Hacettepe Journal of Mathematics and Statistics
Volume 41 (1) (2012), 67 – 75

A GENERAL FRAMEWORK FOR COMPACTNESS IN L-TOPOLOGICAL SPACES

A. Ghareeb*

Received 23 : 08 : 2010 : Accepted 14 : 07 : 2011

Abstract

A general framework for the concepts of compactness, countable compactness, and the Lindelöf property are introduced in L-topological spaces by means of several kinds of open L-sets and their inequalities when L is a complete DeMorgan algebra. The method used in this paper shows that these results are valid for any kind of open L-sets and thus we do not need to repeat it for each kind separately.

Keywords: L-topological space, Compactness, Countable compactness, Lindelöf property.

2000 AMS Classification: 54 A 40.

1. Introduction

The concept of compactness of an I-topological space was first introduced by Chang [6] in terms of open covers. Chang's compactness has been greatly extended to the variable-basis case by Rodabaugh [12], and it can be regarded as a successful definition of compactness in poslat topology from the categorical point of view (see [12, 18]). Moreover, Gantner *et al.* introduced α-compactness [8], Lowen introduced fuzzy compactness, strong fuzzy compactness and ultra-fuzzy compactness [17, 16], Chadwick [5] generalized Lowen's compactness, Liu introduced Q-compactness [15], Li introduced strong Q-compactness [13] which is equivalent to the strong fuzzy compactness in [16], Wang and Zhao introduced N-compactness [29, 31], and Shi introduced S^*-compactness [24].

Recently, Shi presented a new definition of fuzzy compactness in L-topological spaces [20, 25] by means of open L-sets and their inequality where L is a complete DeMorgan algebra. The new definition does not depend on the structure of L. When L is completely distributive, it is equivalent to the notion of fuzzy compactness in [14, 17, 28].

In this paper, following the lines of [20, 24, 25], we will introduce a general framework of compactness in L-topological spaces by means of m-open L-sets and their inequality,

*Department of Mathematics, Faculty of Science, South Valley University, Qena, Egypt.
E-mail: `nasserfuzt@aim.com`

where m means the kind of openness of the L-sets. We also introduce countable m-compactness and the m-Lindelöf property in L-topology.

2. Preliminaries

Throughout this paper $(L, \leq, \bigwedge, \bigvee, \prime)$ is a complete DeMorgan algebra, X a nonempty set. The smallest element and the largest element in L are denoted by 0 and 1, respectively. By L_0 and L_1 we mean $L\backslash\{0\}$ and $L\backslash\{1\}$, respectively. L^X is the set of all L-fuzzy sets (or L-sets, for short) on X. The smallest element and the largest element in L^X are denoted by $\chi_\emptyset$ and χ_X, respectively. We often do not distinguish a crisp subset A of X and its character function χ_A.

A complete lattice L is a complete Heyting algebra if it satisfies the following infinite distributive law: For all $a \in L$ and all $B \subset L$, $a \wedge \bigvee B = \bigvee\{a\wedge \mid b \in B\}$.

An element a in L is called a prime element if $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$. An element a in L is called co-prime if a' is prime [9]. The set of non-unit prime elements in L is denoted by $P(L)$. The set of non-zero co-prime elements in L is denoted by $M(L)$.

The binary relation $\prec$ in L is defined as follows: for a, $b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, the relation $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$ [7]. In a completely distributive DeMorgan algebra L, each element b is a sup of $\{a \in L \mid a \prec b\}$. A set $\{a \in L \mid a \prec b\}$ is called the greatest minimal family of b in the sense of [14, 28], denoted by $\beta(b)$, and $\beta^*(b) = \beta(b) \cap M(L)$. Moreover, for $b \in L$, we define $\alpha(b) = \{a \in L \mid a' \prec b'\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$.

For $a \in L$ and $A \in L^X$, we use the following notations from [26].

$$A_{[a]} = \{x \in X \mid A(x) \geq a\},\ A^{(a)} = \{x \in X \mid A(x) \not\leq a\},$$
$$A_{(a)} = \{x \in X \mid a \in \beta(A(x))\}.$$

An L-topological space (or L-space, for short) is a pair $(X, \mathfrak{T})$, where $\mathfrak{T}$ is a subfamily of L^X which contains $\chi_\emptyset$; χ_X and is closed for any suprema and finite infima. $\mathfrak{T}$ is called an L-topology on X. Members of $\mathfrak{T}$ are called open L-sets and their complements are called closed L-sets.

