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SYSTEM OF GENERALIZED H-RESOLVENT EQUATIONS AND THE CORRESPONDING SYSTEM OF GENERALIZED VARIATIONAL INCLUSIONS[‡]

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Abstract

The aim of this paper is to introduce a new system of generalized H-resolvent equations in uniformly smooth Banach spaces and to mention the corresponding system of variational inclusions. An equivalence relation is established between the system of generalized H-resolvent equations and the system of variational inclusions. We also prove the existence of solutions for the system of generalized H-resolvent equations and the convergence of the iterative sequences generated by the algorithm. Our results are new and generalize many known results appearing in the literature.

Keywords: Generalized H-resolvent equations, System, Variational inclusions, Algorithm, Convergence.

2000 AMS Classification: 47 H 19, 49 J 40.

1. Introduction

Using the concept of resolvent operator technique, Noor and Noor [17] introduced and studied resolvent equations and established the equivalence between the mixed variational inequalities and the resolvent equations. The resolvent equations technique is being used to develop powerful and efficient numerical techniques for solving mixed (quasi) variational inequalities and related optimization problems.

In 2001, Verma [22] introduced and studied a system of variational inequalities and developed some iterative algorithms for approximating the solutions of this system of

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variational inequalities. Pang [18], Cohen and Chaplais [11], Bianchi [6], Ansari and Yao [5] considered a system of scalar variational inequalities and Pang showed that the traffic equilibrium problem, the Nash equilibrium, and the general equilibrium programming problem can be modeled as a system of variational inequalities. As generalizations of a system of variational inequalities, Agarwal *et al.* [1] introduced a system of generalized nonlinear mixed quasi-variational inclusions and investigated the sensitivity analysis of solutions for the system of generalized mixed quasi-variational inclusions in Hilbert spaces. Fang and Huang [12] introduced a new class of H-accretive operators in the setting of Banach spaces and extended the concept of resolvent operators associated with the classical m-accretive operators to H-accretive operators. By using this new resolvent operator technique, they studied the approximate solutions of a class of variational inclusions with H-accretive operators in the setting of Banach spaces. Lan, Cho and Verma [14] solved cocoercive variational inequalities in Banach spaces and Verma [21] introduced a general proximal point algorithm involving an η-maximal accretive framework in Banach spaces.

Very recently J-W. Peng [19] introduced a system of generalized mixed quasi-variational-like inclusions with (H, η)-accretive operators i.e., a family of generalized mixed quasi-variational-like inclusions with (H, η)-accretive operators defined on a product of sets in Banach spaces. Ceng and Yao [10] and Ceng, Wang and Yao [8] studied system of variational inequalities by using the projection method and relaxed extragradient method, respectively. Ahmad and Yao [3] studied and introduced a system of generalized resolvent equations and Ceng and Yao [9] studied mixed equilibrium problems. Ceng, Ansari and Yao [7] applied relaxed viscosity iterative methods for solving variational inequalities in Banach spaces and Hassouni and Moudafi [13] studied a perturbed algorithm for variational inclusions.

Inspired and motivated by the recent research work going on in this field, the aim of this paper is to introduce and study a new system of generalized H-resolvent equations in uniformly smooth Banach spaces. We established an equivalence relation between the system of generalized H-resolvent equations and the corresponding system of variational inclusions. Some iterative algorithms for solving system of generalized H-resolvent equations and convergence criteria are discussed.

2. Formulation and preliminaries

Throughout the paper, unless otherwise specified, we assume that E is a real Banach space with its norm $\|\cdot\|$, E^* is the topological dual of E, d is the metric induced by the norm $\|\cdot\|$, $CB(E)$ (respectively, 2^E) is the family of all nonempty closed and bounded subsets (respectively, all nonempty subsets) of E, $D(\cdot,\cdot)$ is the Hausdorff metric on $CB(E)$ defined by

$$D(A,B) = \max\Big\{ \sup_{x\in A} d(x,B),\ \sup_{y\in B} d(A,y) \Big\},$$

where $d(x,B) = \inf_{y\in B} d(x,y)$ and $d(A,y) = \inf_{x\in A} d(x,y)$.

We also assume that $\langle\cdot,\cdot\rangle$ is the duality pairing between E and $E^\star$ and $\mathfrak{J} : E \to 2^{E^\star}$ is the normalized duality mapping defined by

$$\mathfrak{J}(x) = \{ f \in E^\star : \langle x, f\rangle = \|x\|\|f\|,\ \text{and } \|x\| = \|f\|\},\ \text{for all } x \in E.$$

Now, we recall some definitions, notations and results which will be used throughout the paper.

The uniform convexity of a Banach space E means that for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $x, y \in E$, $\|x\| \leq 1$, $\|y\| \leq 1$, $\|x - y\| = \epsilon$ ensure the following inequality,

$$\|x + y\| \leq 2(1 - \delta).$$

The function

$$\delta_E(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = 1,\ \|y\| = 1,\ \|x - y\| = \epsilon \right\}$$

is called the modulus of convexity of E.

The uniform smoothness of a Banach space E means that for any given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\frac{\|x + y\| + \|x - y\|}{2} - 1 \leq \epsilon \|y\|$$

holds. The function

$$\tau_E(t) = \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1,\ \|y\| = t \right\}$$

is called the modulus of smoothness of E.

We remark that the Banach space E is uniformly convex if and only if $\delta_E(\epsilon) > 0$, for all $\epsilon > 0$, and it is uniformly smooth if and only if $\lim_{t \to 0} \frac{\tau_E(t)}{t} = 0$.

