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# COMPACTNESS IN $L$-FUZZY TOPOLOGICAL SPACES[‡]

Fu-Gui Shi[*†] and Run-Xiang Li[*]

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**Abstract**

The aim of this paper is to introduce a new notion of $L$-fuzzy compactness in $L$-fuzzy topological spaces, which is a generalization of Lowen's fuzzy compactness in $L$-topological spaces. The union of two $L$-fuzzy compact $L$-sets is $L$-fuzzy compact. The intersection of an $L$-fuzzy compact $L$-set $G$ and an $L$-set $H$ with $\mathcal{T}^*(H) = \top$ is $L$-fuzzy compact. The $L$-fuzzy continuous image of an $L$-fuzzy compact $L$-set is $L$-fuzzy compact. The Tychonoff Theorem for $L$-fuzzy compactness is true.

**Keywords:** $L$-fuzzy topology, $\alpha$-fuzzy compactness, $L$-fuzzy compactness.

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## 1. Introduction and preliminaries

The concept of compactness is one of the most important concepts in general topology and it has been generalized to $L$-topological space by many authors (see [3, 6, 10, 12, 13, 17, 18, 20, 23]).

In 1997, H. Aygün, M. W. Warner and S. R. T. Kudri first introduced the concept of smooth compactness in $L$-fuzzy topological spaces [2], which is a generalization of strong compactness in [11, 13, 21]. Subsequently Aygün, Ramadan and S. E. Abbas respectively introduced some weaker and stronger forms of $L$-fuzzy compactness in [1, 14].

The aim of this paper is to introduce a new notion of $L$-fuzzy compactness in $L$-fuzzy topological spaces, which is a generalization of Lowen's compactness in $L$-topological spaces.

Throughout this paper $(L, \bigvee, \bigwedge, ')$ is a completely distributive DeMorgan algebra, $X$ a nonempty set and $L^X$ the set of all $L$-fuzzy sets on $X$. The smallest element and the largest element in $L$ are denoted respectively by $\perp$ and $\top$. The smallest element and the largest element in $L^X$ are denoted respectively by $\underline{\perp}$ and $\underline{\top}$. An $L$-fuzzy set is briefly

[‡]The project is supported by the National Natural Science Foundation of China (10971242)
[*]School of Mathematics, Beijing Institute of Technology, Beijing, 100081, P. R. China.
E-mail: (F.-G. Shi) fuguishi@bit.edu.cn f.g.shi@263.net (R.-X. Li) lirunxiang84@sina.com
[†]Corresponding Author.

written as an $L$-set. We often do not distinguish a crisp subset $A$ from its characteristic function $\chi_A$.

The set of nonunit prime elements in $L$ is denoted by $P(L)$. The set of nonzero co-prime elements in $L$ is denoted by $M(L)$. The set of nonzero co-prime elements in $L^X$ is denoted by $M(L^X)$. The set of all $L$-fuzzy points $x_\lambda$ (i.e., an $L$-fuzzy set $A \in L^X$ such that $A(x) = \lambda \neq 0$ and $A(y) = 0$ for $y \neq x$) is denoted by $\mathrm{pt}(L^X)$.

The binary relation $\prec$ in $L$ is defined as follows: for $a, b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$ [4]. In a completely distributive DeMorgan algebra $L$, each member $b$ is the sup of $\{a \in L \mid a \prec b\}$. In the sense of [11, 21], $\{a \in L \mid a \prec b\}$ is the greatest minimal family of $b$, denoted by $\beta(b)$, and $\beta^*(b) = \beta(b) \cap M(L)$. Moreover for $b \in L$, define $\alpha(b) = \{a \in L \mid a' \prec b'\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$.

For $a \in L$ and $A \in L^X$, we define $A_{[a]} = \{x \in X \mid A(x) \geq a\}$.

