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MORE GENERAL FORMS OF GENERALIZED FUZZY BI-IDEALS IN SEMIGROUPS

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Abstract

More general forms of the paper by O. Kazanci and S. Yamak (*Generalized fuzzy bi-ideals of semigroups*, Soft Comput. **12**, 1119–1124, 2008) are discussed. The notion of $(\in, \in\vee\mathrm{q}_k)$-fuzzy bi-ideals in a semigroup S is introduced, and several properties are investigated. Characterizations of an $(\in, \in\vee\mathrm{q}_k)$-fuzzy bi-ideal in a semigroup S are discussed.

Keywords: $(\in, \in\vee\mathrm{q})$-fuzzy bi-ideal, $(\in, \in\vee\mathrm{q}_k)$-fuzzy bi-ideal.

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1. Introduction

Kuroki initiated the theory of fuzzy semigroups (see [12, 13, 14]). The monograph by Mordeson *et al.* [15] dealt with the theory of fuzzy semigroups and their application in fuzzy coding, fuzzy finite state machines and fuzzy languages.

The idea of fuzzy point and its "belongingness" and "quasicoincidence" with a fuzzy set were given by Pu and Liu [16]. In [5], Bhakat and Das used this idea to define (α, β)-fuzzy subgroups. In [1, 2, 3, 4, 5], (α, β)-fuzzy substructures of algebraic structures are discussed. As a generalization of fuzzy interior ideals of a semigroup, Jun and Song [10] discussed generalized fuzzy interior ideals in semigroups.

Yin *et al.* [17] discussed the $(\in, \in\vee\mathrm{q})$-fuzzy subsemigroups and ideals of an $(\in, \in\vee\mathrm{q})$-fuzzy semigroup. Jun [7] considered a more general form of the notion of quasi-coincidence of a fuzzy point with a fuzzy set, and generalized the results in the papers [8, 9]. As a generalization of fuzzy bi-ideals of a semigroup, Kazanci and Yamak [11] considered the generalized fuzzy bi-ideals of a semigroup.

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The purpose of this article is to obtain more general forms than in Kazanci and Yamak's paper [11]. We introduce the notion of $(\in, \in\vee\mathrm{q}_k)$-fuzzy bi-ideals, and investigate related properties. We provide characterizations of an $(\in, \in\vee\mathrm{q}_k)$-fuzzy bi-ideal. The important achievement of the study with an $(\in, \in\vee\mathrm{q}_k)$-fuzzy bi-ideal is that the notion of an $(\in, \in\vee\mathrm{q})$-fuzzy bi-ideal is a special case of an $(\in, \in\vee\mathrm{q}_k)$-fuzzy bi-ideal, and thus the related results obtained in the paper [11] are corollaries of our results obtained in this paper.

2. Preliminaries

Let S be a semigroup. By a *subsemigroup* of S we mean a nonempty subset G of S such that $G^2 \subseteq G$. A subsemigroup A of a semigroup S is called a *bi-ideal* of S if $ASA \subseteq A$. A semigroup S is said to be *right* (resp., *left*) *zero* if $xy = y$ (resp., $xy = x$) for all $x, y \in S$ (see [13]). A semigroup S is said to be *completely regular* if, for every element $a \in S$, there is an element $x \in S$ such that $a = axa$ and $ax = xa$.

2.1. Definition. [6, 12] A fuzzy set μ in a semigroup S is called a *fuzzy bi-ideal* of S if it is a fuzzy subsemigroup of S and satisfies:

(2.1) $\quad (\forall x, a, y \in S)(\mu(xay) \geq \min\{\mu(x), \mu(y)\}).$

It is well known that a fuzzy set μ in a semigroup S is a fuzzy bi-ideal of S if and only if $U(\mu; t) := \{x \in S \mid \mu(x) \geq t\}$ is a bi-ideal of S for all $t \in (0, 1]$.

2.2. Proposition. [14] *For a non-empty subset I of a semigroup S the following assertions are equivalent:*

(1) *I is a bi-ideal of S.*
(2) *The characteristic function χ_I of I is a fuzzy bi-ideal of S.*

A fuzzy set μ in a set S of the form

$$\mu(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by (x, t).

