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ON A SUM OF THE PSI FUNCTION WITH A LOGARITHM

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Abstract

Let ψ be the psi function, that is the logarithmic derivative of the Euler gamma function. The aim of this paper is to establish an asymptotic formula for the function $\psi(x)+\log\left(e^{1/x}-1\right)$ and to improve some results of Batir (*Some new inequalities for gamma and polygamma functions*, J. Ineq. Pure Appl. Math. **6** (4), Art 103, 2005) and Alzer (*Sharp inequalities for the harmonic numbers*, Expo. Math. **24**, 385–388, 2006). Finally we give a short proof of, respectively, the monotonicity and concavity of the function $\psi(x)+\log\left(e^{1/x}-1\right)$, previously stated by Alzer above, and by Guo and Qi (*Some properties of the psi and polygamma functions*, Hacet. J. Math. Stat. **39** (2), 219–231, 2010).

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1. Introduction

Let $H_n = \sum_{k=1}^{n} \frac{1}{k}$ be the nth harmonic number. Alzer [3] proved

$$a - \log\left(e^{1/(n+1)} - 1\right) \le H_n < b - \log\left(e^{1/(n+1)} - 1\right), \quad (n \ge 1), \tag{1.1}$$

with the best possible constants $a = 1 + \log(\sqrt{e} - 1)$ and $b = \gamma$, where $\gamma = 0.577215\ldots$ is the Euler-Mascheroni constant. One year earlier, Batir [4, Cor. 2.2] established an inequality of type (1.1) with $a = \log\left(\pi^2/6\right)$ and $b = \gamma$.

The harmonic numbers are related to the psi function ψ, which is the logarithmic derivative of the Euler gamma function:

$$\psi(x) = \frac{d}{dx} \log \Gamma(x).$$

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Indeed, $H_n = \psi(n+1) + \gamma$, for $n = 1, 2, 3, \ldots$. See, e.g., [1, p. 258]. As a direct consequence, the proof of inequality (1.1) is reduced by Alzer and Batir to the study of the variation of the function

$$\phi(x) = \psi(x) + \log\left(e^{1/x} - 1\right). \tag{1.2}$$

More precisely, Batir [4, Theorem 2.1] established some inequalities involving ϕ, while Alzer [3, Theorem 1] proved that ϕ is strictly increasing on $(0, \infty)$. Using the monotonicity of ϕ, it follows for all integers $n \geq 1$: $\phi(2) \leq \phi(n+1) < 0 = \phi(\infty)$, which is (1.1).

The function ϕ has attracted also the attention of other researchers and we refer to Guo and Qi [5] who proved moreover that ϕ is strictly concave in $(0, \infty)$, with $\phi(+0) = -\gamma$ and $\phi(\infty) = 0$.

We concentrate in this paper on establishing accurate estimates of ϕ. Having in mind the proof of inequality (1.1) using monotonicity arguments, we can easily realize that for large values of n, the best approximations of the form

$$H_n \approx k - \log\left(e^{1/(n+1)} - 1\right), \ (k \in \mathbb{R}) \tag{1.3}$$

are obtained for $k = \gamma$.

We improve the approximations (1.3) and the inequalities (1.1) as follows:

1.1. Theorem. *For all integers $n \geq 1$, it holds that*

$$\gamma - \log\left(e^{1/(n+1)} - 1\right) + a(n) < H_n < \gamma - \log\left(e^{1/(n+1)} - 1\right) + b(n), \tag{1.4}$$

where

$$a(n) = -\frac{1}{24n^2} + \frac{1}{12n^3} - \frac{337}{2880n^4}, \ b(n) = a(n) + \frac{97}{720n^5}.$$

In terms of the psi function, (1.4) can be written as

$$-\frac{1}{24x^2} + \frac{1}{12x^3} - \frac{337}{2880x^4} < \psi(x+1) + \log\left(e^{1/(x+1)} - 1\right)$$
$$< -\frac{1}{24x^2} + \frac{1}{12x^3} - \frac{337}{2880x^4} + \frac{97}{720x^5}.$$

Theorem 1.1 entitles us to introduce the asymptotic formula as $n \to \infty$,

$$\gamma \sim H_n + \log\left(e^{1/(n+1)} - 1\right) + \frac{1}{24n^2} - \frac{1}{12n^3} + \frac{337}{2880n^4} - \frac{97}{720n^5}, \tag{1.5}$$

but the construction of the complete asymptotic expansion, eventually in terms of Bernoulli numbers, is proposed here as an open problem.

Truncations of (1.5) provide much better estimates than (1.3). Moreover, numerical computations show the superiority of (1.5) over the class of approximations

$$\gamma \approx H_n + \log\left(e^{\alpha/(n+\beta)} - 1\right) - \log\alpha, \ (\alpha, \beta \in \mathbb{R}, \ \alpha > 0), \tag{1.6}$$

recently introduced in [6] The ($\alpha = \beta = 1$ case is (1.1), while it is proven in [6] that the best approximation (1.6) is obtained for $\alpha = \sqrt{2}/2$ and $\beta = \left(2 + \sqrt{2}\right)/4$).