**1.1. Definition.** [8, 9, 15, 19] An $L$-fuzzy topology on a set $X$ is a map $\mathcal{T} : L^X \to L$ such that

(1) $\mathcal{T}(\underline{\top}) = \mathcal{T}(\underline{\bot}) = \top$;
(2) $\forall\, U, V \in L^X$, $\mathcal{T}(U \wedge V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V)$;
(3) $\forall\, U_j \in L^X$, $j \in J$, $\mathcal{T}(\bigvee_{j\in J} U_j) \geq \bigwedge_{j \in J} \mathcal{T}(U_j)$.

$\mathcal{T}(U)$ can be interpreted as the *degree to which $U$ is an open set*.

$\mathcal{T}^*(U) = \mathcal{T}(U')$ will be called the *degree of closedness of $U$*. The pair $(X, \mathcal{T})$ is called an $L$-fuzzy topological space.

A mapping $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is said to be *$L$-fuzzy continuous* if $\mathcal{T}(f_L^{\leftarrow}(B)) \geq \mathcal{U}(B)$ holds for all $B \in L^Y$, where $f_L^{\leftarrow}$ is defined by $f_L^{\leftarrow}(B)(x) = B(f(x))$ [15].

**1.2. Theorem.** [22] *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be $L$-fuzzy topological spaces. Then $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is $L$-fuzzy continuous if and only if $\forall\, a \in M(L)$, $f : (X, \mathcal{T}_{[a]})) \to (Y, \mathcal{U}_{[a]})$ is $L$-continuous, where $\mathcal{T}_{[a]} = \{A \in L^X \mid \mathcal{T}(A) \geq a\}$.* □

**1.3. Definition.** [5]

(1) Let $\mathcal{T}$ be an $L$-fuzzy topology on $X$ and $\mathcal{B} : L^X \to L$ a function with $\mathcal{B} \leq \mathcal{T}$. Then $\mathcal{B}$ is called a *base* of $\mathcal{T}$ if $\mathcal{B}$ satisfies the following condition:

$$\forall\, A \in L^X,\ \forall\, x_\lambda \in \mathrm{pt}(L^X),\ Q_{x_\lambda}(A) \leq \bigvee_{x_\lambda q B \leq A} \mathcal{B}(B),$$

where $Q_{x_\lambda}(A) = \bigvee\limits_{x_\lambda q B \leq A} \mathcal{T}(B)$

(2) Let $\phi : L^X \to L$ be a function. Then $\phi$ is called a *subbase* of $\mathcal{T}$ if and only if $\phi^{(\sqcap)} : L^X \to L$ is a base, where

$$\phi^{(\sqcap)}(A) = \bigvee_{\sqcap_{\lambda\in\Lambda} B_\lambda = A}\ \bigwedge_{\lambda\in\Lambda} \phi(B_\lambda),$$

and $(\sqcap)$ stands for "finite intersection".

**1.4. Definition.** [5] Let $\{(X_j, \mathcal{T}_j)\}_{j\in J}$ be a collection of $L$-fuzzy topological spaces, and $P_i : \prod\limits_{j\in J} X_j \to X_i$ the projection. The $L$-fuzzy topology whose subbase is defined by

$$\forall\, A \in L^{\prod_{j\in J} X_j}, \phi(A) = \bigvee_{j\in J}\ \bigvee_{P_j^{\leftarrow}(U) = A} \mathcal{T}_j(U),$$

is called the *product $L$-fuzzy topology* of $\{\mathcal{T}_j : j \in J\}$, denoted by $\prod\limits_{j\in J} \mathcal{T}_j$, and $(\prod\limits_{j\in J} X_j, \prod\limits_{j\in J} \mathcal{T}_j)$ is called the *product space* of $\{(X_j, \mathcal{T}_j)\}_{j\in J}$.