For a fuzzy point (x, t) and a fuzzy set μ in a set S, Pu and Liu [16] introduced the symbol $(x, t)\,\alpha\,\mu$, where $\alpha \in \{\in, \mathrm{q}, \in\vee\mathrm{q}, \in\wedge\mathrm{q}\}$.

For any fuzzy set μ in a set S, we say that a fuzzy point (x, t) is

(i) *contained* in μ, denoted by $(x, t) \in \mu$, if $\mu(x) \geq t$.
(ii) *quasi-coincident* with μ, denoted by $(x, t)\,\mathrm{q}\,\mu$, if $\mu(x) + t > 1$.

For a fuzzy point (x, t) and a fuzzy set μ in S, we say that

(iii) $(x, t) \in\vee\mathrm{q}\,\mu$ if $(x, t) \in \mu$ or $(x, t)\,\mathrm{q}\,\mu$.
(iv) $(x, t)\,\overline{\alpha}\,\mu$ if $(x, t)\,\alpha\,\mu$ does not hold for $\alpha \in \{\in, \mathrm{q}, \in\vee\mathrm{q}, \in\wedge\mathrm{q}\}$.

3. Generalizations of $(\in, \in\vee\mathrm{q})$-fuzzy bi-ideals

In what follows let S denote a semigroup and k an arbitrary element of $[0, 1)$ unless otherwise specified. For a fuzzy point (x, t) and a fuzzy set μ in S, we say that

(i) $(x, t)\,\mathrm{q}_k\,\mu$ if $\mu(x) + t + k > 1$.
(ii) $(x, t) \in\vee\mathrm{q}_k\,\mu$ if $(x, t) \in \mu$ or $(x, t)\,\mathrm{q}_k\,\mu$.
(iii) $(x, t)\,\overline{\alpha}\,\mu$ if $(x, t)\,\alpha\,\mu$ does not hold for $\alpha \in \{\in, \mathrm{q}_k, \in\vee\mathrm{q}_k\}$.

3.1. Definition. A fuzzy set μ in S is called an $(\in, \in\vee\mathrm{q}_k)$-*fuzzy bi-ideal* of S if it satisfies:

(i) $(x, t_1) \in \mu,\ (y, t_2) \in \mu \implies (xy, \min\{t_1, t_2\}) \in\vee\,\mathrm{q}_k\ \mu$,
(ii) $(x, t_1) \in \mu,\ (y, t_2) \in \mu \implies (xay, \min\{t_1, t_2\}) \in\vee\,\mathrm{q}_k\ \mu$

for all $a, x, y \in S$ and $t_1, t_2 \in (0, 1]$.

An $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy bi-ideal of S with $k = 0$ is called an $(\in, \in\vee\,\mathrm{q})$-*fuzzy bi-ideal* of S (see [11, Definition 3.1]).

3.2. Example. Consider a semigroup $S = \{a, b, c, d\}$ with a multiplication table given by Table 1.

Table 1. Multiplication table

	a	b	c	d
a	a	a	a	a
b	a	a	d	a
c	a	a	a	a
d	a	a	a	a

(1) Let μ be a fuzzy set in S defined by

$$\mu(x) := \begin{cases} 0.3 & \text{if } x \in \{a, b\}, \\ 0.1 & \text{if } x \in \{c, d\}. \end{cases}$$

Then μ is an $(\in, \in\vee\,\mathrm{q}_{0.4})$-fuzzy bi-ideal of S.

(2) Let ν be a fuzzy set in S defined by

$$\nu(x) := \begin{cases} 0.24 & \text{if } x \in \{a, c\}, \\ 0.1 & \text{if } x \in \{b, d\}. \end{cases}$$

Then ν is an $(\in, \in\vee\,\mathrm{q}_{0.52})$-fuzzy bi-ideal of S.

Note that every fuzzy bi-ideal is an $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy bi-ideal. But the converse is not true as seen in the following example.