2.1. Definition. [14, 28] An L-space $(X, \mathfrak{T})$ is called *weakly induced* if $\forall\, a \in L$, $A \in L^X$, it follows that $A^{(a)} \in [\mathfrak{T}]$, where $[\mathfrak{T}]$ denotes the topology formed by all the crisp sets in $\mathfrak{T}$.

2.2. Definition. [14, 28] For a topological space (X, τ), let $\omega_L(\tau)$ denote the family of all lower semi-continuous maps from (X, τ) to L, i.e., $\omega_L(\tau) = \{A \in L^X \mid A^{(a)} \in \tau,\ a \in L\}$. Then $\omega_L(\tau)$ is an L-topology on X; in this case, $(X, \omega_L(\tau))$ is said to be *topologically generated* by (X, τ). A topologically generated L-space is also called an *induced L-space*.

2.3. Definition. [21] Let $(X, \mathfrak{T})$ be an L-space, $a \in L_0$ and $G \in L^X$. A family $\mathfrak{U} \subseteq L^X$ is called a *β_a-cover of* G if for any $x \in X$, it follows that $a \in \beta(G'(x) \vee \bigvee_{A\in\mathfrak{U}} A(x))$. $\mathfrak{U}$ is called a *strong β_a-cover of* G if $a \in \beta(\bigwedge_{x\in X}(G'(x) \vee \bigvee_{A\in\mathfrak{U}} A(x)))$.

2.4. Definition. [21] Let $(X, \mathfrak{T})$ be an L-space, $a \in L_0$ and $G \in L^X$. A family $\mathfrak{U} \subseteq L^X$ is called a *Q_a-cover of* G if for any $x \in X$, it follows that $G'(x) \vee \bigvee_{A\in\mathfrak{U}} A(x) \geq a$.

It is obvious that a strong β_a-cover of G is a β_a-cover of G, and a β_a-cover of G is a Q_a-cover of G. For $a \in L$ and a crisp subset $D \subset X$, we define $a \wedge D$ and $a \vee D$ as follows:

$$(a \wedge D)(x) = \begin{cases} a, & x \in D; \\ 0, & x \notin D. \end{cases} \quad (a \vee D)(x) = \begin{cases} 1, & x \in D; \\ 0, & x \notin D. \end{cases}$$

2.5. Theorem. [26] *For an L-set $A \in L^X$, the following facts are true:*

(1) $A = \bigvee_{a\in L}(a \wedge A_{(a)}) = \bigvee_{a\in L}(a \wedge A_{[a]})$.
(2) $A = \bigwedge_{a\in L}(a \vee A^{(a)}) = \bigwedge_{a\in L}(a \vee A^{[a]})$. □

2.6. Theorem. [26] *Let $(X, \omega_L(\tau))$ be the L-space topologically generated by (X, τ) and $A \in L^X$. Then the following facts hold:*

(1) $\mathrm{cl}(A) = \bigvee_{a\in L}(a \wedge (A_{(a)})^-) = \bigvee_{a\in L}(a \wedge (A_{[a]})^-)$;
(2) $\mathrm{cl}(A)_{(a)} \subset (A_{(a)})^- \subset (A_{[a]})^- \subset \mathrm{cl}(A)_{[a]}$;
(3) $\mathrm{cl}(A) = \bigwedge_{a\in L}(a \vee (A^{(a)})^-) = \bigwedge_{a\in L}(a \vee (A^{[a]})^-)$;
(4) $\mathrm{cl}(A)^{(a)} \subset (A^{(a)})^- \subset (A^{[a]})^- \subset \mathrm{cl}(A)^{[a]}$;
(5) $\mathrm{int}(A) = \bigvee_{a\in L}(a \wedge (A_{(a)})^\circ) = \bigvee_{a\in L}(a \wedge (A_{[a]})^\circ)$;
(6) $\mathrm{int}(A)_{(a)} \subset (A_{(a)})^\circ \subset (A_{[a]})^\circ \subset \mathrm{int}(A)_{[a]}$;
(7) $\mathrm{int}(A) = \bigwedge_{a\in L}(a \vee (A^{(a)})^\circ) = \bigwedge_{a\in L}(a \vee (A^{[a]})^\circ)$;
(8) $\mathrm{int}(A)^{(a)} \subset (A^{(a)})^\circ \subset (A^{[a]})^\circ \subset \mathrm{int}(A)^{[a]}$;

where $(A_{(a)})^-$ and $(A_{(a)})^\circ$ denote respectively the closure and the interior of $A_{(a)}$ in (X, τ) and so on, $\mathrm{cl}(A)$ *and* $\mathrm{int}(A)$ *denote respectively the closure and the interior of A in $(X, \omega_L(\tau))$.* □