**1.5. Theorem.** [5] *A map* $\mathcal{B} : L^X \to L$ *is a base of* $\mathcal{T}$ *if and only if*

$$\mathcal{T}(A) = \bigvee_{\vee_{\lambda\in\Lambda} B_\lambda = A} \bigwedge_{\lambda\in\Lambda} \mathcal{B}(B_\lambda). \qquad \square$$

In [2] the notion of smooth compactness, which is a generalization of strong compactness in [11, 13, 21], was introduced in $L$-fuzzy topological spaces. The following definition gives an equivalent formulation, which will be called $L$-fuzzy strong compactness in the sequel.

**1.6. Definition.** [2] Let $(X, \mathcal{T})$ be an $L$-fuzzy topological space and $G \in L^X$. The $L$-fuzzy subset $G$ is said to be *$L$-fuzzy strongly compact* if and only if $\forall\, p \in P(L)$ and $\forall\, \{F_i\}_{i\in I} \subseteq L^X$ satisfying $\mathcal{T}(F_i) \not\leq p$ $(\forall\, i \in I)$, and $G'(x) \wedge \Big(\bigvee_{i\in I} F_i\Big)(x) \not\leq p$ $(\forall\, x \in X)$, there is a finite subset $I_0$ of $I$ such that $G'(x) \wedge \Big(\bigvee_{i\in I_0} F_i\Big)(x) \not\leq p$ $(\forall\, x \in X)$.

**1.7. Definition.** [18] Let $a \in L\backslash\{\top\}$ and $G \in L^X$. A subfamily $\mathcal{U}$ in $L^X$ is said to be

(1) An *$a$-shading of* $G$ if for any $x \in X$, it follows that $G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x) \not\leq a$.

(2) A *strong $a$-shading of* $G$ if $\bigwedge_{x\in X} \Big(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\Big) \not\leq a$.

**1.8. Definition.** [18] Let $a \in L\backslash\{\bot\}$ and $G \in L^X$. A subfamily $\mathcal{P}$ in $L^X$ is called

(1) An *$a$-remote family of* $G$ if for any $x \in X$, it follows that $G(X) \wedge \bigwedge_{B\in\mathcal{P}} B(x) \not\geq a$.

(2) A *strong $a$-remote family of* $G$ if $\bigvee_{x\in X} \Big(G(x) \wedge \bigwedge_{B\in\mathcal{P}} B(x)\Big) \not\geq a$.

**1.9. Definition.** [18] Let $a \in L\backslash\{\bot\}$ and $G \in L^X$. A subfamily $\mathcal{U}$ in $L^X$ is called

(1) A *$\beta_a$-cover of* $G$ if for any $x \in X$, it follows that $a \in \beta\Big(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\Big)$.

(2) A *strong $\beta_a$-cover of* $G$ if for any $x \in X$, it follows that

$$a \in \beta\Big(\bigwedge_{x\in X} \Big(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\Big)\Big).$$

**1.10. Definition.** [18] Let $a \in L\backslash\{\bot\}$ and $G \in L^X$. A subfamily $\mathcal{U}$ in $L^X$ is called a *$Q_a$-cover of* $G$ if for any $x \in X$ with $G(x) \not\leq a'$, it follows that $\bigvee_{A\in\mathcal{U}} A(x) \geq a$.

It is obvious that $\mathcal{U}$ is a $Q_a$-cover of $G$ if and only if $a \leq \bigwedge_{x\in X} \Big(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\Big)$.

## 2. Compactness in $L$-fuzzy topological spaces

In order to generalize the notion of compactness to $L$-fuzzy topological spaces, first let us research compactness in general topology.

