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3-DIMENSIONAL QUASI-SASAKIAN MANIFOLDS WITH SEMI-SYMMETRIC NON-METRIC CONNECTION

Uday Chand De[*†], Ahmet Yildiz[‡], Mine Turan[‡] and Bilal E. Acet[§]

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Abstract

The object of the present paper is to study semi-symmetric non-metric connections on a 3-dimensional quasi-Sasakian manifold.

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1. Introduction

The notion of quasi-Sasakian structure was introduced by Blair [6] to unify Sasakian and cosymplectic structures. Tanno [28] also added some remarks on quasi-Sasakian structures. The properties of quasi-Sasakian manifolds have been studied by several authors, viz Gonzalez and Chinea [13], Kanemaki [17, 18] and Oubina [21]. Also, Kim [16] studied quasi-Sasakian manifolds and proved that fibred Riemannian spaces with invariant fibres normal to the structure vector field do not admit nearly Sasakian or contact structure but a quasi-Sasakian or cosymplectic structure. Recently, quasi-Sasakian manifolds have been the subject of growing interest due to the discovery of significant applications to physics, in particular to super gravity and magnetic theory. Quasi-Sasakian structures have wide applications in the mathematical analysis of string theory.

Motivated by these studies we propose to study curvature properties of a 3-dimensional quasi-Sasakian manifold with respect to a semi-symmetric non-metric connection. On a 3-dimensional quasi-Sasakian manifold the structure function β was defined by Olszak and with the help of this function he obtained necessary and sufficient conditions for the

[*]Department of Pure Mathematics, University of Calcutta, 35, B. C. Road, Kolkata 700019, West Bengal, India. E-mail: uc_de@yahoo.com

[†]Corresponding Author.

[‡]Art and Science Faculty, Department of Mathematics, Dumlupınar University, Kütahya, Turkey. E-mail: (A. Yıldız) ayildiz44@yahoo.com (M. Turan) minegturan@hotmail.com

[§]Art and Science Faculty, Department of Mathematics, Adıyaman University, Adıyaman, Turkey. E-mail: eacet@adiyaman.edu.tr

manifold to be conformally flat [23]. Then he proved that if the manifold is additionally conformally flat with $\beta =$ constant, then (a) the manifold is locally a product of $\mathbb{R}$ and a two-dimensional Kaehlerian space of constant Gauss curvature (the cosymplectic case), or, (b) the manifold is of constant positive curvature (the non-cosympletic case, here the quasi-Sasakian structure is homothetic to a Sasakian structure). On the other hand Friedmann and Schouten [12, 25] introduced the idea of semi-symmetric linear connection on a differentiable manifold. Hayden [14] introduced a semi-symmetric metric connection on a Riemannian manifold and this was further developed by Yano [29], Imai [15], Nakao [20], Pujar[24], De [10, 11] and many others. The semi-symmetric metric connection in a Sasakian manifold was studied by Yano [31].

Let M be an n-dimensional Riemannian manifold with Riemannian connection ∇. A linear connection $\overset{*}{\nabla}$ on M is said to be a semi-symmetric connection if its torsion tensor $\overset{*}{T}$ satisfies the condition

$$\overset{*}{T}(X,Y) = \eta(Y)X - \eta(X)Y, \tag{1.1}$$

where η is a non zero 1-form.

If moreover $\overset{*}{\nabla}g = 0$ then the connection is called a semi-symmetric metric connection. If $\overset{*}{\nabla}g \neq 0$, then the connection is called a semi-symmetric non-metric connection.

Several authors such as Pravonovic [24], Liang [19], Agashe and Chafle [1], Sengupta, De and Binh [26] and many others introduced semi-symmetric non-metric connections in different ways. In the present paper we study a semi-symmetric non-metric connection in the sense of Agashe and Chafle [1]. In a recent paper Das, De, Singh and Pandey [9] studied Lorentzian manifolds admitting a type of semi-symmetric non-metric connection. They have shown that the semi-symmetric non-metric connection have its application in perfect fluid space-time.

