Hacettepe Journal of Mathematics and Statistics
Volume 41 (1) (2012), 119 – 126

THE GEOMETRY OF COMPLEX CONJUGATE CONNECTIONS

Adara M. Blaga[*†] and Mircea Crasmareanu[‡]

Received 24 : 03 : 2011 : Accepted 22 : 07 : 2011

Abstract

Properties of pairs of conjugate connections are stated with a special view towards the duality of these connections. We express the complex conjugate connections in terms of the structural and the virtual tensors from the almost complex geometry. For a pair of almost complex structures we discuss their mutual recurrence by pointing out that an almost quaternionic structure is implied. The notion of complex conjugate connections is extended in two directions, one called *generalized* obtained by adding a general $(1, 2)$-tensor field and the other called *exponential* since it involves the exponential of the almost complex structure considered.

Keywords: Almost complex structure, (Conjugate) Linear connection, Hermitian manifold, Structural and virtual tensor field.

2000 AMS Classification: 53 B 05, 53 C 15.

1. Introduction

Fix M a smooth, n-dimensional (n = even) manifold for which we denote by $C^\infty(M)$– the algebra of smooth real functions on M, $\mathfrak{X}(M)$–the Lie algebra of vector fields on M, $T^r_s(M)$–the $C^\infty(M)$-module of tensor fields of (r, s)-type on M. Usually $X, Y, Z, \dots$ will be vector fields on M and if $T \to M$ is a vector bundle over M, then $\Gamma(T)$ denotes the $C^\infty(M)$-module of sections of T; e.g. $\Gamma(TM) = \mathfrak{X}(M)$.

Let $\mathcal{C}(M)$ be the set of linear connections on M. Since the difference of two linear connections is a tensor field of $(1, 2)$-type, it follows that $\mathcal{C}(M)$ is a $C^\infty(M)$-affine module associated to the $C^\infty(M)$-linear module $T^1_2(M)$.

Fix now J an almost complex structure on M, i.e. an endomorphism of the tangent bundle such that $J^2 = -I_{\mathfrak{X}(M)}$; then the associated linear connections are:

[*]Department of Mathematics and Computer Science, West University of Timişoara, Bld. V. Pârvan nr. 4, 300223, Timişoara, România. E-mail: adara@math.uvt.ro

[†]Corresponding Author.

[‡]Faculty of Mathematics, University "Al. I. Cuza", 700506, Iaşi, România.
E-mail: mcrasm@uaic.ro

1.1. Definition. $\nabla \in \mathcal{C}(M)$ is a *J-connection* if J is covariant constant with respect to ∇, namely $\nabla J = 0$.

Let $\mathcal{C}_J(M)$ be the set of these connections. In order to find this set, let us consider after [5, p. 105] the maps:

$$\psi_J : \mathcal{C}(M) \to \mathcal{C}(M), \quad \chi_J : T^1_2(M) \to T^1_2(M) \tag{1.1}$$

given by:

$$\psi_J(\nabla) := \frac{1}{2}\left(\nabla - J \circ \nabla \circ J\right), \quad \chi_J(\tau) := \frac{1}{2}(\tau - J \circ \tau \circ J). \tag{1.2}$$

So:

$$\begin{cases} \psi_J(\nabla)_X Y = \frac{1}{2}\left[\nabla_X Y - J(\nabla_X JY)\right] \\ \chi_J(X,Y) = \frac{1}{2}\left[\tau(X,Y) - J(\tau(X,JY))\right]. \end{cases} \tag{1.3}$$

Then, ψ_J is a $C^\infty(M)$-projector on $\mathcal{C}(M)$ associated to the $C^\infty(M)$-linear projector χ_J:

$$\psi_J^2 = \psi_J, \quad \chi_J^2 = \chi_J, \quad \psi_J(\nabla + \tau) = \psi_J(\nabla) + \chi_J(\tau). \tag{1.4}$$