Let $(X, \mathcal{T})$ be a topological space and $G \subseteq X$. Then $G$ is said to be compact if each open cover $\mathcal{U}$ of $G$ has a finite subfamily $\mathcal{V}$ which is an open cover of $G$. By the following fact:

$$\bigwedge_{x\in X}\Big(G'(x)\vee\bigvee_{A\in\mathcal{U}}A(x)\Big)=1 \iff \forall\, x\in X,\ G'(x)\vee\bigvee_{A\in\mathcal{U}}A(x)=1$$
$$\iff \forall\, x\in X,\ G'(x)\neq 1 \text{ implies } \bigvee_{A\in\mathcal{U}}A(x)=1$$
$$\iff \forall\, x\in X,\ G(x)=1 \text{ implies } \bigvee_{A\in\mathcal{U}}A(x)=1$$
$$\iff \forall\, x\in X,\ G(x)\leq\bigvee_{A\in\mathcal{U}}A(x)$$

we know that $G$ is compact if and only if

$$\bigwedge_{x\in X}\Big(G'(x)\vee\bigvee_{A\in\mathcal{U}}A(x)\Big)=1 \text{ implies } \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\Big(G'(x)\vee\bigvee_{A\in\mathcal{V}}A(x)\Big)=1.$$

Based on the above fact, a new definition of fuzzy compactness was presented in $L$-topological spaces when $L$ is a complete DeMorgan algebra in [18]. In fact, it is a generalization of Lowen's fuzzy compactness [12]. Now we generalize it further to $L$-fuzzy topological spaces.

**2.1. Definition.** Let $(X,\mathcal{T})$ be an $L$-fuzzy topological space. Then $G\in L^X$ is said to be $L$-fuzzy compact if for every family $\mathcal{U}\subseteq L^X$, it follows that

$$\bigwedge_{F\in\mathcal{U}}\mathcal{T}(F)\wedge\left(\bigwedge_{x\in X}\Big(G'(x)\vee\bigvee_{F\in\mathcal{U}}F(x)\Big)\right)\leq\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\Big(G'(x)\vee\bigvee_{F\in\mathcal{V}}F(x)\Big).$$

Let $(X,\mathcal{T})$ be an $L$-topological space. Define $\chi_{\mathcal{T}}:L^X\to L$ by

$$\chi_{\mathcal{T}}(A)=\begin{cases}1, & A\in\mathcal{T},\\ 0, & A\notin\mathcal{T}.\end{cases}$$

Obviously, $(X,\chi_{\mathcal{T}})$ is a special type of $L$-fuzzy topological space. We can easily prove the following theorem.

**2.2. Theorem.** *Let $(X,\mathcal{T})$ be an $L$-topological space and $G\in L^X$. Then $G$ is $L$-fuzzy compact in $(X,\chi_{\mathcal{T}})$ if and only if $G$ is fuzzy compact in $(X,\mathcal{T})$.* □

From Definition 2.1 we easily obtain the following theorem by simply using the quasi-complement $'$.

**2.3. Theorem.** *Let $(X,\mathcal{T})$ be an $L$-fuzzy topological space. Then $G\in L^X$ is $L$-fuzzy compact if and only if for every family $\mathcal{P}\subseteq L^X$ it follows that*

$$\bigvee_{F\in\mathcal{P}}(\mathcal{T}^*(F))'\vee\Big(\bigvee_{x\in X}(G(x)\wedge\bigwedge_{F\in\mathcal{P}}F(x))\Big)\geq\bigwedge_{\mathcal{H}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\Big(G(x)\wedge\bigwedge_{F\in\mathcal{H}}F(x)\Big).\quad\square$$

By Definition 2.1 and Theorem 2.3 we immediately obtain some characterizations of $L$-fuzzy compactness as follows.

**2.4. Theorem.** *Let $(X,\mathcal{T})$ be an $L$-fuzzy topological space and $G\in L^X$. Then the following conditions are equivalent to each other.*

(1) *$G$ is $L$-fuzzy compact.*
(2) *For any $a\in M(L)$, each strong $a$-remote family $\mathcal{P}$ of $G$ with $\bigwedge_{F\in\mathcal{P}}\mathcal{T}^*(F)\not\leq a'$ has a finite subfamily $\mathcal{H}$ which is an (a strong) $a$-remote family of $G$.*