2.1. Definition. A mapping $g : E \to E$ is said to be

(i) *accretive*, if for any $x, y \in E$, there exists $j(x - y) \in \mathcal{J}(x - y)$ such that

$$\langle g(x) - g(y), j(x - y) \rangle \geq 0;$$

(ii) *strictly accretive*, if for any $x, y \in E$, there exists $j(x - y) \in \mathcal{J}(x - y)$ such that

$$\langle g(x) - g(y), j(x - y) \rangle \geq 0;$$

and equality holds if and only if $x = y$;

(iii) *strongly accretive*, if for any $x, y \in E$, there exists $j(x - y) \in \mathcal{J}(x - y)$ and a constant $\delta_g > 0$ such that

$$\langle g(x) - g(y), j(x - y) \rangle \geq \delta_g \|x - y\|^2;$$

(iv) *Lipschitz continuous* if for any $x, y \in E$, there exists a constant λ_g such that

$$\|g(x) - g(y)\| \leq \lambda_g \|x - y\|.$$

2.2. Definition. A multi-valued mapping $M : E \to 2^E$ is said to be

(i) *accretive*, if for any $x, y \in E$, there exists $j(x - y) \in \mathcal{J}(x - y)$ such that for all $u \in M(x)$ and $v \in M(y)$,

$$\langle u - v, j(x - y) \rangle \geq 0;$$

(ii) *strongly accretive*, if for any $x, y \in E$, there exists $j(x - y) \in \mathcal{J}(x - y)$ and a constant δ_M such that for all $u \in M(x)$ and $v \in M(y)$,

$$\langle u - v, j(x - y) \rangle \geq \delta_M \|x - y\|^2;$$

(iii) *m-accretive*, if M is accretive and $(I + \rho M)(E) = E$ for every (equivalently, for some) $\rho > 0$, where I is the identity mapping (equivalently, if M is accretive and $(I + M)(E) = E$).

2.3. Definition. Let $H : E \to E$ be an operator. A multivalued mapping $M : E \to 2^E$ is said to be *H-accretive* if M is accretive and $(H + \rho M)(E) = E$ for all $\rho > 0$.

2.4. Remark. If $H = I$, then Definition 2.3 reduces to the usual definition of m-accretive operator.

2.5. Definition. Let $H : E \to E$ be a strictly accretive operator and $M : E \to 2^E$ an H-accretive multivalued mapping. The H-resolvent operator $J^M_{H,\rho} : E \to E$ associated with H and M is defined by

$$J^M_{H,\rho}(x) = (H + \rho M)^{-1}(x), \text{ for all } x \in E.$$

2.6. Theorem. [12] *Let $H : E \to E$ be a strongly accretive operator with constant r and $M : E \to 2^E$ an H-accretive multivalued mapping. Then the H-resolvent operator $J^M_{H,\rho} : E \to E$ associated with H and M is Lipschitz continuous with constant $\frac{1}{\mathrm{r}}$, that is,*

$$\|J^M_{H,\rho}(x) - J^M_{H,\rho}(y)\| \le \frac{1}{\mathrm{r}}\|x - y\|, \text{ for all } x, y \in E. \qquad \square$$

2.7. Definition. [2] The H-resolvent operator $J^M_{H,\rho} : E \to E$ is said to be a *retraction* if

$$[J^M_{H,\rho}(x)]^2 = J^M_{H,\rho}(x), \text{ for all } x \in E.$$

2.8. Definition. A multivalued mapping $G : E \to CB(E)$ is said to be D-Lipschitz continuous if for any $x, y \in E$, there exist a constant $\lambda_{D_G} > 0$, such that

$$D(G(x), G(y)) \le \lambda_{D_G}\|x - y\|.$$

2.9. Proposition. [4, 20] *Let E be a uniformly smooth Banach space and $\mathcal{J} : E \to 2^{E^\star}$ a normalized duality mapping. Then, for any $x, y \in E$,*

(i) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x + y)\rangle$, *for all* $j(x + y) \in \mathcal{J}(x + y)$,

(ii) $\langle x - y, j(x) - j(y)\rangle \le 2C^2\tau_E(4\|x - y\|/\mathcal{D})$, *where* $\mathcal{D} = \sqrt{(\|x\|^2 + \|y\|^2)/2}$. $\square$

Let E_1 and E_2 be any two real Banach spaces, $S : E_1 \times E_2 \to E_1$, $T : E_1 \times E_2 \to E_2$, $p : E_1 \to E_1$, $q : E_2 \to E_2$, $H_1 : E_1 \to E_1$ and $H_2 : E_2 \to E_2$ single-valued mappings, $G : E_1 \to CB(E_1)$, $F : E_2 \to CB(E_2)$ multi-valued mappings. Let $M : E_1 \times E_1 \to 2^{E_1}$ be H_1-accretive and $N : E_2 \times E_2 \to 2^{E_2}$ be H_2-accretive mappings. let $f : E_1 \to E_1$ and $g : E_2 \to E_2$ be nonlinear mappings with $f(E_1) \cap D(M(\cdot, x)) \ne \emptyset$ and $g(E_2) \cap D(N(\cdot, y)) \ne \emptyset$, respectively. Then we consider the following system of generalized H-resolvent equations:

Find $(x, y) \in E_1 \times E_2$, $u \in G(x)$, $v \in F(y)$, $z' \in E_1$, $z'' \in E_2$ such that

$$\begin{aligned} &S(x - p(x), v) + \rho^{-1}R^{M(\cdot,x)}_{H_1,\rho}(z') = 0, \ \rho > 0, \\ &T(u, y - q(y)) + \gamma^{-1}R^{N(\cdot,y)}_{H_2,\gamma}(z'') = 0, \ \gamma > 0, \end{aligned} \tag{2.1}$$

where $R^{M(\cdot,x)}_{H_1\rho} = I - H_1(J^{M(\cdot,x)}_{H_1,\rho})$, $R^{N(\cdot,y)}_{H_2,\gamma} = I - H_2(J^{N(\cdot,y)}_{H_2\gamma})$ and $J^{M(\cdot,x)}_{H_1,\rho}$, $J^{N(\cdot,y)}_{H_2,\gamma}$ are the resolvent operators associated with M and N, respectively.