Apart from the conformal curvature tensor, the concircular curvature tensor is another important tensor from a differential geometry point of view. The concircular curvature tensor in a Riemannian manifold of dimension n is defined by [29] as

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{\tau}{n-1}\{g(Y,Z)X - g(X,Z)Y\}.$$

From the definition it follows that the concircular curvature tensor deviates from that of a space of constant curvature. The concircular curvature tensor in a contact metric manifold has been studied by Blair, Kim and Tripathi [7]. The concircular curvature tensor has its applications in fluid spacetimes. In [3] Ahsan and Siddiqui prove that for a perfect fluid spacetime to possesses a divergence free concircular curvature tensor, a necessary and sufficient condition can be obtained in terms of the Friedmann-Robertson-Walker model.

The paper is organized as follows:

After some preliminaries we prove the existence of a semi-symmetric non-metric connection by giving an example. Then we recall the notion of 3-dimensional quasi-Sasakian manifold in Section 4. In the next section we establish the relation between the Riemannian connection and the semi-symmetric non-metric connnection on a 3-dimensional quasi-Sasakian manifold. Section 6 deals with locally ϕ-symmetric 3-dimensional quasi-Sasakian manifold with respect to the semi-symmetric non-metric connection. Finally we study ξ-concircularly flat and ϕ-concircularly flat 3-dimensional quasi-Saskian manifolds. We prove that ξ-concircularly flatness with respect to the semi-symmetric non-metric connnection and the Riemannian connection coincide. Also we prove that a 3-dimensional ϕ-concircularly flat quasi-Sasakian manifold is a cosymplectic manifold.

2. Preliminaries

Let M be an $(2n+1)$-dimensional connected differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g), where ϕ, ξ, η are tensor fields on M of types $(1,1)$, $(1,0)$, $(0,1)$ respectively, such that [4, 5, 30]

$$\phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \ X, Y \in T(M), \tag{2.2}$$

where $T(M)$ is the Lie algebra of vector fields of the manifold M.

Also

$$\phi\xi = 0, \ \eta \circ \phi = 0, \ \eta(X) = g(X, \xi).$$

Let Φ be the fundamental 2-form of M defined by

$$\Phi(X,Y) = g(X, \phi Y) \ X, Y \in T(M).$$

Then $\Phi(X, \xi) = 0$, $X \in T(M)$. M is said to be quasi-Sasakian if the almost contact structure (ϕ, ξ, η) is normal and the fundamental 2-form Φ is closed, that is, for every $X, Y \in T(M)$,

$$[\phi, \phi](X,Y) + d\eta(X,Y)\xi = 0,$$
$$d\Phi = 0, \quad \Phi(X,Y) = g(X, \phi Y).$$

This was first introduced by Blair [6]. There are many types of quasi-Sasakian structure ranging from the cosymplectic case, $d\eta = 0$ (rank$\eta = 1$), to the Sasakian case, $\eta \wedge (d\eta)^n \neq 0$ (rank$\eta = 2n+1$, $\Phi = d\eta$). The 1-form η has rank $r' = 2p$ if $d\eta^p \neq 0$ and $\eta \wedge (d\eta)^p = 0$, and has rank $r' = 2p+1$ if $d\eta^p = 0$ and $\eta \wedge (d\eta)^p \neq 0$. We also say that r' is the rank of the quasi-Sasakian structure.

Blair [6] also proved that there are no quasi-Sasakian manifold of even rank. In order to study the properties of quasi-Sasakian manifolds Blair [6] proved some theorems regarding Kaehlerian manifolds and the existence of quasi-Sasakian manifolds.