(3) *For any $a \in M(L)$, and any strong $a$-remote family $\mathcal{P}$ of $G$ with $\bigwedge_{F\in\mathcal{P}} \mathcal{T}^*(F) \nleq a'$, there exists a finite subfamily $\mathcal{H}$ of $\mathcal{P}$ and $b \in \beta^*(a)$ such that $\mathcal{H}$ is a (strong) $b$-remote family of $G$.*

(4) *For any $a \in P(L)$, each strong $a$-shading $\mathcal{U}$ of $G$ with $\bigwedge_{F\in\mathcal{U}} \mathcal{T}(F) \nleq a$ has a finite subfamily $\mathcal{V}$ which is an (a strong) $a$-shading of $G$.*

(5) *For any $a \in P(L)$ and any strong $a$-shading $\mathcal{U}$ of $G$ with $\bigwedge_{F\in\mathcal{U}} \mathcal{T}(F) \nleq a$, there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in \alpha^*(a)$ such that $\mathcal{V}$ is a (strong) $b$-shading of $G$.*

(6) *For any $a \in M(L)$ and any $b \in \beta^*(a)$, each $Q_a$-cover $\mathcal{U}$ of $G$ with $\mathcal{T}(F) \geq a$ ($\forall\, F \in \mathcal{U}$) has a finite subfamily $\mathcal{V}$ which is a $Q_b$-cover of $G$.*

(7) *For any $a \in M(L)$ and any $b \in \beta^*(a)$, each $Q_a$-cover $\mathcal{U}$ of $G$ with $\mathcal{T}(F) \geq a$ ($\forall\, F \in \mathcal{U}$) has a finite subfamily $\mathcal{V}$ which is a (strong) $\beta_b$-cover of $G$.* □

**2.5. Theorem.** *Let $(X, \mathcal{T})$ be an $L$-fuzzy topological space and $G \in L^X$. If $\beta(c \wedge d) = \beta(c) \cap \beta(d)$ ($\forall\, c, d \in L$), then the following conditions are equivalent to each other.*

(1) *$G$ is $L$-fuzzy compact.*

(2) *For any $a \in M(L)$, each strong $\beta_a$-cover $\mathcal{U}$ of $G$ with $a \in \beta\Big(\bigwedge_{F\in\mathcal{U}} \mathcal{T}(F)\Big)$ has a finite subfamily $\mathcal{V}$ which is a (strong) $\beta_a$-cover of $G$.*

(3) *For any $a \in M(L)$ and any strong $\beta_a$-cover $\mathcal{U}$ of $G$ with $a \in \beta\Big(\bigwedge_{F\in\mathcal{U}} \mathcal{T}(F)\Big)$, there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in M(L)$ with $a \in \beta^*(b)$ such that $\mathcal{V}$ is a (strong) $\beta_b$-cover of $G$.*

The following theorem gives the relation between $L$-fuzzy strong compactness and $L$-fuzzy compactness. It can be obtained from Theorem 2.4.

**2.6. Theorem.** *$L$-fuzzy strong compactness implies $L$-fuzzy compactess.* □

**2.7. Remark.** In general, $L$-fuzzy compactness need not imply $L$-fuzzy strong compactness. This can be seen for $L$-topologies [11].

## 3. Properties of $L$-fuzzy compactness

In order to research properties of $L$-fuzzy compactness, we first introduce the following definition.

**3.1. Definition.** Let $(X, \mathcal{T})$ be an $L$-topological space, $a \in M(L)$ and $G \in L^X$. Then $G$ is said to be *$a$-fuzzy compact* if and only if $\forall\, b \in \beta(a)$, each $Q_a$-open cover $\mathcal{U}$ of $G$ has a finite subfamily $\mathcal{V}$ which is a $Q_b$-open cover of $G$.

By [16, Theorem 4.3] and Definition 3.1 we can obtain the following result.