2.7. Definition. [21] Let $(X, \mathcal{T})$ be an L-space, $a \in L_1$ and $G \in L^X$. A family $\mathcal{A} \subseteq L^X$ is said to be:

(1) An a-shading of G if for any $x \in X$, $\big(G'(x) \vee \bigvee_{A\in\mathcal{A}} A(x)\big) \nleq a$.
(2) A strong a-shading of G if $\bigwedge_{x\in X} \big(G'(x) \vee \bigvee_{A\in\mathcal{A}} A(x)\big) \nleq a$.
(3) An a-remote family of G if for any $x \in X$, $\big(G(x) \wedge \bigwedge_{B\in\mathcal{A}} B(x)\big) \ngeq a$.
(4) A strong a-remote family of G if $\bigvee_{x\in X} \big(G(x) \wedge \bigwedge_{B\in\mathcal{A}} B(x)\big) \ngeq a$.

2.8. Definition. [21] Let $a \in L_0$ and $G \in L^X$. A subfamily $\mathcal{U}$ of L^X is said to have a *weak a-nonempty intersection in* G if $\bigvee_{x\in X} \big(G(x) \wedge \bigwedge_{A\in\mathcal{U}} A(x)\big) \geq a$. $\mathcal{U}$ is said to have the finite (countable) weak a-intersection property in G if every finite (countable) subfamily $\mathcal{P}$ of $\mathcal{U}$ has a weak a-nonempty intersection in G.

2.9. Definition. [21] Let $a \in L_0$ and $G \in L^X$. A subfamily $\mathcal{U}$ of L^X is said to be a *weak a-filter relative to* G if any finite intersection of members in $\mathcal{U}$ is weak a-nonempty in G. A subfamily $\mathcal{B}$ of L^X is said to be a *weak a-filterbase* relative to G if

$$\{A \in L^X \mid \text{there exists } B \in \mathcal{B} \text{ such that } B \leq A\}$$

is a weak a-filter relative to G.

For a subfamily $\Phi \subseteq L^X$, $2^{(\Phi)}$ denotes the set of all finite subfamilies of Φ and $2^{[\Phi]}$ the set of all countable subfamilies of Φ.

2.10. Definition. Let G be an L-set of an L-space $(X, \mathcal{T})$. G is called a *semiopen L-set* [2] (resp. a *preopen L-set* [27], *α-open L-set* [4], *β-open L-set* [3], *γ-open L-set* [11]) if $G \leq \mathrm{cl}(\mathrm{int}(G))$ (resp. $G \leq \mathrm{int}(\mathrm{cl}(G))$, $G \leq \mathrm{int}(\mathrm{cl}(\mathrm{int}(G)))$, $G \leq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(G)))$, $G \leq \mathrm{cl}(\mathrm{int}(G)) \vee \mathrm{int}(\mathrm{cl}(G))$).

The set of all semiopen L-sets (resp. preopen L-sets, α-open L-sets, β-open L-sets, γ-open L-sets) in $(X, \mathcal{T})$ will be denoted by $SO(X, \mathcal{T})$ (resp. $PO(X, \mathcal{T})$, $\alpha O(X, \mathcal{T})$, $\beta O(X, \mathcal{T})$, $\gamma O(X, \mathcal{T})$). Generally, $mO(X, \mathcal{T})$ denotes the set of all m-open L-sets.

2.11. Lemma. [25] *Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L-spaces, where L is a complete Heyting algebra, let $f : X \to Y$ be a mapping, $f_L^{\to} : L^X \to L^Y$ the extension of f. Then for any $P \subset L^Y$, we have that*

$$\bigvee_{y\in Y} \Big(f_L^{\to}(G)(y) \wedge \bigwedge_{B\in\mathcal{P}} B(y)\Big) = \bigvee_{x\in X} \Big(G(x) \wedge \bigwedge_{B\in\mathcal{P}} f_L^{\leftarrow}(B)(x)\Big). \qquad \square$$

3. A notion of m-compactness

3.1. Definition. Let $(X, \mathcal{T})$ be an L-space. $G \in L^X$ is called (*countably*) *m-compact* if for every (countable) family $\mathcal{U} \subseteq L^X$ of m-open L-sets, it follows that

$$\bigwedge_{x\in X}\Big(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\Big) \le \bigvee_{\psi\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\Big(G'(x) \vee \bigvee_{A\in\psi} A(x)\Big).$$