It follows that $\nabla J = 0$ means $\psi_J(\nabla) = \nabla$, which gives that $\mathcal{C}_J(M) = \mathrm{Im}\psi_J$. This determines completely $\mathcal{C}_J(M)$. Fix ∇_0 arbitrary in $\mathcal{C}(M)$ and ∇ in $\mathcal{C}_J(M)$. So, $\nabla = \psi_J(\nabla')$ with $\nabla' = \nabla_0 + \tau$. In conclusion, $\nabla = \psi_J(\nabla_0) + \chi_J(\tau)$; in other words, $\mathcal{C}_J(M)$ is the affine submodule of $\mathcal{C}(M)$ passing through the J-connection $\psi_J(\nabla_0)$ and having the direction given by the linear submodule $\mathrm{Im}\chi_J$ of $T^1_2(M)$.

Since the projector ψ_J is our main tool in finding $\mathcal{C}_J(M)$, a careful study of it is necessary. Let us remark a decomposition (of arithmetic mean type) of it [5, p. 106]:

$$\psi_J(\nabla) = \frac{1}{2}\big(\nabla + C_J(\nabla)\big) \tag{1.5}$$

with *the conjugation map* $C_J : \mathcal{C}(M) \to \mathcal{C}(M)$:

$$C_J(\nabla)_X = -J \circ \nabla_X \circ J. \tag{1.6}$$

Then *the complex conjugate connection* $C_J(\nabla)$ measures how far the connection ∇ is from being a J-connection and as it is pointed out in [5, p. 105], C_J is the affine symmetry of the affine module $\mathcal{C}(M)$ with respect to the affine submodule $\mathcal{C}_J(M)$, made parallel with the linear submodule $\ker \chi_J$.

The present paper is devoted to a careful study of this connection $C_J(\nabla)$, since all the above computations put in evidence its role in the geometry of J; see also the first section of [1] for the meaning of this connection in special complex manifolds. So, the aim of our study is to obtain several properties of it in both the general case and in Riemannian geometry. The second section is devoted to this scope and after a general result connecting ∇ and $C_J(\nabla)$, we treat two items:

i) The behavior of the complex conjugate connection to a linear change of almost complex structures,
ii) The use of two tensor fields previously considered in the almost complex geometry.

With respect to i), we arrive at two particular remarkable cases concerning the recurrence of the given almost complex structures, while for ii), we derive some useful new identities. Also, for the case i), the skew-symmetry of the given almost complex structures yields an almost quaternionic structure. Let us point out that a similar study for almost product geometry is contained in [3].

In the third section we give some generalizations of the results from the first part by adding an arbitrary tensor field of (1, 2)-type. All generalized complex conjugate connections which form a duality with the initial linear connection are determined. The last section is devoted to the exponential conjugate connection, an object introduced in correspondence with a similar one from [1].

2. Properties of the complex conjugate connection

In what follows, for simplicity, we will denote by the superscript J the complex conjugate connection of ∇:

$$\nabla^{(J)} := C_J(\nabla) = \nabla - J \circ \nabla J \tag{2.1}$$

and then:

$$\nabla^{(J)}_X Y = \nabla_X Y - J\big(\nabla_X JY - J(\nabla_X Y)\big) = -J\big(\nabla_X JY\big). \tag{2.2}$$

The first properties of the complex conjugate connection are stated in the next proposition:

2.1. Proposition. *Let J be an almost complex structure, ∇ a linear connection and $\nabla^{(J)}$ the complex conjugate connection of ∇. Then:*