**3.2. Theorem.** *Let $(X, \mathcal{T})$ be an $L$-topological space. Then $G \in L^X$ is fuzzy compact if and only if $\forall\, a \in M(L)$, $G$ is $a$-fuzzy compact.* □

**3.3. Theorem.** *Let $(X, \mathcal{T})$ be an $L$-fuzzy topological space and $G \in L^X$. Then $G$ is $L$-fuzzy compact in $(X, \mathcal{T})$ if and only if $\forall\, a \in M(L)$, $G$ is $a$-fuzzy compact in $(X, \mathcal{T}_{[a]})$.*

*Proof.* (Necessity) Since $G$ is $L$-fuzzy compact in $(X, \mathcal{T})$, by Definition 2.1 we know that for every family $\mathcal{U} \subseteq L^X$, it follows that

$$\bigwedge_{F\in\mathcal{U}} \mathcal{T}(F) \wedge \left(\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{F\in\mathcal{U}} F(x)\right)\right) \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{F\in\mathcal{V}} F(x)\right).$$

Hence $\forall a \in M(L)$ and $\forall \mathcal{U} \subseteq \mathcal{T}_{[a]}$, we have that

$$a \leq \bigwedge_{x\in X}\Big(G'(x) \vee \bigvee_{F\in\mathcal{U}} F(x)\Big) \implies a \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\Big(G'(x) \vee \bigvee_{F\in\mathcal{V}} F(x)\Big).$$

Thus $\forall b \in \beta(a)$, there exists $\mathcal{V} \in 2^{(\mathcal{U})}$ such that $b \leq \bigwedge_{x\in X}\Big(G'(x) \vee \bigvee_{F\in\mathcal{V}} F(x)\Big)$, i.e., $\forall a \in M(L)$, $\forall b \in \beta(a)$, each $Q_a$-cover $\mathcal{U}$ of $G$ in $\big(X, \mathcal{T}_{[a]}\big)$ has a finite subfamily $\mathcal{V}$ which is a $Q_b$-cover of $G$. Therefore, $\forall a \in M(L)$, $G$ is $a$-fuzzy compact in $(X, \mathcal{T}_{[a]})$.

(Sufficency) Suppose that $\forall a \in M(L)$, $G$ is $a$-fuzzy compact in $(X, \mathcal{T}_{[a]})$. Let $\mathcal{U} \subseteq L^X$ and $a \leq \bigwedge_{F\in\mathcal{U}} \mathcal{T}(F) \wedge \Big(\bigwedge_{x\in X}\Big(G'(x) \vee \bigvee_{F\in\mathcal{U}} F(x)\Big)\Big)$. Then $\mathcal{U} \subseteq \mathcal{T}_{[a]}$ and $a \leq \bigwedge_{x\in X}\Big(G'(x) \vee \bigvee_{F\in\mathcal{U}} F(x)\Big)$. Thus $\forall b \in \beta(a)$, there exists $\mathcal{V} \in 2^{(\mathcal{U})}$ such that $b \leq \bigwedge_{x\in X}\Big(G'(x) \vee \bigvee_{F\in\mathcal{V}} F(x)\Big)$. Hence $a \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\Big(G'(x) \vee \bigvee_{F\in\mathcal{V}} F(x)\Big)$. Therefore $G$ is $L$-fuzzy compact in $(X, \mathcal{T})$. □

Analogous to Shi's proof in [18], we can obtain the following lemma.

**3.4. Lemma.** *Let $(X, \mathcal{T})$ be an $L$-topological space, $a \in M(L)$ and $G \in L^X$. If $G$ is $a$-fuzzy compact, then $G \wedge H$ is $a$-fuzzy compact for each $H \in \mathcal{T}'$.* □

**3.5. Theorem.** *Let $(X, \mathcal{T})$ be an $L$-fuzzy topological space and $G \in L^X$. If $G$ is $L$-fuzzy compact, then for each $H \in L^X$ with $\mathcal{T}^*(H) = \top$, $G \wedge H$ is $L$-fuzzy compact.*