3.3. Example. Let $S = \{a, b, c, d, e\}$ be a set with Cayley table given by Table 2.

Table 2. Cayley table

	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	c	c	e
d	a	a	c	d	e
e	a	a	c	c	e

Then S is a semigroup (see [11]). Define a fuzzy set μ in S by

$$\mu(x) := \begin{cases} 0.7 & \text{if } x \in \{a, b, d\}, \\ 0.6 & \text{if } x = c, \\ 0.4 & \text{if } x = e. \end{cases}$$

Then μ is an $(\in, \in\vee\,\mathrm{q}_{0.2})$-fuzzy bi-ideal of S. But it is not a fuzzy bi-ideal of S since

$$\mu(dcd) = \mu(c) = 0.6 < 0.7 = \min\{\mu(d), \mu(d)\}.$$

3.4. Theorem. *Let μ be a fuzzy set in S. Then μ is an $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy bi-ideal of S if and only if it satisfies:*

(1) $(\forall\, x, y \in S)\ \big(\mu(xy) \geq \min\big\{\mu(x), \mu(y), \frac{1-k}{2}\big\}\big)$,
(2) $(\forall\, x, a, y \in S)\ \big(\mu(xay) \geq \min\big\{\mu(x), \mu(y), \frac{1-k}{2}\big\}\big)$.

Proof. Let μ be an $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy bi-ideal of S. Assume that

$$\mu(ab) < \min\{\mu(a), \mu(b), \tfrac{1-k}{2}\}$$

for some $a, b \in S$. If $\min\{\mu(a), \mu(b)\} \geq \frac{1-k}{2}$, then $\mu(ab) < \frac{1-k}{2}$. Hence $\big(a, \frac{1-k}{2}\big) \in \mu$ and $\big(b, \frac{1-k}{2}\big) \in \mu$, but $\big(ab, \frac{1-k}{2}\big) \overline{\in}\, \mu$. Moreover,

$$\mu(ab) + \tfrac{1-k}{2} < \tfrac{1-k}{2} + \tfrac{1-k}{2} = 1 - k,$$

and so $\big(ab, \frac{1-k}{2}\big)\, \overline{\mathrm{q}_k}\, \mu$. Thus $\big(ab, \frac{1-k}{2}\big)\, \overline{\in\vee\,\mathrm{q}_k}\, \mu$. This is a contradiction.

If $\min\{\mu(a), \mu(b)\} < \frac{1-k}{2}$, then $\mu(ab) < \min\{\mu(a), \mu(b)\}$. Hence there exists $t \in (0, 1]$ such that

$$\mu(ab) < t \leq \min\{\mu(a), \mu(b)\}.$$

It follows that $(a, t) \in \mu$ and $(b, t) \in \mu$, but $(ab, t)\, \overline{\in}\, \mu$. Also, $\mu(ab) + t < \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$, i.e., $(ab, t)\, \overline{\mathrm{q}_k}\, \mu$. Thus $(ab, t)\, \overline{\in\vee\,\mathrm{q}_k}\, \mu$, a contradiction. Consequently, (1) is valid. Now suppose that

$$\mu(xay) < \min\big\{\mu(x), \mu(y), \tfrac{1-k}{2}\big\}$$

for some $x, a, y \in S$. We consider the following two cases:

Case 1. $\min\{\mu(x), \mu(y)\} \geq \frac{1-k}{2}$, Case 2. $\min\{\mu(x), \mu(y)\} < \frac{1-k}{2}$.

Case 1 implies that $\big(x, \frac{1-k}{2}\big) \in \mu$, $\big(y, \frac{1-k}{2}\big) \in \mu$, $\mu(xay) < \frac{1-k}{2}$, that is, $(xay, \frac{1-k}{2})\, \overline{\in}\, \mu$, and

$$\mu(xay) + \tfrac{1-k}{2} < \tfrac{1-k}{2} + \tfrac{1-k}{2} = 1 - k,$$

i.e., $(xay, \frac{1-k}{2})\, \overline{\mathrm{q}_k}\, \mu$. Hence $(xay, \frac{1-k}{2})\, \overline{\in\vee\,\mathrm{q}_k}\, \mu$, a contradiction. For Case 2, we have $\mu(xay) < \min\{\mu(x), \mu(y)\}$ and so $\mu(xay) < t \leq \min\{\mu(x), \mu(y)\}$ for some $t \in (0, 1]$. Thus $(x, t) \in \mu$ and $(y, t) \in \mu$, but $(xay, t)\, \overline{\in}\, \mu$. Also,

$$\mu(xay) + t < t + t < \tfrac{1-k}{2} + \tfrac{1-k}{2} = 1 - k,$$

i.e., $(xay, t)\, \overline{\mathrm{q}_k}\, \mu$. Hence $(xay, t)\, \overline{\in\vee\,\mathrm{q}_k}\, \mu$, a contradiction. Therefore (2) is valid.