1. *$\nabla^{(J)} J = -\nabla J$; it follows that $\nabla \in \mathcal{C}_J(M)$ if and only if $\nabla^{(J)} \in \mathcal{C}_J(M)$;*
2. *∇ and $\nabla^{(J)}$ are in duality: $(\nabla^{(J)})^{(J)} = \nabla$;*
3. *$T_{\nabla^{(J)}} = T_\nabla - J\big(d^\nabla J\big)$, where d^∇ is the exterior covariant derivative induced by ∇, namely $\big(d^\nabla J\big)(X,Y) := \big(\nabla_X J\big)Y - \big(\nabla_Y J\big)X$; it follows that for $\nabla \in \mathcal{C}_J(M)$, the connections ∇ and $\nabla^{(J)}$ have the same torsion;*
4. *$R_{\nabla^{(J)}}(X,Y,Z) = -J\big(R_\nabla(X,Y,JZ)\big)$; it follows that ∇ is flat if and only if $\nabla^{(J)}$ is so;*
5. *Assume that (M,g,J) is an almost Hermitian manifold i.e. $g(JX,JY) = g(X,Y)$; then $\big(\nabla^{(J)}_X g\big)(JY,JZ) = (\nabla_X g)(Y,Z)$. It follows that ∇ is a g-metric connection if and only if $\nabla^{(J)}$ is so.*

Proof. 1. Other relations we shall use are:

$$\nabla^{(J)}_X JY = J\big(\nabla_X Y\big), \quad J\big(\nabla^{(J)}_X Y\big) = \nabla_X JY \tag{2.3}$$

and then:

$$\big(\nabla_X J\big)Y = \nabla_X JY - J\big(\nabla_X Y\big) = J\big(\nabla^{(J)}_X Y\big) - \nabla^{(J)}_X JY = -\big(\nabla^{(J)}_X J\big)Y. \tag{2.4}$$

2. Although a direct proof can be provided by the formula (1.6), we prefer to give a proof here, in order to use (2.1):

$$\big(\nabla^{(J)}\big)^{(J)} = \nabla^{(J)} - J \circ \nabla^{(J)} J = \nabla - J \circ \nabla J - J \circ (-\nabla J) = \nabla.$$

3. A direct computation gives:

$$\begin{aligned} T_{\nabla^{(J)}}(X,Y) &= \nabla^{(J)}_X Y - \nabla^{(J)}_Y X - [X,Y] \\ &= -J\big(\nabla_X JY\big) + J\big(\nabla_Y JX\big) - [X,Y] \\ &= -J\big(\nabla_X JY - \nabla_Y JX\big) + T_\nabla(X,Y) - \nabla_X Y + \nabla_Y X \\ &= T_\nabla(X,Y) - J\big(\big(\nabla_X J\big)Y - \big(\nabla_Y J\big)X\big). \end{aligned} \tag{2.5}$$

4.

$$\begin{aligned} R_{\nabla^{(J)}}(X,Y,Z) &= \nabla^{(J)}_X \nabla^{(J)}_Y Z - \nabla^{(J)}_Y \nabla^{(J)}_X Z - \nabla^{(J)}_{[X,Y]} Z \\ &= -\nabla^{(J)}_X J\big(\nabla_Y JZ\big) + \nabla^{(J)}_Y J\big(\nabla_X JZ\big) + J\big(\nabla_{[X,Y]} JZ\big) \\ &= -J\big(\nabla_X \nabla_Y JZ\big) + J\big(\nabla_Y \nabla_X JZ\big) + J\big(\nabla_{[X,Y]} JZ\big) \\ &= -J\big(R_\nabla(X,Y,JZ)\big). \end{aligned} \tag{2.6}$$

5.

$$\begin{aligned} \big(\nabla^{(J)}_X g\big)(V,W) &= X(g(V,W)) - g\big(\nabla^{(J)}_X V, W\big) - g\big(V, \nabla^{(J)}_X W\big) \\ &= X(g(V,W)) - g\big(-J\big(\nabla_X JV\big), W\big) - g\big(V, -J\big(\nabla_X JW\big)\big) \end{aligned}$$

for any X, V and $W \in \mathfrak{X}(M)$. With $V := JY$ and $W := JZ$ we get:

$$\begin{aligned} \big(\nabla^{(J)}_X g\big)(JY,JZ) &= X(g(JY,JZ)) - g\big(J\big(\nabla_X Y\big), JZ\big) - g\big(JY, J\big(\nabla_X Z\big)\big) \\ &= X(g(Y,Z)) - g\big(\nabla_X Y, Z\big) - g\big(Y, \nabla_X Z\big) = \big(\nabla_X g\big)(Y,Z). \end{aligned} \tag{2.7}$$

