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ON BI-IDEALS ON ORDERED Γ-SEMIGROUPS I

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Abstract

In this paper we introduce and give some characterizations of the left and right simple, the completely regular and the strongly regular po-Γ-semigroups by means of bi-ideals.

Keywords: Γ-semigroup, po-Γ-semigroup, Γ-group, Left (right) ideal, Bi-ideal, Left (right) simple, Left (right) regular, Regular, Completely regular, Strongly regular, Duo, B-simple, Semiprime.

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1. Introduction and preliminaries

In 1981, Sen [15] introduced the concept and notion of the Γ-semigroup as a generalization of semigroup and ternary semigroup. Many classical notions and results of the theory of semigroups have been extended and generalized to Γ-semigroups. In this paper we introduce and characterize the left and right simple, the completely regular and the strongly regular po-Γ-semigroups in terms of bi-ideals and study their structure, extending and generalizing results for ordered semigroups (cf. [10, 11]).

We introduce below necessary notions and present a few auxiliary results that will be used throughout the paper.

In 1986, Sen and Saha [16] defined a Γ-semigroup as a generalization of semigroup and ternary semigroup as follows:

1.1. Definition. Let M and Γ be two non-empty sets. Denote by the letters of the English alphabet the elements of M and with the letters of the Greek alphabet the elements of Γ. Then M is called a Γ-semigroup if

(1) $a\gamma b \in M$, for all $\gamma \in \Gamma$.

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(2) If $m_1, m_2, m_3, m_4 \in M, \gamma_1, \gamma_2 \in \Gamma$ are such that $m_1 = m_3,\ \gamma_1 = \gamma_2$ and $m_2 = m_4$, then $m_1\gamma_1 m_2 = m_3\gamma_2 m_4$.
(3) $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and for all $\alpha, \beta \in \Gamma$.

1.2. Example. Let M be a semigroup and Γ any non-empty set. Define a mapping $M \times \Gamma \times M \to M$ by $a\gamma b = ab$ for all $a, b \in M$ and $\gamma \in \Gamma$. Then M is a Γ-semigroup.

1.3. Example. Let M be the set of all negative rational numbers. Obviously M is not a semigroup under the usual product of rational numbers. Let $\Gamma = \{-\frac{1}{p} : p \text{ is prime}\}$. Let $a, b, c \in M$ and $\alpha \in \Gamma$. Now if $a\alpha b$ is equal to the usual product of the rational numbers a, α, b, then $a\alpha b \in M$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$. Hence M is a Γ-semigroup.

1.4. Example. Let $M = \{-i, 0, i\}$ and $\Gamma = M$. Then M is a Γ-semigroup under the multiplication over complex numbers while M is not a semigroup under complex number multiplication.

These examples illustrate that every semigroup is a Γ-semigroup and that Γ-semigroups are a generalization of semigroups.

A Γ-semigroup M is called a *commutative* Γ*-semigroup* if for all $a, b \in M$ and $\gamma \in \Gamma$, $a\gamma b = b\gamma a$. A non-empty subset K of a Γ-semigroup M is called a *sub-*Γ*-semigroup* of M if for all $a, b \in K$ and $\gamma \in \Gamma$, $a\gamma b \in K$.

1.5. Example. Let $M = [0, 1]$ and $\Gamma = \{\frac{1}{n} : n \text{ is a positive integer}\}$. Then M is a Γ-semigroup under usual multiplication. Let $K = [0, 1/2]$. We have that K is a non-empty subset of M and $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$. Then K is a sub-Γ-semigroup of M.

Other examples of Γ-semigroups can be found in [1, 2, 6, 15, 16].

1.6. Definition. A po-Γ-semigroup (:ordered Γ-semigroup) is a partially ordered set M which at the same time is a Γ-semigroup such that for all $a, b, c \in M$ and for all $\gamma \in \Gamma$

$$a \le b \implies a\gamma c \le b\gamma c, c\gamma a \le c\gamma b.$$

Examples of ordered Γ-semigroups can be found in [3, 4, 5, 7, 8, 9, 12, 13, 14].

Let M be a po-Γ-semigroup and A a non-empty subset of M. Then A is called a *right* (resp. *left*) ideal of M if

(1) $A\Gamma M \subseteq A$ (resp. $M\Gamma A \subseteq A$),
(2) $a \in A, b \le a$ for $b \in M \implies b \in A$.

A is called an *ideal* of M if it is right and left ideal of M. It is clear that the intersection of all ideals of a po-Γ-semigroup M is still an ideal of M. We shall call this particular ideal, if exists, the *kernel* of M and denote it by $K(M)$.

A non-empty subset B of a po-Γ-semigroup M is called a *bi-ideal* of M if

(1) $B\Gamma M\Gamma B \subseteq B$
(2) $a \in B$ and $b \le a$ for $b \in M \implies b \in B$

A right, left or bi-ideal A of a po-Γ-semigroup M is called *proper* if $A \neq M$.

A bi-ideal A of M is called *subidempotent* if $A\Gamma A \subseteq A$.

A bi-ideal A of M is called *idempotent* if $(A\Gamma A] = A$.

An element a of a Γ-semigroup M is called *idempotent* if $\exists\, \gamma \in \Gamma, a = a\gamma a$.

Let M be a po-Γ-semigroup. For $\emptyset \neq A \subseteq M$, we denote by $J(A)$ the ideal of M generated by A, by $L(A)$ (respectively $R(A)$) the left (respectively right) ideal of M generated by A, and by $B(A)$ the bi-ideal of M generated by A.