Conversely, suppose that μ satisfies conditions (1) and (2). Let $x, y \in S$ and $t_1, t_2 \in (0, 1]$ be such that $(x, t_1) \in \mu$ and $(y, t_2) \in \mu$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. Using (1), we have

$$\mu(xy) \geq \min\big\{\mu(x), \mu(y), \tfrac{1-k}{2}\big\} \geq \min\big\{t_1, t_2, \tfrac{1-k}{2}\big\}.$$

If $\min\{t_1, t_2\} > \frac{1-k}{2}$, then $\mu(xy) \geq \frac{1-k}{2}$ and so

$$\mu(xy) + \min\{t_1, t_2\} > \tfrac{1-k}{2} + \tfrac{1-k}{2} = 1 - k,$$

i.e., $(xy, \min\{t_1, t_2\})\ \mathrm{q}_k\ \mu$.

If $\min\{t_1, t_2\} \leq \frac{1-k}{2}$, then $\mu(xy) \geq \min\{t_1, t_2\}$, i.e., $(xy, \min\{t_1, t_2\}) \in \mu$. Hence $(xy, \min\{t_1, t_2\}) \in\vee\,\mathrm{q}_k\ \mu$. Let $x, a, y \in S$ and $t_1, t_2 \in (0, 1]$ be such that $(x, t_1) \in \mu$ and $(y, t_2) \in \mu$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. It follows from (2) that

$$\mu(xay) \geq \min\big\{\mu(x), \mu(y), \tfrac{1-k}{2}\big\} \geq \min\big\{t_1, t_2, \tfrac{1-k}{2}\big\}.$$

If $\min\{t_1, t_2\} > \frac{1-k}{2}$, then $\mu(xay) \geq \frac{1-k}{2}$ and so

$$\mu(xay) + \min\{t_1, t_2\} > \tfrac{1-k}{2} + \tfrac{1-k}{2} = 1 - k,$$

i.e., $(xay, \min\{t_1, t_2\})\ \mathrm{q}_k\ \mu$.

If $\min\{t_1, t_2\} \leq \frac{1-k}{2}$, then $\mu(xay) \geq \min\{t_1, t_2\}$, i.e., $(xay, \min\{t_1, t_2\}) \in \mu$. Thus $(xay, \min\{t_1, t_2\}) \in\vee\, \mathrm{q}_k\ \mu$. Therefore μ is an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy bi-ideal of S. □

If we take $k = 0$ in Theorem 3.4, then we have the following corollary.

3.5. Corollary. [11] *Let μ be a fuzzy set in S. Then μ is an $(\in, \in\vee\, \mathrm{q})$-fuzzy bi-ideal of S if and only if it satisfies:*

(1) $(\forall x, y \in S)\ (\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\})$,
(2) $(\forall x, a, y \in S)\ (\mu(xay) \geq \min\{\mu(x), \mu(y), 0.5\})$. □

The following theorem is a generalization of [11, Theorem 3.5].

3.6. Theorem. *A non-empty subset I of S is a bi-ideal of S if and only if the characteristic function χ_I of I is an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy bi-ideal of S.*

Proof. Assume that χ_I is an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy bi-ideal of S. Let $x, y \in I$ and $a \in S$. Then

$$\chi_I(xy) \geq \min\left\{\chi_I(x), \chi_I(y), \tfrac{1-k}{2}\right\} = \tfrac{1-k}{2}$$

and

$$\chi_I(xay) \geq \min\left\{\chi_I(x), \chi_I(y), \tfrac{1-k}{2}\right\} = \tfrac{1-k}{2}$$

by Theorem 3.4. It follows that $\chi_I(xy) = 1$ and $\chi_I(xay) = 1$ so that $xy \in I$ and $xay \in I$. Hence I is a bi-ideal of S. Since every fuzzy bi-ideal is an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy bi-ideal, the necessity follows from Proposition 2.2. □