The corresponding system of generalized variational inclusions of (2.1) is the following:

Find $(x, y) \in E_1 \times E_2$, $u \in G(x)$, $v \in F(y)$ such that

$$\begin{aligned} &0 \in S(x - p(x), v) + M(f(x), x), \\ &0 \in T(u, y - q(y)) + N(g(y), y). \end{aligned} \tag{2.2}$$

A problem similar to (2.2) is considered by Lan *et al.* [15] in Hilbert spaces.

2.10. Lemma. *$(x, y) \in E_1 \times E_2$, $u \in G(x)$, $v \in F(y)$ is a solution of system of generalized variational inclusions (2.2) if and only if (x, y, u, v) satisfies*

$$f(x) = J^{M(\cdot,x)}_{H_1,\rho}[H_1(f(x)) - \rho S(x - p(x), v)],$$

$$g(y) = J^{N(\cdot,y)}_{H_2,\gamma}[H_2(g(y)) - \gamma T(u, y - q(y))],$$

where $\rho > 0$ and $\gamma > 0$ are constants.

Proof. The proof is a direct consequence of the definition of H-resolvent operator, and hence, is omitted. □

3. Iterative algorithms and a convergence result

In this section, we first establish an equivalence relation between system of generalized H-resolvent equations (2.1) and the system of generalized variational inclusions (2.2). Finally, we prove the existence of a solution of (2.1) and the convergence of sequences generated by the proposed algorithms.

3.1. Proposition. *The system of generalized variational inclusions (2.2) has a solution (x, y, u, v) with $(x, y) \in E_1 \times E_2$, $u \in G(x)$, $v \in F(y)$ if and only if the system of generalized H-resolvent equations (2.1) has a solution (z', z'', x, y, u, v) with $(x, y) \in E_1 \times E_2$, $u \in G(x)$, $v \in F(y)$, $z' \in E_1$, $z'' \in E_2$ where*

(3.1) $$f(x) = J_{H_1,\rho}^{M(\cdot,x)}(z'),$$

(3.2) $$g(y) = J_{H_2,\gamma}^{N(\cdot,y)}(z''),$$

where $z' = H_1(f(x)) - \rho S(x - p(x), v)$ and $z'' = H_2(g(y)) - \gamma T(u, y - q(y))$.

Proof. Let (x, y, u, v) be a solution of the system of generalized variational inclusion (2.2). Then by Lemma 2.10, it satisfies the following equations

$$f(x) = J_{H_1,\rho}^{M(\cdot,x)}[H_1(f(x)) - \rho S(x - p(x), v)],$$
$$g(y) = J_{H_2,\gamma}^{N(\cdot,y)}[H_2(g(y)) - \gamma T(u, y - q(y))].$$

Let $z' = H_1(f(x)) - \rho S(x - p(x), v)$ and $z'' = H_2(g(y)) - \gamma T(u, y - q(y))$. Then we have

$$f(x) = J_{H_1,\rho}^{M(\cdot,x)}(z'),$$
$$g(y) = J_{H_2,\gamma}^{N(\cdot,y)}(z''),$$

$z' = H_1(J_{H_1,\rho}^{M(\cdot,x)}(z')) - \rho S(x - p(x), v)$ and $z'' = H_2(J_{H_2,\gamma}^{N(\cdot,y)}(z'')) - \gamma T(u, y - q(y))$. It follows that

$$\begin{aligned}(I - H_1(J_{H_1,\rho}^{M(\cdot,x)}))(z') &= z' - H_1(J_{H_1,\rho}^{M(\cdot,x)}(z')) \\ &= H_1(J_{H_1,\rho}^{M(\cdot,x)}(z')) - \rho S(x - p(x), v) - H_1(J_{H_1,\rho}^{M(\cdot,x)}(z')) \\ &= -\rho S(x - p(x), v),\end{aligned}$$

and similarly

$$(I - H_2(J_{H_2,\gamma}^{N(\cdot,y)}))(z'') = -\gamma T(u, y - q(y)),$$

i.e.

$$S(x - p(x), v) + \rho^{-1} R_{H_1,\rho}^{M(\cdot,x)}(z') = 0,$$
$$T(u, y - q(y)) + \gamma^{-1} R_{H_2,\gamma}^{N(\cdot,y)}(z'') = 0.$$

Thus, (z', z'', x, y, u, v) is a solution of the system of generalized H-resolvent equations (2.1).

Conversely, let (z', z'', x, y, u, v) be a solution of the system of generalized H-resolvent equations (2.1), then

(3.3) $$\rho S(x - p(x), v) = -R_{H_1,\rho}^{M(\cdot,x)}(z'),$$

(3.4) $$\gamma T(u, y - q(y)) = -R_{H_2,\gamma}^{N(\cdot,y)}(z'').$$

Now

$$\begin{aligned}\rho S(x-p(x),v) &= -R_{H_1,\rho}^{M(\cdot,x)}(z') \\ &= -(I-H_1(J_{H_1,\rho}^{M(\cdot,x)}))(z') \\ &= (H_1(J_{H_1,\rho}^{M(\cdot,x)}))(z')-z' \\ &= (H_1(J_{H_1,\rho}^{M(\cdot,x)}))[H_1(f(x))-\rho S(x-p(x),v)] \\ &\qquad -[H_1(f(x))-\rho S(x-p(x),v)]\end{aligned}$$

