only $N \in \mathbb{N} \setminus \{1\}$.

For obtaining our main results we will make use of the following lemma from the theory of differential subordinations which is a special case of [2, Theorem 2.3h].

1.1. Lemma. *Let Ω be a subset of the complex plane $\mathbb{C}$ and let the function $\psi : \mathbb{C}^2 \times \mathcal{U} \to \mathbb{C}$ satisfy $\psi(Me^{i\theta}, Ke^{i\theta}; z) \notin \Omega$ for all real θ, $K \geq M$ and for all $z \in \mathcal{U}$. If the function $p(z)$ is analytic in $\mathcal{U}$, $p(0) = 0$ and $\psi(p(z), zp'(z); z) \in \Omega$ for all $z \in \mathcal{U}$ then $|p(z)| < M$, $z \in \mathcal{U}$.* $\square$

2. Main results and consequences

2.1. Lemma. *Let $f \in \mathcal{A}$, $N \in \mathbb{N} \setminus \{1\}$ and $-1 \leq B < A \leq 1$. Also, let $\Omega = \mathbb{C} \setminus \Omega_1$, where*

$$\Omega_1 = \left\{ 1 + K(A-B) \cdot \frac{f_N(z)}{zf_N'(z)} \cdot \frac{e^{i\theta}}{(1+Ae^{i\theta})(1+Be^{i\theta})} \;:\; z \in \mathcal{U},\ \theta \in \mathbb{R}, K \geq 1 \right\}.$$

If

$$\frac{[zf'(z)]'/f_N'(z)}{zf'(z)/f_N(z)} \in \Omega, \quad z \in \mathcal{U},$$

then $f \in S_N^*[A,B]$.

Proof. Let us define functions $p(z) = \frac{zf'(z)}{f_N(z)}$ and $p_1(z)$, such that $p(z) = \frac{1+Ap_1(z)}{1+Bp_1(z)}$. Both functions are analytic in $\mathcal{U}$ and $p(0) - 1 = p_1(0) = 0$. Using Lemma 1.1 with

$$\psi(r, s; z) = 1 + (A-B) \cdot \frac{f_N(z)}{zf_N'(z)} \cdot \frac{s}{(1+Ar)(1+Br)}$$

and $M = 1$, after verifying that

$$\psi(p_1(z), zp_1'(z); z) = \frac{[zf'(z)]'/f_N'(z)}{zf'(z)/f_N(z)},$$

we conclude $|p_1(z)| < 1$, $z \in \mathcal{U}$. This inequality is equivalent to the subordinations

$$p_1(z) = \frac{1-p(z)}{Bp(z)-A} \prec z \text{ and } p(z) \prec \frac{1+Az}{1+Bz},$$

which proves that $f \in S_N^*[A,B]$. $\square$

Using Lemma 2.1, we will prove our main result.

2.2. Theorem. *Let $f \in \mathcal{A}$, $N \in \mathbb{N} \setminus \{1\}$, $-1 \leq B < A \leq 1$ and $\mu > 1$. Also let*

$$\lambda \equiv \begin{cases} \dfrac{2\mu\sqrt{A|B|}}{1-AB}, & AB < 0 \text{ and } 4AB \leq (A+B)(1+AB) \leq 4A|B|, \\ \dfrac{(A-B)\mu}{(1+|A|)(1+|B|)}, & \text{otherwise.} \end{cases}$$

If

$$\left| \frac{zf_N'(z)}{f_N(z)} \right| > \frac{1}{\mu}$$

and

$$\left| \frac{[zf'(z)]'/f_N'(z)}{zf'(z)/f_N(z)} - 1 \right| < \lambda$$

for all $z \in \mathcal{U}$ then $f \in S_N^*[A,B]$.

Proof. Let us define a set of complex numbers $\Sigma = \{w : |w-1| < \lambda\}$. In view of Lemma 2.1, to prove this theorem it is enough to show that $\Sigma \subseteq \Omega$, i.e. $\Sigma \cap \Omega_1 = \emptyset$.

If $w \in \Omega_1$ then for some $z \in \mathcal{U}$, $\theta \in \mathbb{R}$ and $K \geq 1$, we have

$$\begin{aligned}|w-1| &= \left| K(A-B)\cdot\frac{f_N(z)}{zf_N'(z)}\cdot\frac{e^{i\theta}}{(1+Ae^{i\theta})(1+Be^{i\theta})}\right| \\ &> \frac{(A-B)\cdot\mu}{|1+Ae^{i\theta}|\cdot|1+Be^{i\theta}|} = \frac{(A-B)\cdot\mu}{\sqrt{1+A^2+2At}\cdot\sqrt{1+B^2+2Bt}} \equiv h(t),\end{aligned}$$

where $t = \cos\theta \in [-1,1]$. If we show that $h(t) \geq \lambda$ for all $t \in [-1,1]$, it will mean that $w \notin \Sigma$ and the proof will be completed.