3.7. Theorem. *Let I be a bi-ideal of S. For every $t \in (0, \frac{1-k}{2}]$, there exists an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy bi-ideal μ of S such that $U(\mu; t) = I$.*

Proof. Let μ be a fuzzy set in S defined by

$$\mu(x) = \begin{cases} t & \text{if } x \in I, \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in S$, where $t \in (0, \frac{1-k}{2}]$. Obviously, $U(\mu; t) = I$. Assume that

$$\mu(ab) < \min\left\{\mu(a), \mu(b), \tfrac{1-k}{2}\right\}$$

for some $a, b \in S$. Since $\#\mathrm{Im}(\mu) = 2$, it follows that $\min\left\{\mu(a), \mu(b), \frac{1-k}{2}\right\} = t$ and $\mu(ab) = 0$. Hence $\mu(a) = t = \mu(b)$, and so $a, b \in I$. Since I is a bi-ideal of S, $ab \in I$. Thus $\mu(ab) = t$, which is a contradiction. Therefore

$$\mu(xy) \geq \min\left\{\mu(x), \mu(y), \tfrac{1-k}{2}\right\}$$

for all $x, y \in S$. Similarly we have

$$\mu(xay) \geq \min\left\{\mu(x), \mu(y), \tfrac{1-k}{2}\right\}$$

for all $x, a, y \in S$. Using Theorem 3.4, we know that μ is an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy bi-ideal of S. □

Taking $k = 0$ in Theorem 3.7, we have the following corollary.

3.8. Corollary. *Let I be a bi-ideal of S. For every $t \in (0, 0.5]$, there exists an $(\in, \in\vee\, \mathrm{q})$-fuzzy bi-ideal μ of S such that $U(\mu; t) = I$.* □

3.9. Theorem. *If $\{\mu_i \mid i \in \Lambda\}$ is a family of $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy bi-ideals of S, then $\bigcap_{i\in\Lambda} \mu_i$ is an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy bi-ideal of S, where $\left(\bigcap_{i\in\Lambda} \mu_i\right)(x) = \inf_{i\in\Lambda} \mu_i(x)$.*

Proof. Let $x, a, y \in S$. Then

$$\begin{aligned}\Big(\bigcap_{i\in\Lambda}\mu_i\Big)(xy) &= \inf_{i\in\Lambda}\mu_i(xy)\\ &\geq \inf_{i\in\Lambda}\min\left\{\mu_i(x),\mu_i(y),\tfrac{1-k}{2}\right\}\\ &= \min\{\inf_{i\in\Lambda}\min\left\{\mu_i(x),\mu_i(y)\right\},\tfrac{1-k}{2}\}\\ &= \min\Big\{\Big(\bigcap_{i\in\Lambda}\mu_i\Big)(x),\Big(\bigcap_{i\in\Lambda}\mu_i\Big)(y),\tfrac{1-k}{2}\Big\}\end{aligned}$$

and

$$\begin{aligned}\Big(\bigcap_{i\in\Lambda}\mu_i\Big)(xay) &= \inf_{i\in\Lambda}\mu_i(xay)\\ &\geq \inf_{i\in\Lambda}\min\left\{\mu_i(x),\mu_i(y),\tfrac{1-k}{2}\right\}\\ &= \min\{\inf_{i\in\Lambda}\min\left\{\mu_i(x),\mu_i(y)\right\},\tfrac{1-k}{2}\}\\ &= \min\Big\{\Big(\bigcap_{i\in\Lambda}\mu_i\Big)(x),\Big(\bigcap_{i\in\Lambda}\mu_i\Big)(y),\tfrac{1-k}{2}\Big\}.\end{aligned}$$

Therefore $\bigcap_{i\in\Lambda}\mu_i$ is an $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy bi-ideal of S. □

Taking $k = 0$ in Theorem 3.9 induces the following corollary.