The above substitutions hold for $Y = -JV$ and $Z = -JW$. □

There are some direct consequences of these formulae:

i) If the pair (∇, J) is *special* i.e. $\big(\nabla_X J\big)Y = \big(\nabla_Y J\big)X$ (according to [1] or [10, p. 1003]), then $d^\nabla J = 0$ and again, the connections ∇ and $\nabla^{(J)}$ have the same torsion. If (M, J, ∇) is *nearly Kähler*, which means $\big(\nabla_X J\big)Y + \big(\nabla_Y J\big)X = 0$ (see [10, p. 1003]), then $J\big(d^\nabla J\big) = \nabla - \nabla^{(J)}$.

ii) If ∇ is the Levi-Civita connection of g, then $\nabla^{(J)}$ is also metric with respect to g.

iii) If ∇ is the Levi-Civita connection of g and in addition, $\nabla \in \mathcal{C}_J(M)$, then $\nabla^{(J)} = \nabla$ is the unique symmetric g-metric connection.

More generally, let $f \in Diff(M)$ be an automorphism of the G-structure defined by J i.e. $f_* \circ J = J \circ f_*$. If f is an affine transformation for ∇, namely $f_*\big(\nabla_X Y\big) = \nabla_{f_* X} f_* Y$, then f is also an affine transformation for $\nabla^{(J)}$.

Let us also recall that in Hermitian geometry, various choices of nice connections are obtained by requiring additional less stringent conditions on the torsion; for example, the Chern and Bismut connections are discussed in detail in [6]. Two natural generalizations of the case $\nabla \in \mathcal{C}_J(M)$ are given in our framework by:

2.2. Proposition. *Let ∇ be a symmetric linear connection.*

i) *Assume that J is ∇-recurrent i.e. $\nabla J = \eta \otimes J$, where η is a 1-form. Then $\nabla^{(J)}$ is a semi-symmetric connection.*

ii) *Assume that $\nabla J = (-\eta) \otimes I_{\mathfrak{X}(M)}$. Then $\nabla^{(J)}$ is a quarter-symmetric connection.*

Proof. Recall that a non-torsionfree linear connection is called:

-*semi-symmetric* if there exists a 1-form π such that its torsion is, [2]:

$$T(X,Y) = \pi(Y)X - \pi(X)Y,$$

-*quarter-symmetric* if in addition there exists a tensor field F of $(1,1)$-type such that, [7]:

$$T(X,Y) = \pi(Y)FX - \pi(X)FY.$$

i) We have $\nabla^{(J)} = \nabla + \eta \otimes I$ and from Proposition 2.1 (3) we get $T_{\nabla^{(J)}} = \eta \otimes I - I \otimes \eta = \eta \wedge I$.

ii) It follows that $\nabla^{(J)} = \nabla + \eta \otimes J$ and, as above, we get $T_{\nabla^{(J)}} = \eta \otimes J - J \otimes \eta = \eta \wedge J$. □

The next subject concerns the behavior of $\nabla^{(\cdot)}$ for families of almost complex structures. Let J_1 and J_2 be two almost complex structures and consider the pencil of $(1,1)$-tensor fields $J_{\alpha,\beta} := \alpha J_1 + \beta J_2$, with α and $\beta \in \mathbb{R}$. In order that $J_{\alpha,\beta}$ to be an almost complex structure, there are two necessary conditions:

1) J_1 and J_2 be skew-commuting structures: $J_1 J_2 = -J_2 J_1$,
2) (α, β) belongs to the unit circle S^1: $\alpha^2 + \beta^2 = 1$.