For non-empty subsets A and B of M and a non-empty subset Γ' of Γ, let $A\Gamma' B = \{a\gamma b : a \in A, b \in B \text{ and } \gamma \in \Gamma'\}$. If $A = \{a\}$, then we also write $\{a\}\Gamma' B$ as $a\Gamma' B$, and similarly if $B = \{b\}$ or $\Gamma' = \{\gamma\}$. For the sake of simplicity, we let M be a po-Γ-semigroup and T its sub-Γ-semigroup. For $A \subseteq T$ we write

$$(A]_T = \{t \in T : t \leq a, \text{ for some } a \in A\}$$

If $T = M$, then we always write $(A]$ instead of $(A]_M$. Clearly, $A \subseteq (A]_T \subseteq (A]$ and $A \subseteq B$ implies that $(A]_T \subseteq (B]_T$ for any non-empty subsets A, B of T. For $A = \{a\}$, we write $(a]$ instead of $(\{a\}]$. We denote by $L(a)$ (respectively $R(a)$) the left (respectively right) ideal of M generated by $a \in M$, by $B(a)$ the bi-ideal of M generated by $a \in M$, by $J(a)$ the ideal of M generated by $a \in M$. One can easily prove that

$$\begin{aligned} L(a) &= M\Gamma a = \{a\} \cup M\Gamma\{a\} = (a \cup M\Gamma a] = (a] \cup (M\Gamma a], \\ R(a) &= a\Gamma M = \{a\} \cup \{a\}\Gamma M = (a \cup a\Gamma M] = (a] \cup (a\Gamma M], \\ B(a) &= (a \cup a\Gamma M\Gamma a] = (a] \cup (a\Gamma M\Gamma a], \\ J(a) &= M\Gamma a\Gamma M = \{a\} \cup M\Gamma\{a\} \cup \{a\}\Gamma M \cup M\Gamma\{a\}\Gamma M \\ &= (a \cup M\Gamma a \cup a\Gamma M \cup M\Gamma a\Gamma M]. \end{aligned}$$

The authors of [7, 14] proved the following:

1.7. Lemma. *Let M be a po-Γ-semigroup. The following statements hold true:*

(1) $A \subseteq (A]$ *for any* $A \subseteq M$
(2) *If* $A \subseteq B \subseteq M$, *then* $(A] \subseteq (B]$.
(3) $(A]\Gamma(B] \subseteq (A\Gamma B]$ *for all subsets A and B of M.*
(4) $((A]] \subseteq (A]$ *for all* $A \subseteq M$.
(5) *For every left (resp. right, two-sided, bi-) ideal T of M,* $(T] = T$.
(6) *If L is a left ideal and R a right ideal of M, then the set* $(L\Gamma R]$ *is an ideal of M.*
(7) *If A, B are ideals of M, then* $(A\Gamma B], (B\Gamma A], A \cup B, A \cap B$ *are ideals of M.*
(8) $(M\Gamma a]$ *(resp.*$(a\Gamma M]$*) is a left (resp. right) ideal of M for every* $a \in M$.
(9) $(M\Gamma a\Gamma M]$ *is an ideal of M for every* $a \in M$.
(10) $((A]\Gamma(B]] = (A\Gamma B]$, *for any* $A, B \subseteq M$.

A po-Γ-semigroup M is called *left* (resp. *right*) *simple* if it does not contain proper left (respectively, right) ideals or equivalently, if for every left (respectively, right) ideal A of M, we have $A = M$.

A subset T of a po-Γ-semigroup M is called *semiprime* if for every $a \in M$ such that $a\Gamma a \subseteq T$, we have $a \in T$. **Equivalent Definition:** For each subset A of M such that $A\Gamma A \subseteq T$, we have $A \subseteq T$.

An element a of a po-Γ-semigroup M is called *regular* if there exists $x \in M$ such that $a \leq a\alpha x\beta a$ for some $\alpha, \beta \in \Gamma$.

A po-Γ-semigroup M is called *regular* if every element of M is regular. The following are equivalent definitions:

(1) For every $A \subseteq M, A \subseteq (A\Gamma M\Gamma A]$.
(2) For every element $a \in M, a \in (a\Gamma M\Gamma a]$.

Let M be a po-Γ-semigroup and $a \in M$. For the sake of simplicity, throughout the paper we write $a^2 = a\gamma a,\ a^3 = (a\gamma)^2 a, \ldots, a^n = (a\gamma)^{n-1}a, \ldots$ for $\gamma \in \Gamma$ and $n \in Z^+$.

2. Characterizations of some classes of po-Γ-semigroups by means of bi-ideals

In [7], the authors proved the following lemma.

2.1. Lemma. *Let M be a po-Γ-semigroup. Then M is left (resp. right) simple if and only if $(M\Gamma a] = M$(resp. $(a\Gamma M] = M$) for all $a \in M$.*

Proof. We prove the lemma only for the left simple case, the other case can be proved analogously.

$\Longrightarrow$ Let M be left simple and $a \in M$. Since $(M\Gamma a]$ is a left ideal of M, by the definition we have $(M\Gamma a] = M$.

$\Longleftarrow$ Let $(M\Gamma a] = M, \forall a \in M$. Let A be a left ideal of M and $x \in A (A \neq \emptyset)$. Since $M = (M\Gamma x] \subseteq (M\Gamma A] \subseteq (A] = A$, we have $A = M$. This implies that M is left simple. □

2.2. Theorem. *Let M be a po-Γ-semigroup. The following are equivalent:*

(1) *M is left and right simple.*
(2) *$M = (a\Gamma M\Gamma a], \forall a \in M$.*
(3) *M is regular, left and right simple.*

Proof. (1) $\Longrightarrow$ (2). Let $a \in M$. By Lemma 2.1, $M = (M\Gamma a] = (a\Gamma M]$. Then $M = (a\Gamma M] = (a\Gamma(M\Gamma a]] = (a\Gamma M\Gamma a]$.

(2) $\Longrightarrow$ (3). Let $a \in M$. By (ii), we have $M = (a\Gamma M\Gamma a] \subseteq (M\Gamma a]$, $(a\Gamma M] \subseteq M$. It follows that $a \in (a\Gamma M\Gamma a]$ and $M = (M\Gamma a] = (a\Gamma M]$.

(3) $\Longrightarrow$ (1). Clear. □

2.3. Proposition. *Let M be a regular po-Γ-semigroup. Then the bi-ideals and the subidempotent bi-ideals of M are the same.*

Proof. If B is a bi-ideal of M, then $B\Gamma M\Gamma B \subseteq B$, $(B\Gamma M\Gamma B] \subseteq (B] = B$. Since M is regular, $B \subseteq (B\Gamma M\Gamma B]$. Thus, $B = (B\Gamma M\Gamma B]$ and

$$B\Gamma B = (B\Gamma M\Gamma B]\Gamma(B\Gamma M\Gamma B] \subseteq (B\Gamma M\Gamma B\Gamma B\Gamma M\Gamma B] \subseteq (B\Gamma M\Gamma B] = B. \qquad \square$$

2.4. Theorem. *A po-Γ-semigoup M is left and right simple if and only if M does not contain proper bi-ideals.*

Proof. $\Longrightarrow$ Let A be a bi-ideal of M. Let $a \in M$ and $b \in A$, $(A \neq \emptyset)$.