Let $\overset{*}{\nabla}$ be a linear connection and ∇ a Riemannian connection of a 3-dimensional quasi-Sasakian manifold M such that

$$\overset{*}{\nabla}_X Y = \nabla_X Y + u(X,Y),$$

where u is a tensor of type $(1,2)$. For $\overset{*}{\nabla}$ to be a semi-symmetric non-metric connection in M we have [1]

$$u(X,Y) = \frac{1}{2}\{\overset{*}{T}(X,Y) + \overset{*}{\grave{T}}(X,Y) + \overset{*}{\grave{T}}(Y,X)\} + g(X,Y)\xi, \tag{2.3}$$

where $g(X,\xi) = \eta(X)$ and $\overset{*}{\grave{T}}$ is a tensor type of $(1,2)$ defined on M:

$$g(\overset{*}{T}(Z,X),Y) = g(\overset{*}{\grave{T}}(X,Y),Z). \tag{2.4}$$

From (1.1) and (2.4) we get

$$\overset{*}{\grave{T}}(X,Y) = \eta(X)Y - g(X,Y)\xi. \tag{2.5}$$

Using (1.1) and (2.5) in (2.3) we get

$$u(X,Y) = \eta(Y)X.$$

Hence a semi-symmetric non-metric connection on a 3-dimensional quasi-Sasakian manifold is given by

$$\overset{*}{\nabla}_X Y = \nabla_X Y + \eta(Y)X.$$

Conversely we show that a linear connection $\overset{*}{\nabla}$ on a 3-dimensional quasi-Sasakian manifold defined by

$$\overset{*}{\nabla}_X Y = \nabla_X Y + \eta(Y)X, \tag{2.6}$$

denotes semi-symmetric non-metric connection.

Using (2.6) the torsion tensor of the connection $\overset{*}{\nabla}$ is given by

$$\overset{*}{T}(X,Y) = \eta(Y)X - \eta(X)Y. \tag{2.7}$$

The above equation shows that the connection $\overset{*}{\nabla}$ is a semi-symmetric connection. Also we have

$$(\overset{*}{\nabla}_X g)(Y,Z) = -\eta(Y)g(X,Z) - \eta(Z)g(Y,X). \tag{2.8}$$

In particular from (2.7) and (2.8) we conclude that $\overset{*}{\nabla}$ is a semi-symmetric non-metric connection. Therefore equation (2.6) is the relation between the Riemannian connection ∇ and semi-symmetric non-metric connection $\overset{*}{\nabla}$ on a 3-dimensional quasi-Sasakian manifold.

3. Example of a semi-symmetric non-metric connection on a Riemannian manifold

If in a local coordinate system the Riemannian-Christoffel symbols Γ^h_{ij} and $\left\{ {h \atop i \quad j} \right\}$ correspond to the semi-symmetric connection and the Levi-Civita connection respectively, then we can express (1.1) as follows:

$$\Gamma^h_{ij} = \left\{ {h \atop i \quad j} \right\} + \eta_j \delta^h_i - g_{ij}\eta^h. \tag{3.1}$$

Let us consider a Riemannian metric g on $\mathbb{R}^4$ given by

$$ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + (x^1)^2(dx^2)^2 + (dx^3)^2 + (dx^4)^2, \tag{3.2}$$

$(i, j = 1, 2, 3, 4)$. Then the only non-vanishing components of the Christoffel symbols with respect to the Levi-Civita connection are

$$\left\{ {1 \atop 2 \quad 2} \right\} = -x^1, \qquad \left\{ {2 \atop 1 \quad 2} \right\} = \left\{ {2 \atop 2 \quad 1} \right\} = \frac{1}{x^1}. \tag{3.3}$$

Now let us define η^i by $\eta^i = (0, -\frac{1}{(x^1)^2}, 0, 0)$. If Γ^h_{ij} corresponds to the semisymmetric connection, then from (3.1) we have the non-zero components of Γ^h_{ij}

$$\Gamma^1_{22} = \left\{ {1 \atop 2 \quad 2} \right\} + \eta^2\delta^1_2 - g^1_{22}\eta^1 = -x^1. \tag{3.4}$$