In fact, 1) above implies that the dimension of M is $4m$ and the triple $\big(J_1, J_2, J_3 := J_1 J_2\big)$ is a *quaternionic structure* on M [8]. Then:

$$\nabla_X^{\left(J_{\alpha,\beta}\right)} Y = \alpha^2 \nabla_X^{(J_1)} Y + \beta^2 \nabla_X^{(J_2)} Y - \alpha\beta\big[J_1(\nabla_X J_2 Y) + J_2(\nabla_X J_1 Y)\big] \tag{2.8}$$

and there are two remarkable particular cases:

i) If J_1 and J_2 are recurrent with respect to ∇ with the same 1-form of recurrence: $\nabla J_i = \eta \otimes J_i$, then the complex conjugate connections coincide $\nabla^{(J_1)} \equiv \nabla^{(J_2)} =: \nabla^{(J_{12})}$, and the invariance of $\nabla^{(J\cdot)}$:

$$\nabla^{(J_{\alpha,\beta})} = \nabla^{(J_{12})}; \tag{2.9}$$

follows.

ii) Assume that the triple (∇, J_1, J_2) is a mixed-recurrent structure: $\nabla J_i = \eta \otimes J_j$ with $i \neq j$. Then ∇ is the average of the two complex conjugate connections, $\nabla = \frac{1}{2}\big(\nabla^{(J_1)} + \nabla^{(J_2)}\big)$ and:

$$\nabla^{(J_{\alpha,\beta})} = \nabla + \big(\alpha^2 - \beta^2\big)\eta \otimes J_1 J_2 + 2\alpha\beta\eta \otimes I. \tag{2.10}$$

The last subject of this section treats two tensor fields associated with a pair (almost complex structure, linear connection) in [9]:

1) *The structural tensor field*:

$$C_\nabla^J(X,Y) := \frac{1}{2}\big[\big(\nabla_{JX} J\big)Y + \big(\nabla_X J\big)JY\big]. \tag{2.11}$$

2) *The virtual tensor field*:

$$B_\nabla^J(X,Y) := \frac{1}{2}\big[\big(\nabla_{JX} J\big)Y - \big(\nabla_X J\big)JY\big]. \tag{2.12}$$

From Proposition 2.1 (1) it follows that both these tensor fields are skew-symmetric with respect to the complex conjugation of connections:

$$C_{\nabla^{(J)}}^J = -C_\nabla^J,\ B_{\nabla^{(J)}}^J = -B_\nabla^J. \tag{2.13}$$

Also

$$C_\nabla^J(JX,JY) = -C_\nabla^J(X,Y),\ B_\nabla^J(JX,JY) = B_\nabla^J(X,Y). \tag{2.14}$$

The importance of these tensor fields for our study is given by the following straightforward relation:

$$\nabla^{(J)} = \nabla + C_\nabla^J - B_\nabla^J. \tag{2.15}$$

Recall after [4] that two linear connections are called *projectively equivalent* if there exists a 1-form τ such that:

$$\nabla' = \nabla + \tau \otimes I + I \otimes \tau. \tag{2.16}$$

A straightforward calculation gives that C^J is invariant for projective changes (2.16), while for B^J we have:

$$\big(B_{\nabla'}^J - B_\nabla^J\big)(X,Y) = \tau(JY)JX + \tau(Y)X. \tag{2.17}$$

Unfortunately, the complex conjugation of connections is not invariant under projective equivalence since:

(2.18) $\nabla'^{(J)} = \nabla^{(J)} + \tau \otimes I + J \otimes (-\tau \circ J)$.

3. Generalized complex conjugate connections

In this section we present a natural generalization of the complex conjugate connection.