3.2. Definition. Let $(X, \mathcal{T})$ be an L-space. $G \in L^X$ is said to have the *m-Lindelöf property* (or to be an *m-Lindelöf L-set*) if for every family $\mathcal{U}$ of m-open L-sets, it follows that

$$\bigwedge_{x\in X}\Big(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\Big) \le \bigvee_{\psi\in 2^{[\mathcal{U}]}} \bigwedge_{x\in X}\Big(G'(x) \vee \bigvee_{A\in\psi} A(x)\Big).$$

3.3. Remark. m-compactness implies countable m-compactness and the m-Lindelöf property. Moreover, an L-set having the m-Lindelöf property is m-compact if and only if it is countably m-compact.

3.4. Theorem. *Let $(X, \mathcal{T})$ be an L-space. Then $G \in L^X$ is (countably) m-compact if and only if for every (countable) family $\mathcal{B}$ of m-closed L-sets, it follows that*

$$\bigvee_{x\in X}\Big(G(x) \wedge \bigwedge_{B\in\mathcal{B}} B(x)\Big) \ge \bigwedge_{\vartheta\in 2^{(\mathcal{B})}} \bigvee_{x\in X}\Big(G(x) \wedge \bigwedge_{B\in\vartheta} B(x)\Big).$$

Proof. Straightforward. □

3.5. Theorem. *Let $(X, \mathcal{T})$ be an L-space. Then $G \in L^X$ has the m-Lindelöf property if and only if for every family $\mathcal{B}$ of m-closed L-sets, it follows that*

$$\bigvee_{x\in X}\Big(G(x) \wedge \bigwedge_{B\in\mathcal{B}} B(x)\Big) \ge \bigwedge_{\vartheta\in 2^{[\mathcal{B}]}} \bigvee_{x\in X}\Big(G(x) \wedge \bigwedge_{B\in\vartheta} B(x)\Big).$$

Proof. Straightforward. □

3.6. Theorem. *Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then the following conditions are equivalent:*

(1) *G is a (countably) m-compact.*
(2) *For any $a \in L_1$, each (countable) m-open strong a-shading $\mathcal{U}$ of G has a finite subfamily which is a strong a-shading of G.*
(3) *For any $a \in L_0$, each (countable) m-closed strong a-remote family $\mathcal{P}$ of G has a finite subfamily which is a strong a-remote family of G.*
(4) *For any $a \in L_0$, each (countable) family of m-closed L-sets which has the finite weak a-intersection property in G has a weak a-nonempty intersection in G.*
(5) *For each $a \in L_0$, every m-closed (countable) weak a-filterbase relative to G has a weak a-nonempty intersection in G.* □

3.7. Theorem. *Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then the following conditions are equivalent:*

(1) *G has the m-Lindelöf property.*
(2) *For any $a \in L_1$, each m-open strong a-shading $\mathcal{U}$ of G has a countable subfamily which is a strong a-shading of G.*
(3) *For any $a \in L_0$, each m-closed strong a-remote family $\mathcal{P}$ of G has a countable subfamily which is a strong a-remote family of G.*
(4) *For any $a \in L_0$, each family of m-closed L-sets which has the countable weak a-intersection property in G has a weak a-nonempty intersection in G.* □

4. Properties of (countable) m-compactness

4.1. Theorem. *Let L be a complete Heyting algebra. If both G and H are (countably) m-compact, then $G \vee H$ is (countably) m-compact.*

Proof. For any (countable) family $\mathcal{B}$ of m-closed L-sets, we have by Theorem 3.4 that

$$\begin{aligned}
&\bigvee_{x\in X}\left((G\vee H)(x)\wedge \bigwedge_{B\in\mathcal{B}} B(x)\right)\\
&\quad = \left\{\bigvee_{x\in X}\left(G(x)\wedge \bigwedge_{B\in\mathcal{B}} B(x)\right)\right\}\vee\left\{\bigvee_{x\in X}\left(H(x)\wedge \bigwedge_{B\in\mathcal{B}} B(x)\right)\right\}\\
&\quad \geq \left\{\bigwedge_{\vartheta\in 2^{(\mathcal{B})}}\bigvee_{x\in X}\left(G(x)\wedge \bigwedge_{B\in\vartheta} B(x)\right)\right\}\vee\left\{\bigwedge_{\vartheta\in 2^{(\mathcal{B})}}\bigvee_{x\in X}\left(H(x)\wedge \bigwedge_{B\in\vartheta} B(x)\right)\right\}\\
&\quad = \bigwedge_{\vartheta\in 2^{(\mathcal{B})}}\bigvee_{x\in X}\left((G\vee H)(x)\wedge \bigwedge_{B\in\vartheta} B(x)\right).
\end{aligned}$$

This shows that $G \vee H$ is (countably) m-compact. □

Analogously we have the following result.