Let $L(b) = (b \cup M\Gamma b]$ be the left ideal of M generated by b. Since M is left simple, we have $M = L(b)$. Since $a \in L(b)$, we have $a \leq b$ or $a \leq x\gamma b$ for some $x \in M$ and $\gamma \in \Gamma$. Let $a \leq b$. Then, since $M \ni a \leq b \in A, A$ is a bi-ideal of M, so we have $a \in A$. Let $a \leq x\gamma b$ for some $x \in M$ and $\gamma \in \Gamma$. Let $R(b) = (b \cup b\Gamma M]$ be the right ideal of M generated by b. Since M is right simple, we have $M = R(b)$.

Since $x \in R(b)$, we have $x \leq b$ or $x \leq b\rho y$ for some $y \in M$ and $\rho \in \Gamma$. Let $x \leq b$. Then we have $a \leq x\gamma b \leq b\gamma b \in A$ and $a \in A$. Let $x \leq b\rho y$ for some $y \in M$ and $\rho \in \Gamma$. Then $a \leq x\gamma b \leq b\rho y\gamma b \in A\Gamma M\Gamma A \subseteq A$ and $a \in A$.

$\Longleftarrow$ Let L be a left ideal of M. Then L is a bi-ideal of M. By hypothesis, $L = M$. Similarly, M is right simple. □

In [1], the authors gave the following definition.

2.5. Definition. Let M be a Γ-semigroup. Then M is called a Γ-*group* if for every $a, a_1 \in M$ and $\alpha, \alpha_1 \in \Gamma$, there exist $b, b_1 \in M$ and $\beta, \beta_1 \in \Gamma$ such that for all $s \in M$ and $\gamma \in \Gamma$,

$$s = b\beta a\alpha s, \gamma = \gamma b\beta a\alpha, \gamma = \beta_1 b_1\alpha_1 a_1\gamma, s = s\beta_1 b_1\alpha_1 a_1.$$

Also, they proved the following proposition.

2.6. Proposition. *Let M be a Γ-semigroup. Then M is a Γ-group if and only if both M and Γ are left simple as well as right simple.* □

2.7. Theorem. *Let M be an ordered Γ-group. Then M does not contain proper bi-ideals.*

Proof. An ordered Γ-group is a Γ-group. From Proposition 2.6 and Theorem 2.4, it follows that the ordered Γ-groups do not contain proper bi-ideals, as well. Another independent proof is given as follows:

Let A be a bi-ideal of M and $a \in M$. Let $b \in A(A \neq \emptyset)$. Since $b \in M$, there exist $b_1, b_2 \in M$ and $\beta, \alpha, \beta_1, \alpha_1 \in \Gamma$ such that $a = b\beta b_1 \alpha a$, $a = a\beta_1 b_2 \alpha_1 b$. Then we have

$$a = b\beta b_1 \alpha a \beta_1 b_2 \alpha_1 b = b\beta(b_1 \alpha a \beta_1 b_2)\alpha_1 b \in A\Gamma M\Gamma A \subseteq A \text{ and } a \in A.$$ □

2.8. Definition. A po-Γ-semigroup M is called *left* (resp. *right*) *regular* if for every $a \in M$, there exist $x \in M$, $\gamma, \mu \in \Gamma$ such that $a \leq x\gamma(a\mu a)$ (resp. $a \leq (a\mu a)\gamma x$).

The following are equivalent definitions:

(1) $a \in (M\Gamma a\Gamma a]$ (resp. $a \in (a\Gamma a\Gamma M]), \forall\, a \in M$
(2) $A \subseteq (M\Gamma A\Gamma A]$ (resp. $A \in (A\Gamma A\Gamma M]), \forall\, A \subseteq M$.

2.9. Lemma. *A po-Γ-semigroup M is left regular if and only if every left ideal of M is semiprime.*

Proof. $\Longrightarrow$. Let L be a left ideal of M and $a \in M, a\Gamma a \subseteq L$. Since $M\Gamma(a\Gamma a) \subseteq M\Gamma L \subseteq L$, we have $(M\Gamma(a\Gamma a)] \subseteq (L] = L$. Since M is left regular, we have $a \in (M\Gamma(a\Gamma a)] \subseteq L$, so $a \in L$.

$\Longleftarrow$. Let $a \in M$. The set $(M\Gamma(a\Gamma a)]$, as a left ideal of M, is semiprime. Since

$$a\Gamma a\Gamma a\Gamma a = (a\Gamma a)\Gamma(a\Gamma a) \in M\Gamma(a\Gamma a) \subseteq (M\Gamma(a\Gamma a)],$$

we have $a\Gamma a \in (M\Gamma(a\Gamma a)]$ and $a \in (M\Gamma(a\Gamma a)]$. □

2.10. Definition. A po-Γ-semigroup M is called *completely regular* if it is regular, left regular and right regular.

If M is a po-Γ-semigroup and $\emptyset \neq A \subseteq M$, then one can easily proves that the set $(A \cup A\Gamma A \cup A\Gamma M\Gamma A]$ is the bi-ideal of M generated by A. In particular, for $A = \{a\}$, $(a \in M)$, we write $B(a) = (a \cup a\Gamma a \cup a\Gamma M\Gamma a]$ for the bi-ideal generated by a. If M is regular, then it is clear that: $B(a) = (a\Gamma M\Gamma a]$. We define a relation $\mathcal{B}$ on M as follows:

$$a\mathcal{B}b \iff B(a) = B(b).$$

Clearly, $\mathcal{B}$ is an equivalence relation on M.