which implies that

$$f(x)=J_{H_1,\rho}^{M(\cdot,x)}[H_1(f(x))-\rho S(x-p(x),v)],$$

and

$$\begin{aligned}\gamma T(u,y-q(y)) &= -R_{H_2,\gamma}^{N(\cdot,y)}(z'') \\ &= -(I-H_2(J_{H_2,\gamma}^{N(\cdot,y)}))(z'') \\ &= (H_2(J_{H_2,\gamma}^{N(\cdot,y)}))(z'')-z'' \\ &= (H_2(J_{H_2,\gamma}^{N(\cdot,y)}))[H_2(g(y))-\gamma T(u,y-q(y))] \\ &\qquad -[H_2(g(y))-\gamma T(u,y-q(y))]\end{aligned}$$

which implies that

$$g(y)=J_{H_2,\gamma}^{N(\cdot,y)}[H_2(g(y))-\gamma T(u,y-q(y))].$$

Thus, we have

$$\begin{aligned}f(x)&=J_{H_1,\rho}^{M(\cdot,x)}[H_1(f(x))-\rho S(x-p(x),v)],\\ g(y)&=J_{H_2,\gamma}^{N(\cdot,y)}[H_2(g(y))-\gamma T(u,y-q(y))],\end{aligned}$$

so, by Lemma 2.10, (x,y,u,v) is a solution of the system of generalized variational inclusions (2.2). □

Alternative Proof. Let

$$z'=H_1(f(x))-\rho S(x-p(x),v) \text{ and } z''=H_2(g(y))-\gamma T(u,y-q(y)).$$

Using (3.1) and (3.2), we can write

$$z'=(H_1(J_{H_1,\rho}^{M(\cdot,x)}))(z')-\rho S(x-p(x),v) \text{ and } z''=(H_2(J_{H_2,\gamma}^{N(\cdot,y)}))(z'')-\gamma T(u,y-q(y)),$$

which implies that

$$\begin{aligned}S(x-p(x),v)+\rho^{-1}R_{H_1,\rho}^{M(\cdot,x)}(z')=0,\ \rho>0,\\ T(u,y-q(y))+\gamma^{-1}R_{H_2,\gamma}^{N(\cdot,y)}(z'')=0,\ \gamma>0,\end{aligned}$$

is the required system of generalized H-resolvent equations. □

3.2. Algorithm. *For given $(x_o,y_o)\in E_1\times E_2$, $u_o\in G(x_o)$, $v_o\in F(y_o)$, $z'_o\in E_1$, $z''_o\in E_2$, compute $\{z'_n\}$, $\{z''_n\}$, $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ by the iterative schemes,*

$$f(x_n)=J_{H_1,\rho}^{M(\cdot,x_n)}(z'_n) \tag{3.5}$$

$$g(y_n)=J_{H_2,\gamma}^{N(\cdot,y_n)}(z''_n). \tag{3.6}$$

Using Nadler's theorem [16], *we have*

(3.7) $u_n \in G(x_n) : \|u_{n+1} - u_n\| \leq D(G(x_{n+1}), G(x_n)),$

(3.8) $v_n \in F(y_n) : \|v_{n+1} - v_n\| \leq D(F(y_{n+1}), F(y_n)),$

(3.9) $z'_{n+1} = [H_1(f(x_n)) - \rho S(x_n - p(x_n), v_n)],$

(3.10) $z''_{n+1} = [H_2(g(y_n)) - \gamma T(u_n, y_n - q(y_n))],$

$n = 0, 1, 2, \ldots.$

The system of generalized resolvent equations (2.1) can also be written as

$$z' = H_1(f(x)) - S(x - p(x), v) + (I - \rho^{-1}) R^{M(\cdot,x)}_{H_1,\rho}(z'),$$
$$z'' = H_2(g(y)) - T(u, y - q(y)) + (I - \gamma^{-1}) R^{N(\cdot,y)}_{H_2,\gamma}(z'').$$

We use this fixed-point formulation to suggest the following iterative method.

3.3. Algorithm. *For given* $(x_o, y_o) \in E_1 \times E_2$, $u_o \in G(x_o)$, $v_o \in F(y_o)$, $z'_o \in E_1$, $z''_o \in E_2$, *compute* $\{z'_n\}$, $\{z''_n\}$, $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ *and* $\{v_n\}$ *by the iterative schemes,*

$$f(x_n) = J^{M(\cdot,x_n)}_{H_1,\rho}(z'_n),$$
$$g(y_n) = J^{N(\cdot,y_n)}_{H_2,\gamma}(z''_n),$$
$$u_n \in G(x_n) : \|u_{n+1} - u_n\| \leq D(G(x_{n+1}), G(x_n)),$$
$$v_n \in F(y_n) : \|v_{n+1} - v_n\| \leq D(F(y_{n+1}), F(y_n)),$$
$$z'_{n+1} = H_1(f(x_n)) - S(x_n - p(x_n), v_n) + (I - \rho^{-1}) R^{M(\cdot,x_n)}_{H_1,\rho}(z'_n),$$
$$z''_{n+1} = H_2(g(y_n)) - T(u_n, y_n - q(y_n)) + (I - \gamma^{-1}) R^{N(\cdot,y_n)}_{H_2,\gamma}(z''_n),$$