3.10. Corollary. [11] *If $\{\mu_i \mid i \in \Lambda\}$ is a family of $(\in, \in\vee\,\mathrm{q})$-fuzzy bi-ideals of S, then $\bigcap_{i\in\Lambda}\mu_i$ is an $(\in, \in\vee\,\mathrm{q})$-fuzzy bi-ideal of S, where $\big(\bigcap_{i\in\Lambda}\mu_i\big)(x) = \inf_{i\in\Lambda}\mu_i(x)$.qed*

Let S be the semigroup given in Example 3.2. Consider the $(\in, \in\vee\,\mathrm{q}_{0.4})$-fuzzy bi-ideal μ of S in Example 3.2 (1) and let ν be a fuzzy set in S defined by

$$\nu(x) := \begin{cases} 0.3 & \text{if } x \in \{a, c\},\\ 0.1 & \text{if } x \in \{b, d\}.\end{cases}$$

Then ν is an $(\in, \in\vee\,\mathrm{q}_{0.4})$-fuzzy bi-ideal of S. Note that

$$\begin{aligned}(\mu\cup\nu)(bc) &= (\mu\cup\nu)(d) = \max\{\mu(d),\nu(d)\} = 0.1\\ &< 0.3 = \min\{(\mu\cup\nu)(b),(\mu\cup\nu)(c),\tfrac{1-0.4}{2}\}.\end{aligned}$$

This shows that a union of $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy bi-ideals may not be an $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy bi-ideal.

Using Theorem 3.4, we provide a condition for an $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy bi-ideal to be a fuzzy bi-ideal.

3.11. Theorem. *Let μ be an $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy bi-ideal of S such that $\mu(x) < \frac{1-k}{2}$ for all $x \in S$. Then μ is a fuzzy ideal of S.*

Proof. Straightforward by using Theorem 3.4. □

Taking $k = 0$ in Theorem 3.11, we have the following corollary.

3.12. Corollary. *Let μ be an $(\in, \in\vee\,\mathrm{q})$-fuzzy bi-ideal of S such that $\mu(x) < 0.5$ for all $x \in S$. Then μ is a fuzzy ideal of S.*

3.13. Theorem. *For a fuzzy set μ in S, the following are equivalent:*

(1) *μ is an $(\in, \in\vee\,\mathrm{q}_k)$-fuzzy bi-ideal of S.*
(2) *$\big(\forall t \in \big(0, \frac{1-k}{2}\big]\big)$, $(U(\mu; t) \neq \emptyset \implies U(\mu; t)$ is a bi-ideal of $S)$.*

Proof. (1) $\Longrightarrow$ (2) Assume that μ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy bi-ideal of S. Let $t \in (0, \frac{1-k}{2}]$ be such that $U(\mu;t) \neq \emptyset$. Let $x, y \in U(\mu;t)$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$, i.e., $(x,t) \in \mu$ and $(y,t) \in \mu$. It follows from Definition 3.1 (i) that $(xy,t) \in \vee \mathrm{q}_k \, \mu$, i.e., $(xy,t) \in \mu$ or $(xy,t) \, \mathrm{q}_k \, \mu$. If $(xy,t) \in \mu$, then $\mu(xy) \geq t$ and so $xy \in U(\mu;t)$. If $(xy,t) \, \mathrm{q}_k \, \mu$, then $\mu(xy) + t > 1-k$. Thus if $\mu(xy) < t$, then

$$1 - k < \mu(xy) + t < t + t \leq \tfrac{1-k}{2} + \tfrac{1-k}{2} = 1 - k.$$

This is a contradiction. Hence $\mu(xy) \geq t$, i.e., $xy \in U(\mu;t)$. Therefore $U(\mu;t)$ is a subsemigroup of S. Let $x, y \in U(\mu;t)$ and $a \in S$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$, i.e., $(x,t) \in \mu$ and $(y,t) \in \mu$. It follows from Definition 3.1 (ii) that $(xay,t) \in \vee \mathrm{q}_k \, \mu$, that is, $(xay,t) \in \mu$ or $(xay,t) \, \mathrm{q}_k \, \mu$. If $(xay,t) \in \mu$, then $\mu(xay) \geq t$ and thus $xay \in U(\mu;t)$. If $(xay,t) \, \mathrm{q}_k \, \mu$, then $\mu(xay) + t > 1-k$. Thus if $\mu(xay) < t$, then

$$1 - k < \mu(xay) + t < t + t \leq \tfrac{1-k}{2} + \tfrac{1-k}{2} = 1 - k.$$

This is impossible, and so $\mu(xay) \geq t$. Hence $xay \in U(\mu;t)$. Consequently, $U(\mu;t)$ is a bi-ideal of S.