2.11. Lemma. *Let M be a po-Γ-semigroup and $B(x)$, $B(y)$ the bi-ideals of M generated by the elements $x, y \in M$, respectively. Then we have $B(x)\Gamma M\Gamma B(y) \subseteq (x\Gamma M\Gamma y]$.*

Proof. We have

$$\begin{aligned} B(x)\Gamma M\Gamma B(y) &= (x \cup x^2 \cup x\Gamma M\Gamma x]\Gamma(M]\Gamma(y \cup y^2 \cup y\Gamma M\Gamma y] \\ &\subseteq ((x \cup x^2 \cup x\Gamma M\Gamma x)\Gamma M\Gamma(y \cup y^2 \cup y\Gamma M\Gamma y)] = (x\Gamma M\Gamma y]. \end{aligned}$$ □

2.12. Lemma. *A po-Γ-semigroup M is completely regular if and only if for any $a \in M$, there exist $x \in M$, $\mu, \alpha, \beta \in \Gamma$ such that $a \leq (a\mu a)\alpha x\beta(a\mu a)$.*

Proof. Assume that M is completely regular. Then for any $a \in M$, there exist $x, y, z \in M, \alpha, \beta, \mu, \rho, \gamma \in \Gamma$ such that

$$a \leq a\alpha x\beta a \leq (a\mu a\gamma y)\alpha x\beta(z\rho a\mu a) = a\mu a\gamma(y\alpha x\beta z)\rho a\mu a.$$

Conversely, suppose that, for any $a \in M$, there exist $x \in M$, $\mu, \alpha, \beta \in \Gamma$ such that $a \leq (a\mu a)\alpha x\beta(a\mu a)$. Then

(1) $a \leq (a\mu a)\alpha x\beta(a\mu a) = a\mu(a\alpha x\beta a)\mu a$.
(2) $a \leq (a\mu a)\alpha x\beta(a\mu a) = (a\mu a\alpha x)\beta a\mu a$.
(3) $a \leq (a\mu a)\alpha x\beta(a\mu a) = a\mu a\alpha(x\beta a\mu a)$.

Thus, M is regular, left regular and right regular. Therefore M is completely regular. □

From Lemma 2.12, it is obvious that the following lemma holds true:

2.13. Lemma. *A po-Γ-semigroup M is completely regular if and only if for every $A \subseteq M$, $A \subseteq ((A\Gamma A)\Gamma M\Gamma(A\Gamma A)]$.*

Equivalently, if for every $a \in M$, $a \in ((a\Gamma a)\Gamma M\Gamma(a\Gamma a)]$. □

2.14. Theorem. *A po-Γ-semigroup M is completely regular if and only if every bi-ideal B of M is semiprime.*

Proof. $\Longrightarrow$. Let B be a bi-ideal, $a \in M$ and $a\Gamma a \subseteq B$. Then for some $x, y, z \in M$, $\alpha, \beta, \gamma, \rho, \mu \in \Gamma$,

$$a \leq a\alpha x\beta a \leq (a\gamma a\rho y)\alpha x\beta(z\mu a\gamma a) = a\gamma(a\rho y\alpha x\beta z\mu a)\gamma a \in B\Gamma M\Gamma B \subseteq B.$$

Thus B is semiprime.

$\Longleftarrow$. Let $a \in M$. Then $(a\Gamma a\Gamma M\Gamma a\Gamma a]$ is a non-empty subset of M. Let $x, y \in (a\Gamma a\Gamma M\Gamma a\Gamma a]$ and $z \in M$. Then for some $u, v \in M$, $\alpha, \beta, \gamma, \rho, \mu, \delta, \sigma \in \Gamma$,

$$\begin{aligned} x\alpha z\beta y &\leq (a\gamma a\rho u\mu a\gamma a)\alpha z\beta(a\gamma a\delta v\sigma a\gamma a) \\ &= a\gamma a\rho(u\mu a\gamma a\alpha z\beta a\gamma a\delta v)\sigma a\gamma a \in a\Gamma a\Gamma M\Gamma a\Gamma a \end{aligned}$$

Thus

$$x\alpha z\beta y \in (a\Gamma a\Gamma M\Gamma a\Gamma a] \text{ and } (a\Gamma a\Gamma M\Gamma a\Gamma a]\Gamma M\Gamma(a\Gamma a\Gamma M\Gamma a\Gamma a] \subseteq (a\Gamma a\Gamma M\Gamma a\Gamma a].$$

Furthermore, if $x \in (a\Gamma a\Gamma M\Gamma a\Gamma a]$ and $y \leq x$, for some $y \in M$, then

$$y \leq x \in (a\Gamma a\Gamma M\Gamma a\Gamma a], \text{ so } y \in (a\Gamma a\Gamma M\Gamma a\Gamma a].$$

Hence $(a\Gamma a\Gamma M\Gamma a\Gamma a]$ is a bi-ideal of M, for all $a \in M$. Since $(a\Gamma a\Gamma a\Gamma a\Gamma a\Gamma a\Gamma a\Gamma a) = (a\Gamma a)\Gamma(a\Gamma a\Gamma a\Gamma a)\Gamma(a\Gamma a) \subseteq (a\Gamma a\Gamma M\Gamma a\Gamma a]$ and $(a\Gamma a\Gamma M\Gamma a\Gamma a]$ is semiprime, we get $(a\Gamma a\Gamma a\Gamma a), (a\Gamma a) \subseteq (a\Gamma a\Gamma M\Gamma a\Gamma a]$ and so $a \in (a\Gamma a\Gamma M\Gamma a\Gamma a]$. By Lemma 2.10, M is completely regular. □

2.15. Lemma. *Let M be a po-Γ-semigroup. Then the following are equivalent:*

(1) *M is completely regular;*
(2) *$B(a) = B(a\Gamma a) = B(a\Gamma a\Gamma M\Gamma a\Gamma a), \forall\, a \in M$;*
(3) *$a\mathcal{B}a\Gamma a$.*

Proof. (1) $\Longrightarrow$ (2). Since M is completely regular, M is regular and we have $B(a) = (a\Gamma M\Gamma a]$. Then $B(a\Gamma a) = (a\Gamma a\Gamma M\Gamma a\Gamma a]$. Since M is regular, left regular and right regular, we have for all $a \in M$:

$$a \in (a\Gamma M\Gamma a] \subseteq ((a\Gamma a\Gamma M]\Gamma M\Gamma(M\Gamma a\Gamma a]] \subseteq (a\Gamma a\Gamma M\Gamma a\Gamma a] \subseteq (a\Gamma M\Gamma a].$$

Therefore

$$B(a) = (a\Gamma M\Gamma a] = (a\Gamma a\Gamma M\Gamma a\Gamma a] = B(a\Gamma a).$$

Also, since for all $a \in M$,

$$a \in (a\Gamma M\Gamma a] \subseteq ((a\Gamma a\Gamma M\Gamma a\Gamma a]\Gamma M\Gamma(a\Gamma a\Gamma M\Gamma a\Gamma a]] \subseteq (a\Gamma a\Gamma M\Gamma a\Gamma a] \subseteq (a\Gamma M\Gamma a],$$

we have $B(a) = B(a\Gamma a) = B(a\Gamma a\Gamma M\Gamma a\Gamma a)$.