$n = 0, 1, 2, \ldots.$

3.4. Theorem. *Let* E_1 *and* E_2 *be any two real uniformly smooth Banach spaces with module of smoothness* $\tau_{E_1}(t) \leq C_1 t^2$ *and* $\tau_{E_2}(t) \leq C_2 t^2$ *for* $C_2, C_2 > 0$, *respectively. Let* $G : E_1 \to CB(E_1)$ *and* $F : E_2 \to CB(E_2)$ *be* D*-Lipschitz continuous mappings with constants* λ_{D_G} *and* λ_{D_F}, *respectively. Let* $H_1 : E_1 \to E_1$ *and* $H_2 : E_2 \to E_2$ *be strongly accretive and Lipschitz continuous mappings with constants* r_1, r_2 *and* λ_{H_1}, λ_{H_2}, *respectively. Let* $M : E_1 \times E_1 \to 2^{E_1}$ *be an* H_1*-accretive operator and let* $N : E_2 \times E_2 \to 2^{E_2}$ *be an* H_2*-accretive operator such that the* H_1*-resolvent operator associated with* M *and the* H_2*-resolvent operator associated with* N *are retractions. Let* $f, p : E_1 \to E_1$, $g, q : E_2 \to E_2$ *be strongly accretive mappings with constants* δ_f, δ_p, δ_g *and* δ_q, *respectively, and Lipschitz continuous with constants* λ_f, λ_p, λ_g *and* λ_q, *respectively. Let* $S : E_1 \times E_2 \to E_1$ *and* $T : E_1 \times E_2 \to E_2$ *be Lipschitz continuous in the first and second arguments, with constants* λ_{S_1}, λ_{S_2} *and* λ_{T_1}, λ_{T_2}, *respectively.*

If there exist constants $\rho > 0$ *and* $\gamma > 0$, *such that*

(3.11)
$$0 < \frac{\mathrm{B}'_1/2 + 1 + \rho\lambda_{\mathrm{S}_1}\sqrt{1 - 2\delta_{\mathrm{p}} + 64\mathrm{C}_1\lambda^2_{\mathrm{p}}} + \gamma\lambda_{\mathrm{T}_1}\lambda_{\mathrm{D_G}}}{\mathrm{r}_1(1 - \frac{\mathrm{B}'_2}{2})} < 1,$$
$$0 < \frac{\mathrm{B}''_1/2 + 1 + \gamma\lambda_{\mathrm{T}_2}\sqrt{1 - 2\delta_{\mathrm{q}} + 64\mathrm{C}_2\lambda^2_{\mathrm{q}}} + \rho\lambda_{\mathrm{S}_2}\lambda_{\mathrm{D_F}}}{\mathrm{r}_2(1 - \frac{\mathrm{B}''_2}{2})} < 1,$$

where

$$B'_1 = 2\sqrt{1 - 2r_1\lambda^2_f + 64C_1\lambda^2_{H_1}\lambda^2_f}, \qquad B'_2 = 2\sqrt{1 - 2\delta_f + 64C_1\lambda^2_f},$$
$$B''_1 = 2\sqrt{1 - 2r_2\lambda^2_g + 64C_2\lambda^2_{H_2}\lambda^2_f}, \qquad B''_2 = 2\sqrt{1 - 2\delta_g + 64C_2\lambda^2_g},$$

then there exist $(x, y) \in E_1 \times E_2$, $u \in G(x)$, $v \in F(y)$, $z' \in E_1$, $z'' \in E_2$ *satisfying the system of generalized* H*-resolvent equations (2.1) and the iterative sequences* $\{z'_n\}$, $\{z''_n\}$, $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ *and* $\{v_n\}$ *generated by Algorithm 3.2 converge strongly to* z', z'', x, y, u *and* v, *respectively.*

Proof. From Algorithm 3.2, we have

$$
\begin{aligned}
\|z'_{n+1} - z'_n\| &= \|H_1(f(x_n)) - \rho S(x_n - p(x_n), v_n) \\
&\qquad - [H_1(f(x_{n-1})) - \rho S(x_{n-1} - p(x_{n-1}), v_{n-1})]\| \\
&\le \|x_n - x_{n-1} - (H_1(f(x_n)) - H_1(f(x_{n-1})))\| + \|x_n - x_{n-1}\| \\
&\qquad + \rho\|S(x_n - p(x_n), v_n) - S(x_{n-1} - p(x_{n-1}), v_{n-1})\|.
\end{aligned} \tag{3.12}
$$

Since H_1 is strongly accretive with constant r_1 and Lipschitz continuous with constant λ_{H_1}, f is Lipschitz continuous with constant λ_f and by Proposition 2.9, we have

$$
\begin{aligned}
&\|x_n - x_{n-1} - (H_1(f(x_n)) - H_1(f(x_{n-1})))\|^2 \\
&\quad \le \|x_n - x_{n-1}\|^2 + 2\langle -(H_1(f(x_n)) - H_1(f(x_{n-1}))), \\
&\qquad\qquad j(x_n - x_{n-1} - (H_1(f(x_n)) - H_1(f(x_{n-1}))))\rangle \\
&\quad = \|x_n - x_{n-1}\|^2 + 2\langle -(H_1(f(x_n)) - H_1(f(x_{n-1}))), j(x_n - x_{n-1})\rangle \\
&\qquad + 2\langle -(H_1(f(x_n)) - H_1(f(x_{n-1}))), \\
&\qquad\qquad j(x_n - x_{n-1} - (H_1(f(x_n)) - H_1(f(x_{n-1}))) - j(x_n - x_{n-1}))\rangle \\
&\quad \le \|x_n - x_{n-1}\|^2 - 2r_1\|f(x_n) - f(x_{n-1})\|^2 \\
&\qquad\qquad + 4d^2\tau_E\Big(\frac{4\|H_1(f(x_n)) - H_1(f(x_{n-1}))\|}{d}\Big) \\
&\quad \le \|x_n - x_{n-1}\|^2 - 2r_1\lambda_f^2\|x_n - x_{n-1}\|^2 + 64C_1\lambda_{H_1}^2\lambda_f^2\|x_n - x_{n-1}\|^2 \\
&\quad \le (1 - 2r_1\lambda_f^2 + 64C_1\lambda_{H_1}^2\lambda_f^2)\|x_n - x_{n-1}\|^2.
\end{aligned} \tag{3.13}
$$