4.2. Theorem. *Let L be a complete Heyting algebra. If both G and H have the m-Lindelöf property, then $G \vee H$ has the m-Lindelöf property.* □

4.3. Theorem. *If G is (countably) m-compact and H is m-closed, then $G \wedge H$ is (countably) m-compact.*

Proof. For any (countable) family $\mathcal{B}$ of m-closed L-sets, we have by Theorem 3.4 that

$$\begin{aligned}
&\bigvee_{x\in X}\left((G\wedge H)(x)\wedge \bigwedge_{B\in\mathcal{B}} B(x)\right)\\
&\quad = \bigvee_{x\in X}\left(G(x)\wedge \bigwedge_{B\in\mathcal{B}\cup\{H\}} B(x)\right)\\
&\quad \geq \bigwedge_{\vartheta\in 2^{(\mathcal{B}\cup\{H\})}}\bigvee_{x\in X}\left(G(x)\wedge \bigwedge_{B\in\vartheta} B(x)\right)\\
&\quad = \left\{\bigwedge_{\vartheta\in 2^{(\mathcal{B})}}\bigvee_{x\in X}\left(G(x)\wedge \bigwedge_{B\in\vartheta} B(x)\right)\right\}\\
&\qquad\qquad \wedge\left\{\bigwedge_{\vartheta\in 2^{(\mathcal{B})}}\bigvee_{x\in X}\left(G(x)\wedge H(x)\wedge \bigwedge_{B\in\vartheta} B(x)\right)\right\}\\
&\quad = \left\{\bigwedge_{\vartheta\in 2^{(\mathcal{B})}}\bigvee_{x\in X}\left(G(x)\wedge H(x)\wedge \bigwedge_{B\in\vartheta} B(x)\right)\right\}\\
&\quad = \left\{\bigwedge_{\vartheta\in 2^{(\mathcal{B})}}\bigvee_{x\in X}\left((G\wedge H)(x)\wedge \bigwedge_{B\in\vartheta} B(x)\right)\right\}.
\end{aligned}$$

This shows that $G \wedge H$ is (countably) m-compact. □

4.4. Theorem. *If G has the m-Lindelöf property and* H *is m-closed, then $G \wedge H$ has the m-Lindelöf property.*

Proof. Similar to Theorem 4.3. □

4.5. Definition. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L-spaces. A map $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is called *m-irresolute* if $f_L^{\leftarrow}(G)$ is m-open for each m-open L-set G.

4.6. Theorem. *Let L be a complete Heyting algebra and let $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ be an m-irresolute map. If G is an m-compact (or, countably m-compact, m- Lindelöf) L-set in $(X, \mathcal{T}_1)$, then so is $f_L^{\rightarrow}(G)$ in $(Y, \mathcal{T}_2)$.*

Proof. Suppose that $\mathcal{P}$ is a family of m-closed L-sets, then

$$\begin{aligned}\bigvee_{y\in Y}\Big(f_L^{\rightarrow}(G)(y) \wedge \bigwedge_{B\in\mathcal{P}} B(y)\Big) &= \bigvee_{x\in X}\Big(G(x) \wedge \bigwedge_{B\in\mathcal{P}} f_L^{\leftarrow}(B)(x)\Big) \\ &\geq \bigwedge_{\vartheta\in 2^{(\mathcal{P})}} \bigvee_{x\in X}\Big(G(x) \wedge \bigwedge_{B\in\mathcal{P}} f_L^{\leftarrow}(B)(x)\Big) \\ &= \bigwedge_{\vartheta\in 2^{(\mathcal{P})}} \bigvee_{y\in Y}\Big(f_L^{\leftarrow}(G)(y) \wedge \bigwedge_{B\in\mathcal{P}} B(y)\Big).\end{aligned}$$

Therefore $f_L^{\rightarrow}(G)$ is m-compact. □

4.7. Theorem. *Let L be a complete Heyting algebra and let $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ be an m-continuous map. If G is an m-compact (a countably m-compact, m-Lindelöf) L-set in $(X, \mathcal{T}_1)$, then $f_L^{\rightarrow}(G)$ is a compact (countably compact, Lindelöf) L-set in $(Y, \mathcal{T}_2)$.*

Proof. Straightforward. □

4.8. Definition. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L-spaces. A map $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is called *strongly m-irresolute* if $f_L^{\leftarrow}(G)$ is open in $(X, \mathcal{T}_1)$ for every m-open L-set G in $(Y, \mathcal{T}_2)$.