(2) $\Longrightarrow$ (3). Clear since $B(a) = B(a\Gamma a),\ \forall\, a \in M$.

(3) $\Longrightarrow$ (1). If $a\mathcal{B}a\Gamma a$, then $B(a) = B(a\Gamma a)$. We have

$$a \in B(a) = B(a\Gamma a) = (a\Gamma a \cup a\Gamma a\Gamma a\Gamma a \cup a\Gamma a\Gamma M\Gamma a\Gamma a].$$

Then $a \leq y$ for some $y \in a\Gamma a \cup a\Gamma a\Gamma a\Gamma a \cup a\Gamma a\Gamma M\Gamma a\Gamma a$.

If $y \in a\Gamma a$, then for some $\gamma \in \Gamma$,

$$a \leq y = a\gamma a \leq (a\gamma a)\gamma(a\gamma a) = a\gamma a\gamma(a\gamma a) \leq (a\gamma a)\gamma a\gamma(a\gamma a) \in (a\Gamma a)\Gamma M\Gamma(a\Gamma a)$$

and $a \in (a\Gamma a\Gamma M\Gamma a\Gamma a]$.

If $y \in a\Gamma a\Gamma a\Gamma a$, then for some $\gamma \in \Gamma$,

$$a \leq y = a^4 = ((a\gamma)^3 a) = a\gamma a\gamma(a\gamma a) \leq a^4\gamma a\gamma(a\gamma a) \in a\Gamma a\Gamma M\Gamma a\Gamma a,$$

and $a \in (a\Gamma a\Gamma M\Gamma a\Gamma a]$.

If $y \in a\Gamma a\Gamma M\Gamma a\Gamma a$, then $a \in (a\Gamma a\Gamma M\Gamma a\Gamma a]$. □

2.16. Theorem. *A po-Γ-semigroup M is completely regular if and only if for each bi-ideal B of M, we have*

$$B = (B\Gamma B].$$

Proof. $\Longrightarrow$. Since B is a sub-Γ-semigroup of M, we have $B\Gamma B \subseteq B$. Then since B is an ideal of M, we have $(B\Gamma B] \subseteq (B] = B$. By Lemma 2.13, we have

$$B \subseteq ((B\Gamma B)\Gamma M\Gamma(B\Gamma B)].$$

Then, by the definition of bi-ideal, we have

$$B \subseteq (B\Gamma(B\Gamma M\Gamma B)\Gamma B] \subseteq (B\Gamma B\Gamma B] \subseteq (B\Gamma B].$$

Thus, $B = (B\Gamma B]$.

$\Longleftarrow$ Let L be a left ideal of M. Then $M\Gamma L \subseteq L$ and $L\Gamma(M\Gamma L) \subseteq M\Gamma L \subseteq L$. Thus L is a bi-ideal of M. Let $x \in M$ be such that $x^2 \in L$. Then $x \in L$. Indeed: We consider the bi-ideal of M generated by x. That is, the set $B(x) = (x \cup x^2 \cup x\Gamma M\Gamma x]$. By hypothesis, we have $x \in B(x) = (B(x)\Gamma B(x)]$. On the other hand, $(B(x)\Gamma B(x)] \subseteq L$. In fact: We have

$$\begin{aligned} B(x)\Gamma B(x) &= (x \cup x^2 \cup x\Gamma M\Gamma x]\Gamma(x \cup x^2 \cup x\Gamma M\Gamma x] \\ &\subseteq ((x \cup x^2 \cup x\Gamma M\Gamma x)\Gamma(x \cup x^2 \cup x\Gamma M\Gamma x)] \\ &= (x^2 \cup x^3 \cup x\Gamma M\Gamma x^2 \cup x^4 \cup x\Gamma M\Gamma x^3 \cup x^2\Gamma M\Gamma x \cup x^3\Gamma M\Gamma x \\ &\qquad \cup x\Gamma M\Gamma x^2\Gamma M\Gamma x] \end{aligned}$$

Since $x^2 \in L, x^3 \in M\Gamma L \subseteq L, (x\Gamma M)\Gamma x^2 \subseteq M\Gamma L \subseteq L, x^4 \in M\Gamma L \subseteq L$, we have

$$B(x)\Gamma B(x) \subseteq (L \cup L\Gamma M] = (L] = L.$$

Then we have $(B(x)\Gamma B(x)] \subseteq ((L]] = (L] = L$ and $x \in L$. In a similar way, we prove that every right ideal of M is semiprime. That is, by Lemma 2.9, M is left and right regular. So, M is a regular po-Γ-semigroup. Indeed: Now let $a \in M$. Let $B(a)$ be the bi-ideal generated by a. Then by hypothesis and Lemma 2.11, we have

$$\begin{aligned} a \in B(a) = (B(a)\Gamma B(a)] &= ((B(a)\Gamma B(a)]\Gamma B(a)] = (B(a)\Gamma B(a)\Gamma B(a)] \\ &\subseteq (B(a)\Gamma M\Gamma B(a)] \subseteq (a\Gamma M\Gamma a] \end{aligned}$$

That is, M is completely regular. □

Let M be a po-Γ-semigroup. Then M is called a B-*simple* po-Γ-semigroup if M does not contain proper bi-ideals.

2.17. Theorem. *Let M be a po-Γ-semigroup. Then the following are equivalent:*

(1) *M is completely regular;*
(2) *$\forall a \in M, a \in (a\Gamma M\Gamma a] = (a\Gamma a\Gamma M\Gamma a\Gamma a]$;*
(3) *Every $\mathcal{B}$-class of M is a B-simple sub-Γ-semigroup of M;*
(4) *Every $\mathcal{B}$-class of M is a sub-Γ-semigroup of M;*
(5) *M is a union of disjoint B-simple sub-Γ-semigroups of M;*
(6) *M is a union of disjoint sub-Γ-semigroups of M;*
(7) *Every bi-ideal of M is semiprime;*
(8) *The set $\{(x)_{\mathcal{B}} | x \in M\}$ coincides with the set of all maximal B-simple sub-Γ-semigroups of M.*

Proof. (1) $\Longrightarrow$ (2) Let M be completely regular. Then M is regular, left regular and right regular. We have for all $a \in M$:

$$a \in (a\Gamma M\Gamma a] \subseteq ((a\Gamma a\Gamma M]\Gamma M\Gamma (M\Gamma a\Gamma a]] \subseteq (a\Gamma a\Gamma M\Gamma a\Gamma a] \subseteq (a\Gamma M\Gamma a].$$