3.1. Definition. A *generalized complex conjugate connection* of ∇ is:

(3.1) $\nabla^{(J,C)} := \nabla^{(J)} + C$

with $C \in T^1_2(M)$ arbitrary.

Since the duality $\nabla \leftrightarrow \nabla^{(J)}$ is one of the main feature of $\nabla^{(J)}$, let us search for tensor fields C such that $\left(\nabla^{(J,C)}\right)^{(J,C)} = \nabla$. From:

(3.2) $\left(\nabla^{(J,C)}\right)^{(J,C)}_X Y = \nabla_X Y - J(C(X, JY)) + C(X, Y)$,

it follows that we are interested in finding solutions C to:

(3.3) $J(C(X, JY)) = C(X, Y)$.

Let us remark that:

i) $C_0 = \nabla J$ is a particular solution of (3.3),
ii) if C is a solution, then $J \circ C$ is again a solution.

So, let us search the duality property for:

(3.4) $\nabla^{(J,\lambda,\mu)} = \nabla^{(J)} + \lambda\nabla J + \mu J(\nabla J) = (1-\mu)\nabla^{(J)} + \lambda\nabla J$

with $\lambda, \mu \in \mathbb{R}$.

3.2. Proposition. *The duality* $\nabla \leftrightarrow \nabla^{(J,\lambda,\mu)}$ *holds only for the pairs* $(\lambda, \mu) \in \{(0,0), (0,2)\}$.

Proof. From:

$$\left(\nabla^{(J,\lambda,\mu)}\right)^{(J,\lambda,\mu)}_X Y = \left[(1-\mu)^2 - \lambda^2\right]\nabla_X Y + 2\lambda(1-\mu)J\big(\nabla_X Y\big) + 2\lambda(\mu-1)\nabla_X JY - 2\lambda^2 J\big(\nabla_X JY\big)$$

we obtain the system:

$$\begin{cases} (1-\mu)^2 - \lambda^2 = 1, \\ \lambda(1-\mu) = \lambda^2 = 0, \end{cases}$$

which has the above solutions.

Let us point out that:

$$\begin{cases} \nabla^{(J,0,0)} = \nabla^{(J)}, \\ \nabla^{(J,0,2)} = -\nabla^{(J)}, \end{cases}$$

which confirms our result. □

Returning to the general case (3.1), let us present the generalizations of some relations from Proposition 2.1:

1. $\nabla^{(J,C)}J = -\nabla J + C(\cdot, J\cdot) - J \circ C$. Then $\nabla \in \mathcal{C}_J(M)$ if and only if $\nabla^{(J,\lambda\nabla J + \mu J \circ \nabla J)} \in \mathcal{C}_J(M)$, with λ and μ arbitrary real numbers;
2. The discussion above;
3. $T_{\nabla^{(J,C)}} = T_\nabla - J\big(d^\nabla J\big) + 2C_{\text{skew}}$, where C_{skew} is the skew-symmetric part of C i.e. $2C_{\text{skew}}(X,Y) = C(X,Y) - C(Y,X)$. So, if C is symmetric and $\nabla \in \mathcal{C}_J(M)$, then ∇ and $\nabla^{(J,C)}$ have the same torsion;

4.

$$R_{\nabla^{(J,C)}}(X,Y)Z = -J\big(R_\nabla(X,Y)JZ\big) - C\big(X, J\big(\nabla_Y JZ\big)\big) + C\big(Y, J\big(\nabla_X JZ\big)\big)$$
$$- C([X,Y],Z) - J(\nabla_X J(C(Y,Z))) + J(\nabla_Y J(C(X,Z)))$$
$$+ C(X, C(Y,Z)) - C(Y, C(X,Z)).$$

3.3. Example. After [10, Theorem 1, p. 1003], the case $C = -\frac{1}{2}J(\nabla J)$ is involved in the tt^*-geometry of (M, J).