It is obvious that a strongly m-irresolute map is m-irresolute and m-continuous. Analogously we have the following result.

4.9. Theorem. *Let L be a complete Heyting algebra and $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ a strongly m-irresolute map. If G is a compact (countably compact, Lindelöf) L-set in $(X, \mathcal{T}_1)$, then $f_L^{\rightarrow}(G)$ is an m-compact (a countably m-compact, m-Lindelöf) L-set in $(Y, \mathcal{T}_2)$.*

Proof. Straightforward. □

5. Good extensions

5.1. Theorem. *Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then the following conditions are equivalent:*

(1) *G is m-compact.*
(2) *For any $a \in L_0$ ($a \in M(L)$), each m-closed strong a-remote family of G has a finite subfamily which is an a-remote (a strong a-remote) family of G.*
(2) *For any $a \in L_0$ ($a \in M(L)$) and any m-closed strong a-remote family $\mathcal{P}$ of G, there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ and $b \in \beta(a)$ ($b \in \beta^*(a)$) such that $\mathcal{F}$ is a (strong) b-remote family of G.*
(3) *For any $a \in L_1$ ($a \in P(L)$), each m-open strong a-shading of G has a finite subfamily which is an a-shading (a strong a-shading) of G.*
(4) *For any $a \in L_1$ ($a \in P(L)$) and any m-open strong a-shading $\mathcal{U}$ of G, there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in \beta(a)$ ($b \in \beta^*(a)$) such that $\mathcal{V}$ is a (strong) b-shading of G.*
(5) *For any $a \in L_0$ ($a \in M(L)$), each m-open strong β_a-cover of G has a finite subfamily which is a (strong) β_a-cover of G.*

(6) *For any $a \in L_0$ ($a \in M(L)$) and any m-open strong β_a-cover $\mathcal{U}$ of G, there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in L$ ($b \in M(L)$) with $a \in \beta(b)$ such that $\mathcal{V}$ is a (strong) β_b-cover of G.*

(7) *For any $a \in L_0$ ($a \in M(L)$) and any $b \in \beta(a) \setminus \{0\}$, each m-open Q_a-cover of G has a finite subfamily which is a Q_b-cover of G.*

(8) *For any $a \in L_0$ ($a \in M(L)$) and any $b \in \beta(a) \setminus \{0\}$ ($b \in \beta^*(a)$), each m-open Q_a-cover of G has a finite subfamily which is a (strong) Q_b-cover of G.* □

Analogously we also can present characterizations of countable m-compactness and the m-Lindelöf property.

If $mO(X,\mathfrak{T})$ denotes the set of m-open L-sets in $(X,\mathfrak{T})$, we will denote the corresponding set in (X,τ) by $\mathcal{M}O(X,\tau)$. The following lemma can be proved separately using Theorem 2.6 for the special cases of $mO(X,\mathfrak{T})$ and $\mathcal{M}O(X,\tau)$.

5.2. Lemma. *Let $(X,\omega(L))$ be generated topologically by (X,τ). If A is an $\mathcal{M}$-open set in (X,τ), then χ_A is an m-open L-set in $(X,\omega_L(\tau))$. If B is an m-open L-set in $(X,\omega_L(\tau))$, then $B_{(a)}$ is an $\mathcal{M}$-open set in (X,τ) for every $a \in L$.* □

The next two theorems show that m-compactness, countable m-compactness and the m-Lindelöf property are good extensions.