Similarly we obtain

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{x^1}, \qquad \Gamma^3_{32} = \Gamma^4_{42} = \Gamma^1_{12} = -1, \tag{3.5}$$

$$\Gamma^2_{11} = \Gamma^2_{44} = \Gamma^2_{33} = \frac{1}{(x^1)^2}. \tag{3.6}$$

Now we have

$$g_{22,1} = \frac{\partial g_{22}}{\partial x^1} - g_{2h}\Gamma^h_{21} - g_{2h}\Gamma^h_{21}$$
$$= 2x^1 - 2g_{22}\Gamma^2_{21} = 2x^1 - 2(x^1)^2 \times \frac{1}{x^1} = 0,$$

where ',' denotes the covariant derivative with respect to the semi-symmetric connection Γ. Similarly, the covariant derivative of all components of the metric tensor g, except $g_{11,2}, g_{33,2}, g_{44,2}, g_{12,1}, g_{32,3}, g_{42,4}$, with respect to the semi-symmetric connection Γ are zero, because

$$g_{11,2} = \frac{\partial g_{11}}{\partial x^2} - g_{1h}\Gamma^h_{12} - g_{1h}\Gamma^h_{12}$$
$$= -2g_{11}\Gamma^1_{12} = -2 \times 1 \times (-1)$$
$$= 2 \neq 0,$$

and

$$g_{11,2} = \frac{\partial g_{12}}{\partial x^1} - g_{1h}\Gamma^h_{21} - g_{2h}\Gamma^h_{11}$$
$$= -g_{22}\Gamma^2_{11} = -(x^1)^2 \times \frac{1}{(x^1)^2}$$
$$= -1 \neq 0.$$

By a similar calculation we can show that $g_{33,2} = g_{44,2} = 2 \neq 0$ and $g_{32,3} = g_{42,4} = -1 \neq 0$. Thus Γ is not a metric connection. So Γ is a semi-symmetric non-metric connection.

4. Quasi-Sasakian structure of dimension 3

Now we consider a 3-dimensional quasi-Sasakian manifold. An almost contact metric manifold M is a 3-dimensional quasi-Sasakian manifold if and only if [22]

$$\nabla_X \xi = -\beta\phi X, \ X \in TM, \tag{4.1}$$

for a certain function β on M such that $\xi\beta = 0$, ∇ being the operator of covariant differentiation with respect to the Riemannian connection of M. Clearly, such a quasi-Sasakian manifold is cosymplectic if and only if $\beta = 0$. Now we are going to show that the assumption $\xi\beta = 0$ is not necessary.

Since in a 3-dimensional quasi-Sasakian manifold (4.1) holds, we have [22]

$$(\nabla_X \phi)Y = \beta(g(X,Y)\xi - \eta(Y)X), \quad X, Y \in TM. \tag{4.2}$$

Because of (4.1) and (4.2), we find

$$\nabla_X(\nabla_Y \xi) = -(X\beta)\phi Y - \beta^2\{g(X,Y)\xi - \eta(Y)X\} - \beta\phi\nabla_X Y,$$

which implies that

$$R(X,Y)\xi = -(X\beta)\phi Y + (Y\beta)\phi X + \beta^2\{\eta(Y)X - \eta(X)Y\}. \tag{4.3}$$

Thus we get from (4.3)

$$R(X,Y,Z,\xi) = (X\beta)g(\phi Y, Z) - (Y\beta)g(\phi X, Z) - \beta^2\{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\}, \tag{4.4}$$

where $R(X,Y,Z,W) = g(R(X,Y)Z,W)$. Putting $X = \xi$ in (4.4), we obtain

$$R(\xi,Y,Z,\xi) = (\xi\beta)g(\phi Y, Z) + \beta^2\{g(Y,Z) - \eta(Y)\eta(Z)\}. \tag{4.5}$$

Interchanging Y and Z of (4.5) yields

$$R(\xi,Z,Y,\xi) = (\xi\beta)g(Y, \phi Z) + \beta^2\{g(Y,Z) - \eta(Y)\eta(Z)\}. \tag{4.6}$$

Since $R(\xi, Y, Z, \xi) = R(Z, \xi, \xi, Y) = R(\xi, Z, Y, \xi)$, from (4.5) and (4.6) we have

$$\{g(\phi Y, Z) - g(Y, \phi Z)\}\xi\beta = 0.$$

Therefore we can easily verify that $\xi\beta = 0$.