4. Exponential complex conjugate connections

Following [1] we consider:

4.1. Definition. The *exponential conjugate connection* of ∇ is:

$$\nabla^{(J,\theta)} := \exp(-\theta J) \circ \nabla \circ \exp(\theta J) \tag{4.1}$$

where:

$$\begin{cases} \exp(\theta J) = \cos\theta \cdot I + \sin\theta \cdot J \\ \exp(-\theta J) = \cos\theta \cdot I - \sin\theta \cdot J \end{cases} \tag{4.2}$$

with θ an arbitrary real number, possible on the 1-dimensional torus $S^1 = \mathbb{R}/\mathbb{Z}$.

4.2. Remarks. i) Our convention in (4.1) is the reverse of the choice of [1] and this fact is motivated by the usual conjugation of $b \in G$ with respect to the element a of a group $(G, \cdot)$ as $a^{-1} \cdot b \cdot a$. Also, in [1] the exponential complex conjugate connection is parametrized by the projective line $P^1 = S^1/\pi$.

ii) The complex conjugate $\nabla^{(J)}$ corresponds to $\theta = \frac{\pi}{2}$.

4.3. Proposition.

i) $\nabla^{(J,\theta)}$ *is in duality with* ∇ *only for* $\theta = \frac{k}{2}\pi$ *with* k *an integer.*
ii) $T_{\nabla^{(J,\theta)}} = T_\nabla + \exp(-\theta J)\big(d^\nabla \exp(\theta J)\big) = T_\nabla - \sin\theta \cdot \exp(-\theta J)\big(d^\nabla J\big)$.

Proof. i) This is a consequence of:

$$\big(\nabla^{(J,\theta)}\big)^{(J,\theta)} = \nabla^{(J,2\theta)}.$$

ii) A straightforward computation. □

We do not add the relationship between the curvature tensor fields of ∇ and $\nabla^{(J,\theta)}$ since it involves very complicated terms.

Acknowledgements.

The first author acknowledges the support given by research grant PN-II-ID-PCE-2011-3-0921.

The authors are thankful to the referees for their valuable comments towards the improvement of the paper.

References

[1] Alekseevsky, D. V., Cortés, V. and Devchand, C. *Special complex manifolds*, J. Geom. Phys. **42** (1-2), 85–105, 2002.

[2] Binh, T. Q. *On semi-symmetric connections*, Period. Math. Hungar. **21** (2), 101–107, 1990.

[3] Blaga, A. M. and Crasmareanu, M. *The geometry of product conjugate connections*, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) **58** (2012), in press.

[4] Calin, O., Matsuzoe, H. and Zhang, J. *Generalizations of conjugate connections*, in Trends in differential geometry, complex analysis and mathematical physics. Proceedings of 9th international workshop on complex structures, integrability and vector fields, Sofia, Bulgaria, August 25-29, 2008. (World Scientific, Hackensack NJ, 2009), 26–34.

[5] Cruceanu, V. *Almost product bicomplex structures on manifolds*, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) **51** (1), 99–118, 2005.

[6] Gauduchon, P. *Hermitian connections and Dirac operators*, Boll. Un. Mat. Ital. B (7) **11** (2) suppl., 257–288, 1997.

[7] Hirică, I. E. and Nicolescu, L. *On quarter-symmetric metric connections on pseudo-Riemannian manifolds*, Balkan J. Geom. Appl. **16** (1), 56–65, 2011.

[8] Ishihara, S. *Quaternion Kählerian manifolds*, J. Diff. Geom. **9**, 483–500, 1974.

[9] Kirichenko, V. F. *Method of generalized Hermitian geometry in the theory of almost contact manifold*, Itogi Nauki i Tekhniki, Problems of geometry **18**, 25–71, 1986; translated in J. Soviet. Math. **42** (5), 1885–1919, 1988.

[10] Schäfer, L. *tt^*-geometry on the tangent bundle of an almost complex manifold*, J. Geom. Phys. **57** (3), 999–1014, 2007.