5.3. Theorem. *Let $(X,\omega_L(\tau))$ be generated topologically by (X,τ). Then $(X,\omega_L(\tau))$ is (countably) m-compact if and only if (X,τ) is (countably) $\mathcal{M}$-compact.*

Proof. Necessity. Let $\mathcal{A}$ be an $\mathcal{M}$-open cover (a countable $\mathcal{M}$-open cover) of (X,τ). Then $\{\chi_A : A \in \mathcal{A}\}$ is a family of m-open L-sets in $(X,\omega_L(\tau))$ with

$$\bigwedge_{x\in X}\Big(\bigvee_{A\in\mathcal{U}}\chi_A(x)\Big)=1.$$

From the (countable) m-compactness of $(X,\omega_L(\tau))$ we know that

$$1 \geq \bigvee_{\psi\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\Big(\bigvee_{A\in\psi}\chi_A(x)\Big) \geq \bigwedge_{x\in X}\Big(\bigvee_{A\in\mathcal{U}}\chi_A(x)\Big)=1.$$

This implies that there exists $\psi \in 2^{(\mathcal{U})}$ such that $\bigwedge_{x\in X}(\bigvee_{A\in\psi}\chi_A(x))=1$. Hence ψ is a cover of (X,τ). Therefore (X,τ) is (countably) $\mathcal{M}$-compact.

Sufficiency. Let $\mathcal{U}$ be a (countable) family of m-open L-sets in $(X,\omega_L(\tau))$ and let $\bigwedge_{x\in X}\Big(\bigvee_{B\in\mathcal{U}}B(x)\Big)=a$. If $a=0$, then we obviously have

$$\bigwedge_{x\in X}\Big(\bigvee_{B\in\mathcal{U}}B(x)\Big) \leq \bigvee_{\psi\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\Big(\bigvee_{A\in\psi}B(x)\Big).$$

Now we suppose that $a \neq 0$. In this case, for any $b \in \beta(a)\setminus\{0\}$ we have

$$b \in \beta\Big(\bigwedge_{x\in X}\Big(\bigvee_{B\in\mathcal{U}}B(x)\Big)\Big) \subseteq \bigcap_{x\in X}\beta\Big(\bigvee_{B\in\mathcal{U}}B(x)\Big)=\bigcap_{x\in X}\bigcup_{B\in\mathcal{U}}\beta(B(x)).$$

By Lemma 5.2 this implies that $\{B_{(b)} \mid B \in \mathcal{U}\}$ is an $\mathcal{M}$-open cover of (X,τ). From the (countable) $\mathcal{M}$-compactness of (X,τ) we know that there exists $\psi \in 2^{(\mathcal{U})}$ such that $\{B_{(b)} \mid B \in \psi\}$ is a cover of (X,τ). Hence $b \leq \bigvee_{x\in X}(\bigwedge_{B\in\psi}B(x))$. Furthermore we have

$$b \leq \bigwedge_{x\in X}\Big(\bigvee_{B\in\psi}B(x)\Big) \leq \bigvee_{\psi\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\Big(\bigvee_{B\in\psi}B(x)\Big).$$

This implies that

$$\bigwedge_{x\in X}\Big(\bigvee_{B\in\mathcal{U}} B(x)\Big) = a = \bigvee\{b : b \in \beta(a)\} \leq \bigvee_{\psi\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\Big(\bigvee_{B\in\psi} B(x)\Big).$$

Therefore $(X, \omega_L(\tau))$ is (countably) m-compact. □

Analogously we have the following theorem.

5.4. Theorem. *Let $(X, \omega_L(\tau))$ be generated topologically by (X, τ). Then $(X, \omega_L(\tau))$ has the m-Lindelöf property if and only if (X, τ) has the $\mathcal{M}$-Lindelöf property.* □

6. Conclusion and remarks

In this paper, we give a general framework for the concept of compactness in L-topological spaces. Instead of studying compactness for each type of open L-sets $O(X, \mathcal{T})$ separately, we examine the compactness for open sets of type $mO(X, \mathcal{T})$.

If $mO(X, \mathcal{T}) = SO(X, \mathcal{T})$, we get the study of Shi [23], when $mO(X, \mathcal{T}) = PO(X, \mathcal{T})$, we get the study of Shi [19]. In the case of $mO(X, \mathcal{T}) = \alpha O(X, \mathcal{T})$ we have the study of Shi [21]. This method can be applied for the cases of $mO(X, \mathcal{T}) = \beta O(X, \mathcal{T})$, $mO(X, \mathcal{T}) = \gamma O(X, \mathcal{T})$, and so on.

We conclude from this that there are no benefits from repeating the same study on other kinds of L-sets where we can get any kind of compactness by choosing a suitable type m.

References

[1] Andrijević, D. *On b-open sets*, Mat. Vesnik **48**, 59û-64, 1996.

[2] Azad, K. K. *On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity*, J. Math. Anal. Appl. **82**, 14–32, 1981.