(2) $\Longrightarrow$ (1) Assume that μ does not satisfy Theorem 3.4 (1). Then

$$\mu(xy) < s \leq \min\left\{\mu(x), \mu(y), \tfrac{1-k}{2}\right\}$$

for some $x, y \in S$ and $s \in (0,1]$. Clearly, $s \in (0, \frac{1-k}{2}]$, $(x,s) \in \mu$ and $(y,s) \in \mu$, but $(xy,s) \overline{\in} \mu$. Moreover,

$$\mu(xy) + s < s + s \leq \tfrac{1-k}{2} + \tfrac{1-k}{2} = 1 - k,$$

and so $(xy,s) \, \overline{\mathrm{q}_k} \, \mu$. Hence $(xy,s) \, \overline{\in \vee \mathrm{q}_k} \, \mu$, a contradiction. Thus Theorem 3.4 (1) is valid.

Now, suppose that μ does not satisfy Theorem 3.4 (2). Then

$$\mu(xay) < r \leq \min\left\{\mu(x), \mu(y), \tfrac{1-k}{2}\right\}$$

for some $x, a, y \in S$ and $r \in (0,1]$. It follows that $(x,r) \in \mu$, $(y,r) \in \mu$, and $r \in (0, \frac{1-k}{2}]$, but $(xay,r) \overline{\in} \mu$. Also,

$$\mu(xay) + r < r + r \leq \tfrac{1-k}{2} + \tfrac{1-k}{2} = 1 - k,$$

which implies that $(xay,r) \, \overline{\mathrm{q}_k} \, \mu$. Hence $(xay,r) \, \overline{\in \vee \mathrm{q}_k} \, \mu$, a contradiction. Therefore μ satisfies Theorem 3.4 (2). Using Theorem 3.4, we know that μ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy bi-ideal of S. □

If we take $k = 0$ in Theorem 3.13, then we have the following corollary.

3.14. Corollary. [11] *For a fuzzy set μ in S, the following are equivalent:*

(1) *μ is an $(\in, \in \vee \mathrm{q})$-fuzzy bi-ideal of S.*
(2) $(\forall\, t \in (0, 0.5])$, *$(U(\mu;t) \neq \emptyset \implies U(\mu;t)$ is a bi-ideal of $S)$.* □

3.15. Theorem. *Let S be a semigroup. If $0 \leq k < r < 1$, then every $(\in, \in \vee \mathrm{q}_k)$-fuzzy bi-ideal of S is an $(\in, \in \vee \mathrm{q}_r)$-fuzzy bi-ideal of S.*

Proof. Straightforward. □

The following example shows that if $0 \leq k < r < 1$, then an $(\in, \in \vee \mathrm{q}_r)$-fuzzy bi-ideal of S may not be an $(\in, \in \vee \mathrm{q}_k)$-fuzzy bi-ideal of S.

3.16. Example. Consider a semigroup $S = \{a, b, c, d, e\}$ with a multiplication table given by Table 3.

Table 3. Multiplication table

	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

Define a fuzzy set μ in S by

$$\mu = \begin{pmatrix} a & b & c & d & e \\ 0.7 & 0.6 & 0.3 & 0.4 & 0.5 \end{pmatrix}.$$

Then μ is an $(\in, \in \vee\, \mathrm{q}_{0.2})$-fuzzy bi-ideal of S. If $t \in \left(0.4, \frac{1-0.14}{2}\right] = (0.4, 0.43]$, then $U(\mu; t) = \{a, b, e\}$ is not a bi-ideal of S. Hence μ is not an $(\in, \in\vee\, \mathrm{q}_{0.14})$-fuzzy bi-ideal of S by Theorem 3.13.