In a 3-dimensional Riemannian manifold, we always have [30]

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y - \frac{\tau}{2}\{g(Y,Z)X - g(X,Z)Y\}, \tag{4.7}$$

where Q is the Ricci operator, that is, $S(X,Y) = g(QX,Y)$ and τ is the scalar curvature of the manifold.

Let M be a 3-dimensional quasi-Sasakian manifold. The Ricci tensor S of M is given by [23]

$$S(Y,Z) = (\frac{\tau}{2} - \beta^2)g(Y,Z) + (3\beta^2 - \frac{\tau}{2})\eta(Y)\eta(Z) - \eta(Y)d\beta(\phi Z) - \eta(Z)d\beta(\phi Y), \tag{4.8}$$

where τ is the scalar curvature of M.

As a consequence of (4.8), we get for the Ricci operator Q

$$QX = (\frac{\tau}{2} - \beta^2)X + (3\beta^2 - \frac{\tau}{2})\eta(X)\xi + \eta(X)(\phi \mathrm{grad}\beta) - d\beta(\phi X)\xi, \tag{4.9}$$

where the gradient of a function f is related to the exterior derivative df by the formula $df(X) = g(\mathrm{grad} f, X)$. From (4.8), we have

$$S(X,\xi) = 2\beta^2\eta(X) - d\beta(\phi X). \tag{4.10}$$

As a consequence of (4.1), we also have

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y) = -\beta g(\phi X, Y). \tag{4.11}$$

Also from (4.8), it follows that

$$S(\phi X, \phi Z) = S(X,Z) - 2\beta^2\eta(X)\eta(Z). \tag{4.12}$$

5. Curvature tensor of M with respect to a semi-symmetric non-metric connection

The curvature tensor $\overset{*}{R}$ of M with respect to the semi-symmetric non-metric connection $\overset{*}{\nabla}$ is defined by

$$\overset{*}{R}(X,Y)Z = \overset{*}{\nabla}_X \overset{*}{\nabla}_Y Z - \overset{*}{\nabla}_Y \overset{*}{\nabla}_X Z - \overset{*}{\nabla}_{[X,Y]} Z.$$

From (2.6) and (2.1), we have

$$\overset{*}{R}(X,Y)Z = R(X,Y)Z + (\nabla_X \eta)(Z)Y - (\nabla_Y \eta)(Z)Y.$$

And using (4.11) we get

$$\overset{*}{R}(X,Y)Z = R(X,Y)Z - \beta g(\phi X, Z)Y + \beta g(\phi Y, Z)X. \tag{5.1}$$

A relation between the curvature tensor of M with respect to the semi-symmetric non-metric connection and the Riemannian connection is given by the relation (5.1).

Taking the inner product of (5.1) with W we have

$$\overset{*}{R}(X,Y,Z,W) = R(X,Y,Z,W) - \beta\{g(\phi X, Z)g(Y,W) - g(\phi Y, Z)g(X,W)\}, \tag{5.2}$$

where $\overset{*}{R}(X,Y,Z,W) = g(\overset{*}{R}(X,Y)Z, W)$.