*Proof.* $\forall a \in M(L)$, since $G$ is $L$-fuzzy compact in $(X, \mathcal{T})$, by Theorem 3.3, $G$ is $a$-fuzzy compact in $(X, \mathcal{T}_{[a]})$. By $\mathcal{T}^*(H) = \top$, we know that $H' \in \mathcal{T}_{[a]}$. Further, by Lemma 3.4, $G \wedge H$ is $a$-fuzzy compact in $(X, \mathcal{T}_{[a]})$. Then by Theorem 3.3, $G \wedge H$ is $L$-fuzzy compact in $(X, \mathcal{T})$. □

Analogous to Shi's proof in [18], we can obtain the following Lemma 3.6.

**3.6. Lemma.** *Let $(X, \mathcal{T})$ be an $L$-topological space, $G, H \in L^X$ and $a \in M(L)$. If $G$ and $H$ are $a$-fuzzy compact, then so is $G \vee H$.* □

**3.7. Theorem.** *Let $(X, \mathcal{T})$ be an $L$-fuzzy topological space and $H, G \in L^X$. If $G$ and $H$ are $L$-fuzzy compact, then so is $G \vee H$.*

*Proof.* Since $G$ and $H$ are $L$-fuzzy compact in $(X, \mathcal{T})$, by Theorem 3.3, $\forall a \in M(L)$, we know that $G$ and $H$ are $a$-fuzzy compact in $(X, \mathcal{T}_{[a]})$. By Lemma 3.6, $G \vee H$ is $a$-fuzzy compact in $(X, \mathcal{T}_{[a]})$. So $G \vee H$ is $L$-fuzzy compact in $(X, \mathcal{T})$. □

Analogous to Shi's proof in [18], we can obtain the following Lemma 3.8.

**3.8. Lemma.** *Let $(X, \mathcal{T})$, $(Y, \mathcal{U})$ be two $L$-topological spaces and $a \in M(L)$. If $G$ is $a$-fuzzy compact in $(X, \mathcal{T})$ and $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is an $L$-continuous mapping, then $f_L^{\to}(G)$ is $a$-fuzzy compact in $(Y, \mathcal{U})$.* □

**3.9. Theorem.** *Let $(X, \mathcal{T})$, $(Y, \mathcal{U})$ be two $L$-fuzzy topological spaces, and $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ an $L$-fuzzy continuous mapping. If $G \in L^X$ is $L$-fuzzy compact in $(X, \mathcal{T})$, then so is $f_L^{\to}(G)$ in $(Y, \mathcal{U})$.*

*Proof.* Since $G$ is $L$-fuzzy compact in $(X, \mathcal{T})$, by Theorem 3.3, $\forall a \in M(L)$, $G$ is $a$-fuzzy compact in $(X, \mathcal{T}_{[a]})$. By Theorem 1.2, $f : (X, \mathcal{T}_{[a]}) \to (Y, \mathcal{U}_{[a]})$ is an $L$-continuous mapping. Hence $f_L^{\to}(G)$ is $a$-fuzzy compact in $(Y, \mathcal{U}_{[a]})$. Therefore $f_L^{\to}(G)$ is $L$-fuzzy compact in $(Y, \mathcal{U})$. □

Analogous to Shi's proof in [18], we can obtain the following Lemma 3.10.

**3.10. Lemma.** *Let $(X,\mathcal{T})$ be the product of a family of $L$-topological spaces $\{(X_i,\mathcal{T}_i)\}_{i\in J}$, and $a\in M(L)$. Then $(X,\mathcal{T})$ is $a$-fuzzy compact if and only if for each $i\in J$, $(X_i,\mathcal{T}_i)$ is $a$-fuzzy compact.* □

From Theorem 2.6 we easily obtain the following Lemma 3.11.