[3] Balasubäsamanian, G. *On fuzzy β-compact spaces and fuzzy β-extremally disconnected spaces*, Kybernetika **33**, 271–277, 1997.

[4] Bin Shahna, A. S. *On fuzzy strong semicontinuity and fuzzy precontinuity*, Fuzzy Sets and Systems **44**, 303–308, 1991.

[5] Chadwick, J. J. *A generalized form of compactness in fuzzy topological spaces*, J. Math. Anal. Appl. **162**, 92–110, 1991.

[6] Chang, C. L. *Fuzzy topological spaces*, J. Math. Anal. Appl. **24**, 39–90, 1968.

[7] Dwinger, P. *Characterizations of the complete homomorphic images of a completely distributive complete lattice I*, Indagationes Mathematicae (Proceedings) **85**, 403–414, 1982.

[8] Gantner, T. E., and Steinlage, R. C. and Warren, R. H. *Compactness in fuzzy topological spaces*, J. Math. Anal. Appl. **62**, 547–562, 1978.

[9] Gierz, G. *et al. A Compendium of Continuous Lattices* (Springer Verlag, Berlin, 1980).

[10] Hanafy, I. M. *βS^*-compactness in L-fuzzy topological spaces*, J. Nonlinear Sci. Appl. **9**, 27–37, 2009.

[11] Hanafy, I. M. *Fuzzy γ-open sets and fuzzy γ-continuity*, J. Fuzzy Math. **7**, 419–430, 1999.

[12] Höhle, U. and Rodabaugh, S. E. S. E. Rodabaugh, Ed. *Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory* **3** (Kluwer Academic Publishers, Boston/Dordrecht/London, 1999).

[13] Li, Z. F. *Compactness in fuzzy topological spaces*, Chinese Kexue Tongbao **6**, 321–323, 1983.

[14] Liu, Y. M. and Luo, M. K. *Fuzzy Topology* (World Scientific, Singapore, 1997).

[15] Liu, Y. M. *Compactness and Tychnoff theorem in fuzzy topological spaces*, Acta Mathematica Sinica **24**, 260–268, 1981.

[16] Lowen, R. *A comparsion of different compactness notions in fuzzy topological spaces*, J. Math. Anal. Appl. **64**, 446–454, 1978.

[17] Lowen, R. *Fuzzy topological spaces and fuzzy compactness*, J. Math. Anal. Appl. **56**, 621–633, 1976.

[18] Rodabaugh, S. E. *Point-set lattice-theoretic topology*, Fuzzy Sets and Systems **40**, 297–345, 1991.

[19] Shi, F.-G. *P-compactness in L-topological spaces*, J. Nonlinear Sci. Appl. **2**, 225–233, 2009.

[20] Shi, F.-G. *A new definition of fuzzy compactness*, Fuzzy Sets and Systems **158**, 1486–1495, 2007.

[21] Shi, F.-G. *A new form of fuzzy α-compactness*, Mathematica Bohemica **131**, 15–28, 2006.

[22] Shi, F.-G. *Semicompactness in L-topological spaces*, International Journal of Mathematics and Mathematical Sciences **12**, 1869–1878, 2005.

[23] Shi, F.-G. *Semicompactness in L-topological spaces*, Int. J. Math. Math. Sci. **12**, 1869–1878, 2005.

[24] Shi, F.-G. *A new notion of fuzzy compactness in L-topological spaces*, Information Sciences **173**, 35–48, 2005.

[25] Shi, F.-G. *Countable compactness and the Lindelöf property of L-fuzzy sets*, Iranian Journal of Fuzzy Systems **1**, 79–88, 2004.

[26] Shi, F.-G. *Theory of L_β-nested sets and L_α-nested and their applications*, Fuzzy Systems and Mathematics (In Chinese) **4**, 65–72, 1995.

[27] Singal, M. K. and Prakash, N. *Fuzzy preopen sets and fuzzy preseparation axioms*, Fuzzy Sets and Systems **44**, 273–281, 1991.

[28] Wang, G.-J. *Theory of L-Fuzzy Topological Space* (in Chinese) (Shaanxi Normal University Press, Xi'an, 1988).

[29] Wang, G. J. *A new fuzzy compactness defined by fuzzy nets*, J. Math. Anal. Appl. **94**, 1–23, 1983.

[30] Zadeh, L. A. *Fuzzy sets*, Inform. Control **8**, 338–353, 1965.

[31] Zhao, D. S. *The N-compactness in L-fuzzy topological spaces*, J. Math. Anal. Appl. **128**, 64–70, 1987.