Therefore,

$$a \in (a\Gamma M\Gamma a] = (a\Gamma a\Gamma M\Gamma a\Gamma a].$$

(2) $\Longrightarrow$ (3) We have that $(x)_{\mathcal{B}}$ is a sub-Γ-semigroup of M. Indeed: First, $\emptyset \neq (x)_{\mathcal{B}} \subseteq M$, $\forall\, x \in (x)_{\mathcal{B}}$. Let $a, b \in (x)_{\mathcal{B}}$. Since $a\mathcal{B}x, b\mathcal{B}x$, we have $B(a) = B(x) = B(b)$. Now, by hypothesis, we have:

$$\begin{aligned} a\Gamma b &\subseteq (((a\Gamma b\Gamma a\Gamma b)\Gamma M\Gamma (a\Gamma b\Gamma a\Gamma b)] \\ &\subseteq (B(a)\Gamma M\Gamma B(b)] = (B(b)\Gamma M\Gamma B(b)] \subseteq B(b). \end{aligned}$$

Then

$$a\Gamma b \subseteq B(b) = B(x).$$

Hence

$$B(a\Gamma b) \subseteq B(b) = B(x).$$

On the other hand, let $y \in B(x)$. By hypothesis, we have:

$$\begin{aligned} (a\Gamma M\Gamma y] &\subseteq ((a\Gamma a\Gamma M\Gamma a\Gamma a]\Gamma M\Gamma y] \\ &\subseteq (a\Gamma a\Gamma M\Gamma a\Gamma a\Gamma M\Gamma y] \subseteq (a\Gamma B(a)\Gamma M\Gamma B(b)] \\ &= (a\Gamma B(b)\Gamma M\Gamma B(b)] \subseteq (a\Gamma B(b)] \subseteq (a\Gamma b\Gamma M\Gamma b] \end{aligned}$$

and

$$\begin{aligned} (y\Gamma M\Gamma b] &\subseteq (y\Gamma M\Gamma (b\Gamma b\Gamma M\Gamma b\Gamma b]] \subseteq (y\Gamma M\Gamma b\Gamma b\Gamma M\Gamma b\Gamma b] \subseteq (B(x)\Gamma M\Gamma B(b)\Gamma b] \\ &= (B(a)\Gamma M\Gamma B(b)\Gamma b] = (B(a)\Gamma M\Gamma B(a)\Gamma b] \subseteq (B(a)\Gamma b] \subseteq (a\Gamma M\Gamma a\Gamma b]. \end{aligned}$$

So

$$
\begin{aligned}
y &\in (y\Gamma y\Gamma M\Gamma y\Gamma y] \subseteq (y\Gamma(y\Gamma y\Gamma M\Gamma y\Gamma y]\Gamma M\Gamma y\Gamma y]\Gamma(y\Gamma y\Gamma y\Gamma M\Gamma y\Gamma y\Gamma M\Gamma y\Gamma y] \\
&\subseteq (B(a)\Gamma M\Gamma y\Gamma y\Gamma M\Gamma B(b)] \subseteq ((B(a)\Gamma M\Gamma y]\Gamma(y\Gamma M\Gamma B(b)]] \\
&\subseteq ((a\Gamma M\Gamma y]\Gamma(y\Gamma M\Gamma b]] \subseteq ((a\Gamma b\Gamma M\Gamma b]\Gamma(a\Gamma M\Gamma a\Gamma b]] \\
&\subseteq (a\Gamma b\Gamma M\Gamma b\Gamma a\Gamma M\Gamma a\Gamma b] \subseteq (B(a\Gamma b)\Gamma M\Gamma b\Gamma a\Gamma M\Gamma B(a\Gamma b)] \\
&\subseteq (B(a\Gamma b)] = B(a\Gamma b).
\end{aligned}
$$

Thus, $y \in B(a\Gamma b)$, so $B(x) \subseteq B(a\Gamma b)$. Therefore $B(x) = B(a\Gamma b)$. Moreover, $(x)_{\mathcal{B}}$ is a sub-Γ-semigroup of M.

Let B be a bi-ideal of $(x)_{\mathcal{B}}$. Then $B = (x)_{\mathcal{B}}$. Indeed: For any $y \in (x)_{\mathcal{B}}$, suppose that $z \in B$. Since $z \in B \subseteq (x)_{\mathcal{B}} = (y)_{\mathcal{B}}$, then we have

$$y \in B(y) = B(z) = B(x).$$

By the hypothesis and Lemma 2.15, then

$$y \in B(z) \subseteq B(z\Gamma z\Gamma z\Gamma z) = (z\Gamma M\Gamma z] \subseteq (B\Gamma M\Gamma B] \subseteq (B] = B.$$

Therefore $B = (x)_{\mathcal{B}}$, that is, $(x)_{\mathcal{B}}$ is a B-simple sub-Γ-semigroup of M.

(3) $\Longrightarrow$ (4) Clear.

(3) $\Longrightarrow$ (5) It is clear that $M = \bigcup\{(x)_{\mathcal{B}} | x \in M\}$, and M is a union of disjoint B-simple sub-Γ-semigroups of M.

(5) $\Longrightarrow$ (6) Clear.

(6) $\Longrightarrow$ (7) Let

$$M = \bigcup\{S_\alpha | \alpha \in Y\},$$

where S_α is a B-simple sub-Γ-semigroup of M for all $\alpha \in Y$, Y is an index set. Then every bi-ideal is semiprime. Indeed: Let B be a bi-ideal of M, $\forall\, \alpha \in M$ such that $a\Gamma a \subseteq B$. Since $a \in M$, then there exists $\alpha \in Y$ such that $a \in S_\alpha$. On the other hand, $B \cap S_\alpha$ is a bi-ideal of S_α. In fact:

$$
\begin{aligned}
\emptyset \neq B \bigcap S_\alpha &\subseteq S_\alpha (a\Gamma a \subseteq B, a\Gamma a \subseteq S_\alpha), \\
\Big(B \bigcap S_\alpha\Big)\Gamma S_\alpha \Gamma \Big(B \bigcap S_\alpha\Big) &\subseteq B\Gamma S_\alpha \Gamma B \bigcap B\Gamma S_\alpha \Gamma S_\alpha \bigcap S_\alpha \Gamma S_\alpha \Gamma B \bigcap S_\alpha \Gamma S_\alpha \Gamma S_\alpha \\
&\subseteq B\Gamma M\Gamma B \bigcap B\Gamma S_\alpha \Gamma S_\alpha \bigcap S_\alpha \Gamma S_\alpha \Gamma B \bigcap S_\alpha \\
&\subseteq (B \bigcap S_\alpha) \bigcap B\Gamma S_\alpha \Gamma S_\alpha \bigcap S_\alpha \Gamma S_\alpha \Gamma B \subseteq B \bigcap S_\alpha.
\end{aligned}
$$