From (5.2), clearly

$$\overset{*}{R}(X,Y,Z,W) = -\overset{*}{R}(Y,X,Z,W), \tag{5.3}$$

$$\overset{*}{R}(X,Y,Z,W) = -\overset{*}{R}(X,Y,W,Z). \tag{5.4}$$

Combining above two relations we have

$$\overset{*}{R}(X,Y,Z,W) = \overset{*}{R}(Y,X,W,Z). \tag{5.5}$$

We also have

$$\begin{aligned}\overset{*}{R}(X,Y)Z + \overset{*}{R}(Y,Z)X + \overset{*}{R}(Z,X)Y \\ = 2\beta\{g(\phi X,Z)Y + g(\phi Y,Z)X + g(\phi Z,Y)X\}.\end{aligned} \tag{5.6}$$

This is the first Bianchi identity for $\overset{*}{\nabla}$.

From (5.6) it is obvious that $\overset{*}{R}(X,Y)Z + \overset{*}{R}(Y,Z)X + \overset{*}{R}(Z,X)Y = 0$ if $\beta = 0$.

Hence we can state that if the manifold is cosympletic then the curvature tensor with respect to the semi-symmetric non-metric satisfies the first Bianchi identity.

Contracting (5.2) over X and W, we obtain

$$\overset{*}{S}(Y,Z) = S(Y,Z) + 2\beta g(\phi Y,Z), \tag{5.7}$$

where $\overset{*}{S}$ and S denote the Ricci tensor of the connections $\overset{*}{\nabla}$ and ∇ respectively.

From (5.7), we obtain a relation between the scalar curvature of M with respect to the Riemannian connection and the semi-symmetric non-metric connection which is given by

$$\overset{*}{\tau} = \tau. \tag{5.8}$$

So we have the following:

5.1. Proposition. *For a 3-dimensional quasi-Sasakian manifold M with the semi-symmetric non-metric connection $\overset{*}{\nabla}$,*

(1) *The curvature tensor $\overset{*}{R}$ is given by (5.1),*
(2) *The Ricci tensor $\overset{*}{S}$ is given by (5.7),*
(3) $\overset{*}{\tau} = \tau$. □

6. Locally ϕ-symmetric 3-dimensional quasi-Sasakian manifolds with respect to the semi-symmetric non-metric connection

6.1. Definition. A quasi-Sasakian manifold is said to be *locally ϕ-symmetric* if

$$\phi^2(\nabla_W R)(X,Y)Z = 0, \tag{6.1}$$

for all vector fields W, X, Y, Z orthogonal to ξ. This notion was introduced for Sasakian manifolds by Takahashi [27].

Analogous to the definition of ϕ-symmetric 3-dimensional quasi-Sasakian manifold with respect to the Riemannian connection, we define locally ϕ-symmetric 3-dimensional quasi-Sasakian manifold with respect to a semi-symmetric non-metric connection by

$$\phi^2(\overset{*}{\nabla}_W \overset{*}{R})(X,Y)Z = 0, \tag{6.2}$$

for all vector fields W, X, Y, Z orthogonal to ξ. Using (2.6) we can write

$$(\overset{*}{\nabla}_W \overset{*}{R})(X,Y)Z = (\nabla_W \overset{*}{R})(X,Y)Z + \eta(\overset{*}{R}(X,Y)Z)W. \tag{6.3}$$

Now differentiating (5.1) with respect to W we obtain

$$(\nabla_W \overset{*}{R})(X,Y)Z = (\nabla_W \overset{*}{R})(X,Y)Z + (W\beta)\{g(\phi Y,Z)X - g(\phi X,Z)Y\}. \tag{6.4}$$

From (5.1) and (4.4) it follows that

$$\begin{aligned}\eta(\overset{*}{R}(X,Y)Z)W = (X\beta)g(\phi Y,Z)W - (Y\beta)g(\phi X,Z)W \\ - \beta^2\{\eta(Y)g(X,Z)W - \eta(X)g(Y,Z)W\} \\ + \beta\{g(\phi Y,Z)\eta(X)W - g(\phi X,Z)\eta(Y)W\}.\end{aligned} \tag{6.5}$$