Since S is Lipschitz continuous in both arguments, F is D-Lipschitz continuous, we have

$$
\begin{aligned}
&\|S(x_n - p(x_n), v_n) - S(x_{n-1} - p(x_{n-1}), v_{n-1})\| \\
&\quad = \|S(x_n - p(x_n), v_n) - S(x_{n-1} - p(x_{n-1}), v_n) \\
&\qquad\qquad + S(x_{n-1} - p(x_{n-1}), v_n) - S(x_{n-1} - p(x_{n-1}), v_{n-1})\| \\
&\quad \le \|S(x_n - p(x_n), v_n) - S(x_{n-1} - p(x_{n-1}), v_n)\| \\
&\qquad\qquad + \|S(x_{n-1} - p(x_{n-1}), v_n) - S(x_{n-1} - p(x_{n-1}), v_{n-1})\| \\
&\quad \le \lambda_{S_1}\|x_n - x_{n-1} - (p(x_n) - p(x_{n-1}))\| + \lambda_{S_2}\|v_n - v_{n-1}\| \\
&\quad \le \lambda_{S_1}\|x_n - x_{n-1} - (p(x_n) - p(x_{n-1}))\| + \lambda_{S_2}D(F(y_n), F(y_{n-1})) \\
&\quad \le \lambda_{S_1}\|x_n - x_{n-1} - (p(x_n) - p(x_{n-1}))\| + \lambda_{S_2}\lambda_{D_F}\|y_n - y_{n-1}\|.
\end{aligned} \tag{3.14}
$$

By Proposition 2.9, we have (see, for example the proof of [4, Theorem 3])

$$
\|x_n - x_{n-1} - (p(x_n) - p(x_{n-1}))\|^2 \le (1 - 2\delta_p + 64C_1\lambda_p^2)\|x_n - x_{n-1}\|^2. \tag{3.15}
$$

Using (3.15), (3.14) becomes

$$
\begin{aligned}
&\|S(x_n - p(x_n), v_n) - S(x_{n-1} - p(x_{n-1}), v_{n-1})\| \\
&\qquad \le \lambda_{S_1}\sqrt{1 - 2\delta_p + 64C_1\lambda_p^2}\|x_n - x_{n-1}\| + \lambda_{S_2}\lambda_{D_F}\|y_n - y_{n-1}\|.
\end{aligned} \tag{3.16}
$$

Using (3.13) and (3.16), (3.12) becomes

$$
\begin{aligned}
&\|z'_{n+1} - z'_n\| \\
&\quad \le \sqrt{1 - 2r_1\lambda_f^2 + 64C_1\lambda_{H_1}^2\lambda_f^2}\|x_n - x_{n-1}\| + \|x_n - x_{n-1}\| \\
&\qquad + \rho(\lambda_{S_1}\sqrt{1 - 2\delta_p + 64C_1\lambda_p^2}\|x_n - x_{n-1}\| + \lambda_{S_2}\lambda_{D_F}\|y_n - y_{n-1}\|) \\
&\quad = \left(\sqrt{1 - 2r_1\lambda_f^2 + 64C_1\lambda_{H_1}^2\lambda_f^2} + 1 + \rho\lambda_{S_1}\sqrt{1 - 2\delta_p + 64C_1\lambda_p^2}\right)\|x_n - x_{n-1}\| \\
&\qquad + \rho\lambda_{S_2}\lambda_{D_F}\|y_n - y_{n-1}\| \\
&\quad = \left(B'_1/2 + 1 + \rho\lambda_{S_1}\sqrt{1 - 2\delta_p + 64C_1\lambda_p^2}\right)\|x_n - x_{n-1}\| \\
&\qquad + \rho\lambda_{S_2}\lambda_{D_F}\|y_n - y_{n-1}\|,
\end{aligned}
\tag{3.17}
$$

where $B'_1 = 2\sqrt{1 - 2r_1\lambda_f^2 + 64C_1\lambda_{H_1}^2\lambda_f^2}$.

Again, from Algorithm 3.2 we have

$$
\begin{aligned}
\|z''_{n+1} - z''_n\| &= \|H_2(g(y_n)) - \gamma T(u_n, y_n - q(y_n)) \\
&\qquad - [H_2(g(y_{n-1})) - \gamma T(u_{n-1}, y_{n-1} - q(y_{n-1}))]\| \\
&\le \|y_n - y_{n-1} - (H_2(g(y_n)) - H_2(g(y_{n-1})))\| + \|y_n - y_{n-1}\| \\
&\qquad + \gamma\|T(u_n, y_n - q(y_n)) - T(u_{n-1}, y_{n-1} - q(y_{n-1}))\|.
\end{aligned}
\tag{3.18}
$$

Since H_2 is strongly accretive with constant r_2 and Lipschitz continuous with constant λ_{H_2}, g is Lipschitz continuous with constant λ_g and by Proposition 2.9, we have

$$
\|y_n - y_{n-1} - (H_2(g(y_n)) - H_2(g(y_{n-1})))\|^2 \le (1 - 2r_2\lambda_g^2 + 64C_2\lambda_{H_2}^2\lambda_g^2)\|y_n - y_{n-1}\|^2. \tag{3.19}
$$

Since T is Lipschitz continuous in both arguments, G is D-Lipschitz continuous, we have

$$
\begin{aligned}
&\|T(u_n, y_n - q(y_n)) - T(u_{n-1}, y_{n-1} - q(y_{n-1}))\| \\
&\qquad \le \|T(u_n, y_n - q(y_n)) - T(u_{n-1}, y_n - q(y_n))\| \\
&\qquad\qquad + \|T(u_{n-1}, y_n - q(y_n)) - T(u_{n-1}, y_{n-1} - q(y_{n-1}))\| \\
&\qquad \le \lambda_{T_1}\|u_n - u_{n-1}\| + \lambda_{T_2}\|y_n - q(y_n) - (y_{n-1} - q(y_{n-1}))\| \\
&\qquad \le \lambda_{T_1}D(G(x_n), G(x_{n-1})) + \lambda_{T_2}\|y_n - y_{n-1} - (q(y_n) - q(y_{n-1}))\| \\
&\qquad \le \lambda_{T_1}\lambda_{D_G}\|x_n - x_{n-1}\| + \lambda_{T_2}\|y_n - y_{n-1} - (q(y_n) - q(y_{n-1}))\|.
\end{aligned}
\tag{3.20}
$$