Let

$$y \in B \bigcap S_\alpha, S_\alpha \ni z \leq y.$$

Since $z \leq y \in B$ and B is a bi-ideal of M, we have $z \in B$. Thus $z \in B \bigcap S_\alpha$. By hypothesis, we have $B \bigcap S_\alpha = S_\alpha$, that is $a \in B$.

(7) $\Longleftrightarrow$ (1) Clear by Theorem 2.14.

(1) $\Longrightarrow$ (8) Let $x \in M$. By (2) $\Longrightarrow$ (3), $(x)_{\mathcal{B}}$ is a B-simple sub-Γ-semigroup of M. By (1) $\Longrightarrow$ (2) and the proof of Theorem 2.14, we have $(x\Gamma x\Gamma M\Gamma x\Gamma x] = (x\Gamma M\Gamma x]$ is

a bi-ideal of M. Let T be a B-simple sub-Γ-semigroup of M such that $T \supseteq (x)_{\mathcal{B}}$, then $(x\Gamma M\Gamma x]\bigcap T$ is a bi-ideal of T. Indeed:

$$\begin{aligned}\emptyset \neq (x\Gamma M\Gamma x]\bigcap T &\subseteq T\big(x\Gamma x\Gamma x \subseteq (x\Gamma M\Gamma x],\ x\Gamma x\Gamma x \subseteq T\big),\\ \Big((x\Gamma M\Gamma x]\bigcap T\Big)\Gamma T\Gamma\Big((x\Gamma M\Gamma x]\bigcap T\Big) &\subseteq (x\Gamma M\Gamma x]\Gamma T\Gamma(x\Gamma M\Gamma x]\\ &\bigcap (x\Gamma M\Gamma x]\Gamma T\Gamma T\\ &\bigcap T\Gamma T\Gamma(x\Gamma M\Gamma x]\bigcap T\Gamma T\Gamma T\\ &\subseteq (x\Gamma M\Gamma x]\bigcap(x\Gamma M\Gamma x]\Gamma T\Gamma T\\ &\bigcap T\Gamma T\Gamma(x\Gamma M\Gamma x]\bigcap T\\ &= ((x\Gamma M\Gamma x]\bigcap T)\bigcap(x\Gamma M\Gamma x]\Gamma T\Gamma T\\ &\bigcap T\Gamma T\Gamma(x\Gamma M\Gamma x]\\ &\subseteq (x\Gamma M\Gamma x]\bigcap T.\end{aligned}$$

If $a \in (x\Gamma M\Gamma x]\bigcap T, T \ni b \leq a \in (x\Gamma M\Gamma x]\bigcap T$, by $b \leq a \in (x\Gamma M\Gamma x]$, $(x\Gamma M\Gamma x]$ is a bi-ideal of M, we have $b \in (x\Gamma M\Gamma x]$, that is $b \in (x\Gamma M\Gamma x]\bigcap T$. Since T is B-simple, we have $(x\Gamma M\Gamma x]\bigcap T = T$. Let $y \in T$. We have

$$B(y) \ni y \in (x\Gamma M\Gamma x] \subseteq B(x)\Gamma M\Gamma B(x) \subseteq B(x),$$

then $B(y) \subseteq B(x)$. Similarly, $y \in T$ implies that $(y\Gamma M\Gamma y]\bigcap T$ is a bi-ideal of T and $(y\Gamma M\Gamma y]\bigcap T = T$. Since $x \in T$, we get

$$B(x) \ni x \in (y\Gamma M\Gamma y] \subseteq B(y)\Gamma M\Gamma B(y) \subseteq B(y) \text{ and } B(y) \subseteq B(x).$$

Therefore, we have $y \in (x)_{\mathcal{B}}$, that is $T = (x)_{\mathcal{B}}$, thus $(x)_{\mathcal{B}}$ is a maximal B-simple sub-Γ-semigroup of M.

On the other hand, let T be a maximal B-simple sub-Γ-semigroup of M and $x \in T$. Since $T \subseteq (x)_{\mathcal{B}}$ (from the above proof), we have $T = (x)_{\mathcal{B}}(x \in M)$. That is $T \subseteq \{(x)_{\mathcal{B}} \mid x \in M\}$.

(8) $\Longrightarrow$ (4) For any $x \in M$, by (8), we have $(x)_{\mathcal{B}}$ is a B-simple sub-Γ-semigroup of M.

(4) $\Longrightarrow$ (1) Since $(x)_{\mathcal{B}}$ is a sub-Γ-semigroup of M, $\forall\, x \in M$, then $x\Gamma x\Gamma x\Gamma x\Gamma x \subseteq (x)_{\mathcal{B}}$ and we have

$$x \in B(x) = B(x\Gamma x\Gamma x\Gamma x\Gamma x) \subseteq B(x\Gamma x\Gamma M\Gamma x\Gamma x).$$

It is easy to see that M is regular, left regular and right regular. □

2.18. Definition. A po-Γ-semigroup M is called *strongly regular* if for every $a \in M$, there exist $x \in M$, $\alpha, \beta \in \Gamma$ such that $a \leq a\alpha x\beta a$ and $a\gamma x = x\gamma a$ for all $\gamma \in \Gamma$.