Now using (6.4) and (6.5) in (6.3) we obtain

$$\begin{aligned}(\overset{*}{\nabla}_W \overset{*}{R})(X,Y)Z = (\nabla_W R)(X,Y)Z + (W\beta)\{g(\phi Y,Z)X - g(\phi X,Z)Y\} \\ + (X\beta)g(\phi Y,Z)W - (Y\beta)g(\phi X,Z)W \\ - \beta^2\{\eta(Y)g(X,Z)W - \eta(X)g(Y,Z)W\} \\ + \beta\{g(\phi Y,Z)\eta(X) - g(\phi X,Z)\eta(Y)\},\end{aligned} \tag{6.6}$$

which implies, in view of (2.1),

$$\begin{aligned}\phi^2(\nabla_W \overset{*}{R})(X,Y)Z = \phi^2(\nabla_W R)(X,Y)Z + (W\beta)\{g(\phi X,Z)Y - g(\phi Y,Z)X\} \\ - (X\beta)g(\phi Y,Z)W + (Y\beta)g(\phi X,Z)W \\ - \beta^2\{\eta(Y)g(X,Z)\phi^2 W - \eta(X)g(Y,Z)\phi^2 W\},\end{aligned} \tag{6.7}$$

where X,Y,Z,W are orthogonal to ξ.

If $(\overset{*}{\nabla}_W \overset{*}{R})(X,Y)Z = (\nabla_W R)(X,Y)Z$, then

$$(W\beta)\{g(\phi X,Z)Y - g(\phi Y,Z)X\} - (X\beta)g(\phi Y,Z)W + (Y\beta)g(\phi X,Z)W = 0. \tag{6.8}$$

On contracting X we have

$$(Y\beta)g(\phi W,Z) = 0. \tag{6.9}$$

If we take $Y = Z = e_i$ in (6.9), $\{i = 1,2,3\}$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, we get

$$(e_i\beta)g(\phi W,e_i) = 0,$$

or,

$$g(\mathrm{grad}\beta,e_i)g(\phi W,e_i) = 0,$$

which implies

$$g(\mathrm{grad}\beta,\phi W) = 0,$$

that is,

$$(\phi W)\beta = 0. \tag{6.10}$$

Putting $W = \phi W$ and using (2.1) in (6.10) we have

$$W\beta = 0. \tag{6.11}$$

for all W. Hence $\beta =$ constant.

Conversely, if $\beta =$ constant then from (6.7) it follows that

$$\phi^2(\nabla_W \overset{*}{R})(X,Y)Z = \phi^2(\nabla_W R)(X,Y)Z.$$

Now we can state the following:

6.2. Theorem. *For a 3-dimensional non-cosympletic quasi-Sasakian manifold, local ϕ-symmetry for the Riemannian connection ∇ and the semi-symmetric non-metric connection $\overset{*}{\nabla}$ coincide if and only if the structure function $\beta =$ constant.* □

7. Concircular curvature tensor on a 3-dimensional quasi-Sasakian manifold with respect to a semi-symmetric non-metric connection

Analogous to the definition of concircular curvature tensor in a Riemannian manifold we define concircular curvature tensor with respect to the semi-symmetric non-metric connection $\overset{*}{\nabla}$ as

$$\overset{*}{C}(X,Y)Z = \overset{*}{R}(X,Y)Z - \frac{\overset{*}{\tau}}{6}\{g(Y,Z)X - g(X,Z)Y\}. \tag{7.1}$$

Using (5.1) and (5.7) in (7.1) we get

$$\overset{*}{C}(X,Y)Z = \tilde{C}(X,Y)Z + \beta\{g(\phi Y,Z)X - g(\phi X,Z)Y\}. \tag{7.2}$$

From (7.2) the following follows easily:

7.1. Proposition. $\overset{*}{C}(X,Y)Z = \tilde{C}(X,Y)Z$ *if and only if the manifold is cosymplectic.* □

The notion of an ξ-conformally flat contact manifold was introduced by Zhen, Cabrerizo and Fernandez [32]. In an analogous way we define an ξ-concircularly flat 3-dimensional quasi-Sasakian manifold.