2. Proofs and further remarks

Recall that a function z is completely monotonic on an interval I if z has derivatives of all orders on I and $(-1)^n z^{(n)}(x) \geq 0$, for every $x \in I$. For more detailed information, see [8].

We use a result of Alzer [2, Theorem 8], who proved that for every integer $n \geq 0$, the functions

$$F_n(x) = \log \Gamma(x) - \left(x - \frac{1}{2}\right) \log x + x - \frac{1}{2} \log 2\pi - \sum_{i=1}^{2n} \frac{B_{2i}}{2i(2i-1)x^{2i-1}}$$

and

$$H_n(x) = -\log \Gamma(x) + \left(x - \frac{1}{2}\right) \log x - x + \frac{1}{2} \log 2\pi + \sum_{i=1}^{2n+1} \frac{B_{2i}}{2i(2i-1)x^{2i-1}}$$

are completely monotonic on $(0, \infty)$. Here B_j denotes the jth Bernoulli number, see e.g. [1, p. 804]. In particular, $F_n' < 0$ and $H_n' < 0$ and these inequalities can be equivalently written as

$$u_{2m+1}(x) < \psi(x) - \log x < u_{2n}(x), \ (x > 0, \ m, n = 1, 2, 3, \ldots), \tag{2.1}$$

where

$$u_k(x) = -\frac{1}{2x} - \sum_{j=1}^{k} \frac{B_{2j}}{2jx^{2j}}.$$

We also use the inequalities [7, Part. I, Chap. 4, Probl. 154],

$$\sum_{j=1}^{2n} \frac{B_{2j}}{(2j)!} x^{2j} < \frac{x}{e^x - 1} - 1 + \frac{x}{2} < \sum_{j=1}^{2m+1} \frac{B_{2j}}{(2j)!} x^{2j}, \ (x > 0)$$

under the form

$$v_{2m+1}(x) < \log \frac{e^x - 1}{x} < v_{2n}(x), \ (x > 0, \ m, n = 1, 2, 3, \ldots), \tag{2.2}$$

where

$$v_k(x) = -\log \left(1 - \frac{x}{2} + \sum_{j=1}^{k} \frac{B_{2j}}{(2j)!} x^{2j}\right).$$

Now we are in a position to give

Proof of Theorem 1. We have

$$\begin{aligned}\psi(x+1) + \log\left(e^{\frac{1}{x+1}} - 1\right) &= \psi(x+1) - \log(x+1) + \log \frac{e^{\frac{1}{x+1}} - 1}{\frac{1}{x+1}} \\ &< u_2(x+1) + v_2\left(\frac{1}{x+1}\right) \\ &< -\frac{1}{24x^2} + \frac{1}{12x^3} - \frac{337}{2880x^4} + \frac{97}{720x^5}.\end{aligned}$$

To prove the last inequality it suffices to show $g < 0$, where

$$g(x) = u_2(x+1) + v_2\left(\frac{1}{x+1}\right) - \left(-\frac{1}{24x^2} + \frac{1}{12x^3} - \frac{337}{2880x^4} + \frac{97}{720x^5}\right).$$

But g is strictly increasing, since

$$\begin{aligned}g'(x) = &\left[1806\,072x + 7011\,015x^2 + 15\,507\,080x^3 + 21\,336\,405x^4\right. \\ &\left.+18\,654\,360x^5 + 10\,086\,181x^6 + 3068\,760x^7 + 399\,480x^8 + 203\,215\right] \\ &\times \left[720x^6(x+1)^5\left(419 + 1920x + 3300x^2 + 2520x^3 + 720x^4\right)\right]^{-1} \\ &> 0,\end{aligned}$$

with $g(\infty) = 0$, so $g(x) < 0$, for all $x \in (0, \infty)$.

Further,

$$\begin{aligned}\psi(x+1)+\log\left(e^{\frac{1}{x+1}}-1\right) &= \psi(x+1)-\log(x+1)+\log\frac{e^{\frac{1}{x+1}}-1}{\frac{1}{x+1}}\\ &> u_3(x+1)+v_3\left(\frac{1}{x+1}\right)\\ &> -\frac{1}{24x^2}+\frac{1}{12x^3}-\frac{337}{2880x^4}.\end{aligned}$$

To prove the last inequality it suffices to show $h>0$, where

$$h(x)=u_3(x+1)+v_3\left(\frac{1}{x+1}\right)-\left(-\frac{1}{24x^2}+\frac{1}{12x^3}-\frac{337}{2880x^4}\right).$$

But h is strictly decreasing, since

$$\begin{aligned}h'(x)=&-\big(41\,516\,041+541\,694\,671x+3239\,782\,371x^2+11\,740\,823\,213x^3\\ &+28\,702\,590\,497x^4+49\,842\,303\,783x^5+63\,001\,042\,993x^6\\ &+58\,363\,076\,023x^7+39\,297\,523\,314x^8+18\,743\,293\,086x^9\\ &+6008\,274\,720x^{10}+1162\,203\,840x^{11}+102\,664\,800x^{12}\big)\\ &\times\left[5040x^5(x+1)^7\left(17\,599+115\,836x+317\,478x^2+463\,680x^3\right.\right.\\ &\left.\left.+380\,520x^4+166\,320x^5+30\,240x^6\right)\right]^{-1}\\ &<0,\end{aligned}$$

with $h(\infty)=0$, so $h(x)>0$, for all $x\in(0,\infty)$. □

Our method using the completely monotonicity of functions F_n and H_n is suitable for establishing other new results. As additional examples, we give new simpler proofs of the monotonicity and concavity of the function ϕ given by (1.2). Although our results can be stated on $[1,\infty)$, we consider that the following proofs are useful because of their simplicity and applicability.