Indeed, if $0 \leq B < A$ then $AB \geq 0$ and $h(t) \geq h(1) = \frac{(A-B)\cdot\mu}{(1+A)\cdot(1+B)} = \lambda$ for all $t \in [-1,1]$. Similarly, if $B < A \leq 0$ then $h(t) \geq h(-1) = \frac{(A-B)\cdot\mu}{(1-A)\cdot(1-B)} = \lambda$ for all $t \in [-1,1]$. Finally, let $B < 0 < A$, i.e. $AB < 0$. Then the function $h(t)$ attains its minimal value for $t_* = -\frac{(A+B)(1+AB)}{4AB}$, which is in $[-1,1]$ if and only if $4AB \leq (A+B)(1+AB) \leq 4A|B|$. That value is $h(t_*) = \frac{2\mu\sqrt{A|B|}}{1-AB} = \lambda$. □

Now we will give several corollaries that can be obtained from Theorem 2.2.

2.3. Corollary. *Let $f \in \mathcal{A}$, $N \in \mathbb{N}\setminus\{1\}$, $-1 \leq B \leq -3+2\sqrt{2} = -0.172\ldots$ and $\mu > 1$. Also, let $|zf_N'(z)/f_N(z)| > 1/\mu$ for all $z \in \mathcal{U}$.*

(i) *If $0 < A \leq |B|$ and*

$$\left|\frac{[zf'(z)]'/f_N'(z)}{zf'(z)/f_N(z)} - 1\right| < \frac{2\mu\sqrt{A|B|}}{1-AB}$$

for all $z \in \mathcal{U}$ then $f \in S_N^*[A,B]$.

(ii) *If $|B| \leq A \leq 1$ and*

$$\left|\frac{[zf'(z)]'/f_N'(z)}{zf'(z)/f_N(z)} - 1\right| < \frac{(A-B)\mu}{(1+|A|)(1+|B|)}$$

for all $z \in \mathcal{U}$ then $f \in S_N^*[A,B]$.

Proof. In both cases, (i) and (ii), $AB < 0$. So, in order to prove (i), it is enough to show that

$$4AB \leq (A+B)(1+AB) \leq 4A|B|,$$

and the rest follows from Theorem 2.2. Indeed, if $-1 \leq B \leq -3+2\sqrt{2}$ and $0 < A \leq |B|$ then $A+B \leq 0$, and the second inequality is obvious. The first inequality, $4AB \leq (A+B)(1+AB)$, is equivalent to $\frac{A}{(1-A)^2} \geq -\frac{B}{(1-B)^2}$. This one is also true because

$$\frac{A}{(1-A)^2} \geq \frac{1}{4} \geq -\frac{B}{(1-B)^2}$$

for all A and B in the specified range. The proof of (ii) goes in a similar way. □

If we put $A = 1-2\alpha$ $(0 \leq \alpha < 1)$ and $B = -1$ in Theorem 2.2 we receive

2.4. Corollary. *Let $f \in \mathcal{A}$, $N \in \mathbb{N}\setminus\{1\}$, $0 \leq \alpha < 1$ and $\mu > 1$. Also let*

$$\lambda_1 \equiv \begin{cases}\mu/2, & 0 \leq \alpha \leq 1/2,\\ \frac{1-\alpha}{2\alpha}\mu, & 1/2 < \alpha < 1.\end{cases}$$

If

$$\left|\frac{zf_N'(z)}{f_N(z)}\right| > \frac{1}{\mu}$$

and

$$\left|\frac{[zf'(z)]'/f_N'(z)}{zf'(z)/f_N(z)} - 1\right| < \lambda_1$$

for all $z \in \mathcal{U}$ *then* $f \in S_N^*(\alpha)$.

Proof. First, let us note that $\lambda_1 = \frac{(1-\alpha)\mu}{1+|1-2\alpha|} = \frac{(A-B)\mu}{(1+|A|)(1+|B|)}$. Further, if $0 \le \alpha \le \frac{1}{2}$ then $A \ge 0$, $AB \ge 0$ and the conclusion of the Corollary follows since $\lambda_1 = \lambda$. In the case when $\frac{1}{2} < \alpha < 1$, we have $A > 0$, $AB < 0$, but $(A+B)(1+AB) = -4\alpha^2 > 4(1-2\alpha) = 4A|B|$. Again the conclusion follows because of $\lambda_1 = \lambda$. □

For $\alpha = 0$ in Corollary 2.4 we obtain

2.5. Corollary. *Let* $f \in \mathcal{A}$, $N \in \mathbb{N} \setminus \{1\}$ *and* $\mu > 1$. *Also, let* $|zf_N'(z)/f_N(z)| > 1/\mu$ *for all* $z \in \mathcal{U}$. *If*

$$\left|\frac{[zf'(z)]'/f_N'(z)}{zf'(z)/f_N(z)} - 1\right| < \frac{\mu}{2}$$

for all $z \in \mathcal{U}$ *then* $f \in S_N^*(0) = S_N^*[1,-1] = S_N^*$. □

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