7.2. Definition. A 3-dimensional quasi-Sasakian manifold M is called *ξ-concircularly flat* if the condition $\tilde{C}(X,Y)\xi = 0$ holds on M.

From (7.2) it is clear that $\overset{*}{C}(X,Y)\xi = \tilde{C}(X,Y)\xi$. So we have following:

7.3. Theorem. *In a 3-dimensional quasi-Sasakian manifold, ξ-concircularly flatness with respect to the semi-symmetric non-metric connection and the Riemannian connection coincide.* □

Analogous to the definition of ϕ-conformally flat contact metric manifold [8], we define a ϕ-concircularly flat 3-dimensional quasi-Sasakian manifold.

7.4. Definition. A 3-dimensional quasi-Sasakian manifold satisfying the condition

$$\phi^2\tilde{C}(\phi X,\phi Y)\phi Z = 0. \tag{7.3}$$

is called *ϕ-concircularly flat*.

Let us suppose that M is a 3-dimensional ϕ-concircularly flat quasi-Sasakian manifold with respect to the semi-symmetric non-metric connection. It can be easily seen that $\phi^2\overset{*}{C}(\phi X,\phi Y)\phi Z = 0$ holds if and only if

$$g(\overset{*}{C}(\phi X,\phi Y)\phi Z,\phi W) = 0, \tag{7.4}$$

for all $X,Y,Z,W \in T(M)$.

Using (7.1), ϕ-concircularly flat means

$$g(\overset{*}{R}(\phi X,\phi Y)\phi Z,\phi W) = \frac{\overset{*}{\tau}}{6}\{g(\phi Y,\phi Z)g(\phi X,\phi W) - g(\phi X,\phi Z)g(\phi Y,\phi W)\}. \tag{7.5}$$

Let $\{e_1, e_2, \xi\}$ be a local orthonormal basis of the vector fields in M. Using the fact that $\{\phi e_1, \phi e_2, \xi\}$ is also a local orthonormal basis, putting $X = W = e_i$ in (7.5) and summing up with respect to i, we have

$$\sum_{i=1}^{2} g(\overset{*}{R}(\phi e_i, \phi Y)\phi Z, \phi e_i) = \frac{\overset{*}{\tau}}{6}\sum_{i=1}^{2}\{g(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)g(\phi Y, \phi e_i)\}, \tag{7.6}$$

which implies

$$S(\phi Y, \phi Z) - \beta g(Y, \phi Z) = \frac{\overset{*}{\tau}}{6} g(\phi Y, \phi Z). \tag{7.7}$$

Putting $Y = \phi Y$ and $Z = \phi Z$ in (7.7) and using (2.1), (4.10) with $\beta =$ constant, we get

$$S(Y, Z) = 3\beta g(\phi Y, Z) + \frac{\overset{*}{\tau}}{6} g(Y, Z) + (2\beta^2 - \frac{\overset{*}{\tau}}{6})\eta(Y)\eta(Z), \tag{7.8}$$

Now interchanging Y and Z and then subtracting yields

$$S(Y, Z) - S(Z, Y) = 3\beta g(\phi Y, Z) - 3\beta g(\phi Z, Y). \tag{7.9}$$

Since the Ricci tensor S is symmetric and $g(\phi Y, Z) = -g(\phi Z, Y)$, we get

$$6\beta g(\phi Y, Z) = 0, \tag{7.10}$$

which implies that

$$\beta = 0.$$

Thus we have:

7.5. Theorem. *If a 3-dimensional quasi-Sasakian manifold with constant structure function β is ϕ-concircularly flat with respect to the semi-symmetric non-metric connection, then the manifold is a cosymplectic manifold.* □

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