Using the same argument as for (3.15), we have

$$
\|y_n - y_{n-1} - (q(y_n) - q(y_{n-1}))\|^2 \le (1 - 2\delta_q + 64C_2\lambda_q^2)\|y_n - y_{n-1}\|^2. \tag{3.21}
$$

Using (3.21), (3.20) becomes

$$
\begin{aligned}
&\|T(u_n, y_n - q(y_n)) - T(u_{n-1}, y_{n-1} - q(y_{n-1}))\| \\
&\qquad \le \lambda_{T_1}\lambda_{D_G}\|x_n - x_{n-1}\| + \lambda_{T_2}\sqrt{1 - 2\delta_q + 64C_2\lambda_q^2}\|y_n - y_{n-1}\|.
\end{aligned}
\tag{3.22}
$$

Using (3.19) and (3.22), (3.18) becomes

$$\begin{aligned}
&\|z''_{n+1} - z''_n\| \\
&\quad \le \sqrt{1 - 2r_2\lambda_g^2 + 64C_2\lambda_{H_2}^2\lambda_g^2}\|y_n - y_{n-1}\| + \|y_n - y_{n-1}\| \\
&\qquad + \gamma\Big(\lambda_{T_1}\lambda_{D_G}\|x_n - x_{n-1}\| + \lambda_{T_2}\sqrt{1 - 2\delta_q + 64C_2\lambda_q^2}\|y_n - y_{n-1}\|\Big) \\
&\quad = \Big(\sqrt{1 - 2r_2\lambda_g^2 + 64C_2\lambda_{H_2}^2\lambda_g^2} + 1 + \gamma\lambda_{T_2}\sqrt{1 - 2\delta_q + 64C_2\lambda_q^2}\Big)\|y_n - y_{n-1}\| \\
&\qquad + \gamma\lambda_{T_1}\lambda_{D_G}\|x_n - x_{n-1}\| \\
&\quad = \Big(B''_1/2 + 1 + \gamma\lambda_{T_2}\sqrt{1 - 2\delta_q + 64C_2\lambda_q^2}\Big)\|y_n - y_{n-1}\| \\
&\qquad + \gamma\lambda_{T_1}\lambda_{D_G}\|x_n - x_{n-1}\|, \qquad (3.23)
\end{aligned}$$

where $B''_1 = 2\sqrt{1 - 2r_2\lambda_g^2 + 64C_2\lambda_{H_2}^2\lambda_g^2}$.

By (3.17) and (3.23), we have

$$\begin{aligned}
&\|z'_{n+1} - z'_n\| + \|z''_{n+1} - z''_n\| \\
&\quad \le (B'_1/2 + 1 + \rho\lambda_{S_1}\sqrt{1 - 2\delta_p + 64C_1\lambda_p^2} + \gamma\lambda_{T_1}\lambda_{D_G})\|x_n - x_{n-1}\| \qquad (3.24) \\
&\qquad + \Big(B''_1/2 + 1 + \gamma\lambda_{T_2}\sqrt{1 - 2\delta_q + 64C_2\lambda_q^2} + \rho\lambda_{S_2}\lambda_{D_F}\Big)\|y_n - y_{n-1}\|.
\end{aligned}$$

Also from (3.5) and (3.6), we have

$$\begin{aligned}
\|x_n - x_{n-1}\| &= \|x_n - x_{n-1} - (f(x_n) - f(x_{n-1})) + J_{H_1,\rho}^M(z'_n) - J_{H_1,\rho}^M(z'_{n-1})\| \\
&\le \|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\| + \|J_{H_1,\rho}^M(z'_n) - J_{H_1,\rho}^M(z'_{n-1})\| \\
&\le \|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\| + \frac{1}{r_1}\|z'_n - z'_{n-1}\|. \qquad (3.25)
\end{aligned}$$

Using the same argument as for (3.15), we have

$$\|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\|^2 \le (1 - 2\delta_f + 64C_1\lambda_f^2)\|x_n - x_{n-1}\|^2. \qquad (3.26)$$

Using (3.26), (3.25) becomes

$$\begin{aligned}
\|x_n - x_{n-1}\| &\le \sqrt{(1 - 2\delta_f + 64C_1\lambda_f^2)}\|x_n - x_{n-1}\| + \frac{1}{r_1}\|z'_n - z'_{n-1}\| \\
&\le \frac{B'_2}{2}\|x_n - x_{n-1}\| + \frac{1}{r_1}\|z'_n - z'_{n-1}\|,
\end{aligned}$$

where $B'_2 = 2\sqrt{(1 - 2\delta_f + 64C_1\lambda_f^2)}$. This implies

$$\|x_n - x_{n-1}\| \le \frac{1}{r_1(1 - \frac{B'_2}{2})}\|z'_n - z'_{n-1}\|, \qquad (3.27)$$

and

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \|y_n - y_{n-1} - (g(y_n) - g(y_{n-1})) + J_{H_2,\gamma}^N(z''_n) - J_{H_2,\gamma}^N(z''_{n-1})\| \\
&\le \|y_n - y_{n-1} - (g(y_n) - g(y_{n-1}))\| + \|J_{H_2,\gamma}^N(z''_n) - J_{H_2,\gamma}^N(z''_{n-1})\| \\
&\le \|y_n - y_{n-1} - (g(y_n) - g(y_{n-1}))\| + \frac{1}{r_2}\|z''_n - z''_{n-1}\|. \qquad (3.28)
\end{aligned}$$