**3.11. Lemma.** *Let $(X,\mathcal{T}_1)$ and $(X,\mathcal{T}_2)$ be two $L$-topological spaces and $a\in M(L)$. If $\mathcal{T}_1\subseteq\mathcal{T}_2$ and $(X,\mathcal{T}_2)$ is $a$-fuzzy compact, then $(X,\mathcal{T}_1)$ is $a$-fuzzy compact as well.* □

**3.12. Theorem.** *Let $(X,\mathcal{T})$ be the product of a family of $L$-fuzzy topological spaces $\{(X_i,\mathcal{T}_i)\}_{i\in J}$. Then $(X,\mathcal{T})$ is $L$-fuzzy compact if and only if for each $i\in J$, $(X_i,\mathcal{T}_i)$ is $L$-fuzzy compact.*

*Proof.* $\forall\, A\in L^{\prod_{i\in J}X_i}$, let

$$\phi(A)=\bigvee_{i\in J}\bigvee_{P_i^{\leftarrow}(U)=A}\mathcal{T}_i(U)\text{ and }\mathcal{B}(B)=\bigvee_{\sqcap\mathcal{A}=B}\bigwedge_{A\in\mathcal{A}}\phi(A),$$

where $(\sqcap)$ stands for "finite intersection". Then $\phi$ is a subbase of $(X,\mathcal{T})$ and $\mathcal{B}:L^X\to L$ is a base of $(X,\mathcal{T})$.

For each $A\in L^X$, by Theorem 1.5 we have $\mathcal{T}(C)=\bigvee_{\bigvee\mathcal{D}=C}\bigwedge_{B\in\mathcal{D}}\mathcal{B}(B)$. Now $\forall\,a\in M(L)$, if $\mathcal{T}(C)\geq a$, then $\forall\,r\in\beta(a)$, there exists $\mathcal{D}$ such that $\bigvee\mathcal{D}=C$ and $\forall\,B\in\mathcal{D}$, $r\prec\mathcal{B}(B)$. By $\mathcal{B}(B)=\bigvee_{\sqcap\mathcal{A}=B}\bigwedge_{A\in\mathcal{A}}\phi(A)$, we know that there exists $\mathcal{A}$ such that $\sqcap\mathcal{A}=B$ and $\forall\,A\in\mathcal{A}$, $r\prec\phi(A)$. Furthermore by $\phi(A)=\bigvee_{i\in J}\bigvee_{P_i^{\leftarrow}(U)=A}\mathcal{T}_i(U)$, there exists $i\in J$ and $U\in L^X$ such that $r\prec\mathcal{T}_i(U)$. By $P_i^{\leftarrow}(U)=A$ and $B=\sqcap\mathcal{A}$, we have that $C=\bigvee\{B:B\in\mathcal{D}\}\in\prod_{i\in J}(\mathcal{T}_i)_{[r]}$. Therefore $\mathcal{T}_{[a]}\subseteq\prod_{i\in J}(\mathcal{T}_i)_{[r]}$.

For each $i\in J$, by the $L$-fuzzy compactness of $(X_i,\mathcal{T}_i)$, Theorem 3.2 and Lemma 3.10 we know that $\forall\,a\in M(L)$, $\forall\,r\in\beta(a)$, $(\prod_i X_i,\prod_i(\mathcal{T}_i)_{[r]})$ is $r$-compact. Hence by Lemma 3.11, $(X,\mathcal{T}_{[a]})$ is $r$-compact. Now, $\forall\,b\in\beta(a)$, take $r\in\beta(a)$ such that $b\in\beta(r)$. Since each $Q_a$-cover $\mathcal{U}$ of $X$ must be a $Q_r$-cover $\mathcal{U}$ of $X$, by the $r$-compactness of $(X,\mathcal{T}_{[a]})$, $\mathcal{U}$ has a finite subfamily $\mathcal{V}$ which is a $Q_b$-cover of $X$. This shows that $\forall\,a\in M(L)$, $(X,\mathcal{T}_{[a]})$ is $a$-fuzzy compact. Thus by Theorem 3.2, $(X,\mathcal{T})$ is $L$-fuzzy compact. □


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