2.19. Lemma. *Let M be a po-Γ-semigroup. The following are equivalent :*

(1) *M is strongly regular*
(2) *M is left regular, right regular, and $(M\Gamma a\Gamma M]$ is a strongly regular sub-Γ-semigroup of M, for every $a \in M$.*
(3) *For every $a \in M$, we have $a \in (M\Gamma a] \cap (a\Gamma M]$ and $(M\Gamma a\Gamma M]$ is a strongly regular sub-Γ-semigroup of M.*

Proof. (1) $\Longrightarrow$ (2) Let $a \in M$. Since M is strongly regular, then there exist $x \in M$, $\alpha, \beta \in \Gamma$ such that $a \leq a\alpha x\beta a = a\alpha a\beta x$. This shows that M is left regular. Similarly, M is right regular. We also have $(M\Gamma a\Gamma M]$ is strongly regular. Indeed:

A) Let $a \in M$. Then there exists $x \in M$, such that $a \in (a\Gamma x\Gamma a]$ and $a\Gamma x = x\Gamma a$. Then

$$a \in (a\Gamma x\Gamma a] \subseteq ((a\Gamma x\Gamma a)\Gamma x\Gamma a] = (a\Gamma(x\Gamma a\Gamma x)\Gamma a].$$

We put $Y = x\Gamma a\Gamma x$. Then we have

$$\begin{aligned} a &\in (a\Gamma Y\Gamma a], \\ Y &= x\Gamma a\Gamma x \\ &\subseteq (x\Gamma(a\Gamma x\Gamma a)\Gamma x] = ((x\Gamma a\Gamma x)\Gamma a\Gamma x] = (Y\Gamma a\Gamma x] \subseteq (Y\Gamma(a\Gamma x\Gamma a)\Gamma x] \\ &= (Y\Gamma a\Gamma(x\Gamma a\Gamma x)] = (Y\Gamma a\Gamma Y], \\ a\Gamma Y &= a\Gamma(x\Gamma a\Gamma x) = a\Gamma(x\Gamma a)\Gamma(x\Gamma a) = (x\Gamma a\Gamma x)\Gamma a = Y\Gamma a \end{aligned}$$

B) Let L be a left ideal and R a right ideal of M. Then $(L\Gamma R]$ is a strongly regular sub-Γ-semigroup of M. Indeed: By Lemma 1.4 (6), we have $(L\Gamma R]$ is an ideal of M, i.e. a sub-Γ-semigroup of M. Let $a \in (L\Gamma R] \subseteq M$. Since M is strongly regular, by A) there exist $z \in Y \subseteq M$ such that $a \le (a\alpha z\beta a), z \le (z\delta a\rho z)$ for some $\alpha, \beta, \delta, \rho \in \Gamma$ and $z\gamma a = a\gamma z$ for all $\gamma \in \Gamma$.

Since $a \in (L\Gamma R]$, there exist $y \in L, x \in R, \mu \in \Gamma$ such that $a \le y\mu x$. Then $z\delta a\rho z \le z\delta y\mu x\rho z$ for some $\mu \in \Gamma$. Since $z\delta y \in M\Gamma L \subseteq L$ and $x\rho z \in R\Gamma M \subseteq R$, we have $z\delta y\mu x\rho zt \in L\Gamma R$ and $z\delta a\rho z \in (L\Gamma R]$. Since $z \le z\delta a\rho z \in (L\Gamma R]$, then $z \in (L\Gamma R]$.

C) Let $a \in M$, then $(M\Gamma a]$ is a left ideal and $(a\Gamma M]$ is a right ideal of M. Moreover, $(M\Gamma a\Gamma M] = ((M\Gamma a]\Gamma(a\Gamma M]]$. Indeed:

$$\begin{aligned} M\Gamma a\Gamma M \subseteq M\Gamma(M\Gamma a\Gamma a]\Gamma M = (M]\Gamma(M\Gamma a\Gamma a]\Gamma(M] \subseteq (M\Gamma M\Gamma a\Gamma a\Gamma M] \\ \subseteq (M\Gamma a\Gamma a\Gamma M] = ((M\Gamma a)\Gamma(a\Gamma M)] = ((M\Gamma a]\Gamma(a\Gamma M]], \end{aligned}$$

hence $(M\Gamma M] \subseteq (((M\Gamma a]\Gamma(a\Gamma M]]] = ((M\Gamma a]\Gamma(a\Gamma M]]$. On the other hand,

$$((M\Gamma a]\Gamma(a\Gamma M]] = ((M\Gamma a)\Gamma(a\Gamma M)] = (M\Gamma a\Gamma a\Gamma M] \subseteq (M\Gamma a\Gamma M].$$

By B), $(M\Gamma a\Gamma M]$ is a strongly regular sub-Γ-semigroup.

(2) $\Longrightarrow$ (3). Let $a \in M$. Since M is left and right regular, then $a \in (M\Gamma a\Gamma a]$ and $a \in (a\Gamma a\Gamma M]$, there exist $x, y \in M$ such that $a \le x\alpha a\gamma a, a \le a\gamma a\beta y$, $\alpha, \beta, \gamma \in \Gamma$. We have:

$$\begin{aligned} a &\le x\alpha(a\gamma a) \le x\alpha(a\gamma a\beta y)\gamma a = (x\alpha(a\gamma a)\beta y)\gamma a \in (M\Gamma a] \\ a &\le (a\gamma a)\beta y \le a\gamma(x\alpha a\gamma a)\beta y = a\gamma(x\alpha(a\gamma a)\beta y) \in (a\Gamma M]. \end{aligned}$$

So, $\forall\, a \in M$, we have $a \in (M\Gamma a] \cap (a\Gamma M]$.

(3) $\Longrightarrow$ (1) Let $a \in M$. Since $a \in (M\Gamma a] \cap (a\Gamma M]$, we have $a \le x\alpha a, a \le a\beta y$ for some $x, y \in M$, $\alpha, \beta \in \Gamma$. Then

$$a \le a\beta y \le (x\alpha a)\beta y = x\alpha(a\beta y) \in M\Gamma a\Gamma M, \text{ and } a \in (M\Gamma a\Gamma M].$$

Since $(M\Gamma a\Gamma M]$ is strongly regular, there exist $t \in (M\Gamma a\Gamma M](\subseteq M)$, $\delta, \rho \in \Gamma$ such that $a \le a\delta t\rho a$ and $a\gamma t = t\gamma a, \forall\, \gamma \in \Gamma$. That is, M is strongly regular. □

It is clear that the strongly regular po-Γ-semigroups are completely regular. By Theorem 2.16 and Lemma 2.19, we have the following:

2.20. Theorem. *A po-Γ-semigroup M is strongly regular if and only if the following conditions hold true:*

(1) *For every bi-ideal B of M, we have $B = (B\Gamma B]$.*
(2) *$(M\Gamma a\Gamma M]$ is a strongly regular sub-Γ-semigroup of M, $\forall\, a \in M$.* □

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