Using the same argument as for (3.15), we have

$$\|y_n - y_{n-1} - (g(y_n) - g(y_{n-1}))\|^2 \le (1 - 2\delta_g + 64C_2\lambda_g^2)\|y_n - y_{n-1}\|^2. \qquad (3.29)$$

Using (3.29), (3.28) becomes

$$\begin{aligned}\|y_n - y_{n-1}\| &\le \sqrt{(1-2\delta_g + 64C_2\lambda_g^2)}\|y_n - y_{n-1}\| + \frac{1}{r_2}\|z''_n - z''_{n-1}\| \\ &\le \frac{B''_2}{2}\|y_n - y_{n-1}\| + \frac{1}{r_2}\|z''_n - z''_{n-1}\|,\end{aligned}$$

where $B''_2 = 2\sqrt{1-2\delta_g + 64C_2\lambda_g^2}$. This implies that

$$\|y_n - y_{n-1}\| \le \frac{1}{r_2(1-\frac{B''_2}{2})}\|z''_n - z''_{n-1}\|. \tag{3.30}$$

Using (3.27) and (3.30), (3.24) becomes

$$\begin{aligned}&\|z'_{n+1} - z'_n\| + \|z''_{n+1} - z''_n\| \\ &\quad\le \left[\frac{B'_1/2 + 1 + \rho\lambda_{S_1}\sqrt{1-2\delta_p + 64C_1\lambda_p^2} + \gamma\lambda_{T_1}\lambda_{D_G}}{r_1(1-\frac{B'_2}{2})}\right]\|z'_n - z'_{n-1}\| \\ &\qquad + \left[\frac{B''_1/2 + 1 + \gamma\lambda_{T_2}\sqrt{1-2\delta_q + 64C_2\lambda_q^2} + \rho\lambda_{S_2}\lambda_{D_F}}{r_2(1-\frac{B''_2}{2})}\right]\|z''_n - z''_{n-1}\| \\ &\quad\le \theta(\|z'_n - z'_{n-1}\| + \|z''_n - z''_{n-1}\|), \qquad (3.31)\end{aligned}$$

where

$$\theta = \max\left\{\frac{B'_1/2 + 1 + \rho\lambda_{S_1}\sqrt{1-2\delta_p + 64C_1\lambda_p^2} + \gamma\lambda_{T_1}\lambda_{D_G}}{r_1(1-\frac{B'_2}{2})}, \frac{B''_1/2 + 1 + \gamma\lambda_{T_2}\sqrt{1-2\delta_q + 64C_2\lambda_q^2} + \rho\lambda_{S_2}\lambda_{D_F}}{r_2(1-\frac{B''_2}{2})}\right\}.$$

By (3.11) we know that $0 < \theta < 1$ and so (3.31) implies that $\{z'_n\}$ and $\{z''_n\}$ are both Cauchy sequences. Thus, there exists $z' \in E_1$ and $z'' \in E_2$ such that $z'_n \to z'$ and $z''_n \to z''$ as $n \to \infty$.

From (3.27) and (3.30) it follows that $\{x_n\}$ and $\{y_n\}$ are also Cauchy sequences, that is, there exists $x \in E_1$ and $y \in E_2$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Also from (3.7) and (3.8), we have

$$\begin{aligned}\|u_{n+1} - u_n\| &\le D(G(x_{n+1}), G(x_n)) \le \lambda_{D_G}\|x_{n+1} - x_n\|, \\ \|v_{n+1} - v_n\| &\le D(F(y_{n+1}), F(y_n)) \le \lambda_{D_F}\|y_{n+1} - y_n\|,\end{aligned}$$

and hence $\{u_n\}$ and $\{v_n\}$ are also Cauchy sequences, so there exist $u \in E_1$ and $v \in E_2$ such that $u_n \to u$ and $v_n \to v$ respectively.

Now, we will show that $u \in G(x)$ and $v \in F(y)$. In fact, since $u_n \in G(x_n)$ and

$$\begin{aligned}d(u_n, G(x)) &\le \max\Big\{d(u_n, G(x)), \sup_{w_1 \in G(x)} d(G(x_n), w_1)\Big\} \\ &\le \max\Big\{\sup_{w_2 \in G(x_n)} d(w_2, G(x)), \sup_{w_1 \in G(x)} d(G(x_n), w_1)\Big\} \\ &= D(G(x_n), G(x)),\end{aligned}$$

we have

$$\begin{aligned}d(u, G(x)) &\le \|u - u_n\| + d(u_n, G(x)) \\ &\le \|u - u_n\| + D(G(x_n), G(x)) \\ &\le \|u - u_n\| + \lambda_{D_G}\|x_n - x\| \to 0 \text{ as } n \to \infty.\end{aligned}$$

which implies that $d(u, G(x)) = 0$. Since $G(x) \in CB(E)$, it follows that $u \in G(x)$. Similarly, we can show that $v \in F(y)$. By continuity of f, g, p, q, H_1, H_2, G, F, M, N, S, T, $J^M_{H_1,\rho}(\cdot, x)$, $J^N_{H_2,\gamma}(\cdot, y)$ and Algorithm 3.1, we have

$$z' = H_1(f(x)) - \rho S(x - p(x), v) = H_1(J^M_{H_1,\rho}(\cdot, x)(z')) - \rho S(x - p(x), v) \in E_1$$

and

$$z'' = H_2(g(y)) - \gamma T(u, y - q(y)) = H_2(J^N_{H_2,\gamma}(\cdot, y)(z'')) - \gamma T(u, y - q(y)) \in E_2.$$

By Proposition 3.1, the required result follows. □

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