3.17. Theorem. *Let μ be an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy bi-ideal of S. If S is completely regular, then $\min\left\{\mu(a), \frac{1-k}{2}\right\} = \min\left\{\mu(a^2), \frac{1-k}{2}\right\}$ for all $a \in S$.*

Proof. Let $a \in S$. Then there exists $x \in S$ such that $a = a^2xa^2$. Hence

$$\begin{aligned}
\min\left\{\mu(a), \tfrac{1-k}{2}\right\} &= \min\left\{\mu(a^2xa^2), \tfrac{1-k}{2}\right\} \\
&\geq \min\left\{ \min\left\{\mu(a^2), \mu(a^2), \tfrac{1-k}{2}\right\}, \tfrac{1-k}{2}\right\} \\
&= \min\left\{\mu(a^2), \tfrac{1-k}{2}\right\} \\
&\geq \min\left\{ \min\left\{\mu(a), \mu(a), \tfrac{1-k}{2}\right\}, \tfrac{1-k}{2}\right\} \\
&= \min\left\{\mu(a), \tfrac{1-k}{2}\right\},
\end{aligned}$$

which implies that $\min\left\{\mu(a), \frac{1-k}{2}\right\} = \min\left\{\mu(a^2), \frac{1-k}{2}\right\}$ for all $a \in S$. □

3.18. Corollary. *Let μ be an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy bi-ideal of S. If S is completely regular and $\mu(a) < \frac{1-k}{2}$ for all $a \in S$, then $\mu(a) = \mu(a^2)$ for all $a \in S$.* □

3.19. Corollary. *Let μ be an $(\in, \in\vee\, \mathrm{q})$-fuzzy bi-ideal of S. If S is completely regular, then $\min\{\mu(a), 0.5\} = \min\{\mu(a^2), 0.5\}$ for all $a \in S$.* □

For any fuzzy set μ in S and $t \in (0, 1]$, we define

$$Q(\mu; t) := \{x \in X \mid (x; t)\, \mathrm{q}\, \mu\}, \quad Q^k(\mu; t) := \{x \in X \mid (x; t)\ \mathrm{q}_k\ \mu\}.$$

3.20. Theorem. *If μ is an $(\in, \in\vee\, \mathrm{q}_k)$-fuzzy bi-ideal of S, then*

$$\left(\forall\, t \in \left(\tfrac{1-k}{2}, 1\right]\right) \left(Q^k(\mu; t) \neq \emptyset \implies Q^k(\mu; t) \text{ is a bi-ideal of } S\right)$$

Proof. Let $t \in \left(\frac{1-k}{2}, 1\right]$ be such that $Q^k(\mu; t) \neq \emptyset$. Let $x, a, y \in S$. If $x, y \in Q^k(\mu; t)$, then $(x, t)\, \mathrm{q}_k\, \mu$ and $(y, t)\, \mathrm{q}_k\, \mu$, i.e., $\mu(x) + t + k > 1$ and $\mu(y) + t + k > 1$. It follows from Theorem 3.4 that

$$\begin{aligned}
\mu(xy) &\geq \min\left\{\mu(x), \mu(y), \tfrac{1-k}{2}\right\} \geq 1 - t - k, \\
\mu(xay) &\geq \min\left\{\mu(x), \mu(y), \tfrac{1-k}{2}\right\} \geq 1 - t - k
\end{aligned}$$

so that $(xy, t)\, \mathrm{q}_k\, \mu$ and $(xay, t)\, \mathrm{q}_k\, \mu$. Hence $xy, xay \in Q^k(\mu; t)$, and therefore $Q^k(\mu; t)$ is a bi-ideal of S. □

3.21. Corollary. *Let μ be an $(\in, \in\vee\mathrm{q}_k)$-fuzzy bi-ideal of S. If $k < r < 1$, then*

$$\big(\forall t \in \big(\tfrac{1-r}{2}, 1\big]\big)\ \big(Q^r(\mu;t) \neq \emptyset \implies Q^r(\mu;t) \text{ is a bi-ideal of } S\big)$$

Proof. Straightforward by Theorems 3.15 and 3.20. □

If we take $k = 0$ in Theorem 3.20, then we have the following corollary.

3.22. Corollary. *If μ is an $(\in, \in\vee\mathrm{q})$-fuzzy bi-ideal of S, then*

$$(\forall t \in (0.5, 1])\ \big(Q(\mu;t) \neq \emptyset \implies Q(\mu;t) \text{ is a bi-ideal of } S\big)$$ □

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