2.1. Theorem. (Alzer [3]) *The function ϕ is strictly increasing on $[1,\infty)$.*

Proof. Inequality $\psi'>0$ is equivalent to

$$\frac{1}{x}+\log\left(1-\frac{1}{x^2\psi'(x)}\right)>0,$$

while from $F_2''>0$ we get

$$\psi'(x)>\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{6x^3}-\frac{1}{30x^5},\ (x>0).$$

Now it suffices to show $t>0$, where

$$t(x)=\frac{1}{x}+\log\left(1-\frac{1}{x^2\left(\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{6x^3}-\frac{1}{30x^5}\right)}\right).$$

We have $t'(x)=-P(x)\left[x^2Q(x)R(x)\right]^{-1}$, where

$$\begin{aligned}P(x)&=91+830(x-1)+2615(x-1)^2+4000(x-1)^3\\ &\quad+3250(x-1)^4+1350(x-1)^5+225(x-1)^6,\\ Q(x)&=19+85(x-1)+140(x-1)^2+105(x-1)^3+30(x-1)^4,\\ R(x)&=49+175(x-1)+230(x-1)^2+135(x-1)^3+30(x-1)^4.\end{aligned}$$

Finally, t is strictly decreasing on $[1,\infty)$, with $t(\infty)=0$, so $t(x)>0$, for all $x\in[1,\infty)$, and the theorem is proved. □

The concavity of ϕ, first established in [5] using a quite complicated method, can be also proved in the following way.

2.2. Theorem. (Guo and Qi [5]) *The function ϕ is strictly concave on $[1,\infty)$.*

Proof. We have

$$\phi''(x)=\psi''(x)+\left(\frac{2}{x^3}+\frac{1}{x^4}\right)g(x)-\frac{1}{x^4}g^2(x)<0,$$

where $g(x)=\left(1-e^{-1/x}\right)^{-1}$. By using the following inequalities arising from the expansion of e^{-t},

$$a(x)<\frac{1}{1-e^{-\frac{1}{x}}}<b(x),$$

where

$$a(x)=\frac{1}{1-\left(1-\frac{1}{x}+\frac{1}{2x^2}-\frac{1}{6x^3}+\frac{1}{24x^4}-\frac{1}{120x^5}+\frac{1}{720x^6}-\frac{1}{5040x^7}\right)},$$
$$b(x)=\frac{1}{1-\left(1-\frac{1}{x}+\frac{1}{2x^2}-\frac{1}{6x^3}+\frac{1}{24x^4}-\frac{1}{120x^5}+\frac{1}{720x^6}\right)},$$

and the inequality

$$\psi''(x)<-\frac{1}{x^2}-\frac{1}{x^3}-\frac{1}{2x^4},$$

arising from $F_1'''<0$, we get

$$\begin{aligned}\phi''(x)&<\left(-\frac{1}{x^2}-\frac{1}{x^3}-\frac{1}{2x^4}\right)+\left(\frac{2}{x^3}+\frac{1}{x^4}\right)b(x)-\frac{1}{x^4}a^2(x)\\&=-\frac{P(x)}{2x^4Q(x)R^2(x)}<0,\end{aligned}$$

where

$$\begin{aligned}R(x)&=1-7x+42x^2-210x^3+840x^4-2520x^5+5040x^6,\\Q(x)&=455+2466(x-1)+5370(x-1)^2+5880(x-1)^3+3240(x-1)^4\\&\quad+720(x-1)^5,\end{aligned}$$

$$\begin{aligned}P(x)&=2357\,487\,180+42\,988\,345\,380(x-1)+369\,160\,818\,451(x-1)^2\\&+1983\,061\,528\,596(x-1)^3+7464\,108\,845\,416(x-1)^4\\&+20\,892\,515\,249\,544(x-1)^5+45\,036\,465\,002\,040(x-1)^6\\&+76\,373\,345\,608\,128(x-1)^7+103\,158\,858\,030\,480(x-1)^8\\&+111\,625\,536\,718\,080(x-1)^9+96\,770\,286\,007\,920(x-1)^{10}\\&+66\,833\,707\,469\,760(x-1)^{11}+36\,324\,643\,320\,000(x-1)^{12}\\&+15\,211\,763\,280\,000(x-1)^{13}+4740\,319\,584\,000(x-1)^{14}\\&+1035\,877\,248\,000(x-1)^{15}+141\,740\,928\,000(x-1)^{16}\\&+9144\,576\,000(x-1)^{17}.\end{aligned}$$

Now $\phi'' < 0$, and consequently, ϕ is strictly concave. □

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