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EXTRAGRADIENT METHOD FOR VARIATIONAL INEQUALITIES

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Abstract

In this paper, we suggest and analyze a new extragradient iterative method, which is suggested by combining a modified extragradient method with the viscosity approximation method, for finding the common element of the set of fixed points of a countable family of nonexpansive mappings, and the solution set of the variational inequality in a Hilbert space. This new method includes the extragradient and viscosity methods as special cases. We also consider the strong convergence of the proposed method under some mild conditions. Several special cases are also discussed. Results proved in this paper may be viewed as an improvement and refinement of the previously known results.

Keywords: Variational inequalities, Extragradient method, Fixed point, Viscosity approximation method, α-inverse-strongly-monotone.

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1. Introduction

Variational inequality theory, which was introduced in 1960's by Stampacchia [28], has had a great impact and influence in the development of several branches of pure and applied sciences. The ideas and techniques of this theory are being used in a variety of diverse fields and have proved to be productive and innovative, see [1-32] and the references therein. It is now well-known that variational inequalities are equivalent to fixed-point problems, the origin of which can be traced back to Lions and Stampacchia [13]. This alternative formulation has been used to suggest and analyze projection iterative methods for solving variational inequalities under the conditions that the involved operator must be strongly monotone and Lipschitz continuous. These conditions are very strict and rule out its application in several important problems. To overcome this drawback, Korpelevich [12] suggested and analyzed the extragradient method by using the technique of updating the solution. It has been shown that if the underlying operator is only monotone and Lipschitz continuous, then the approximate solution converges to the exact solution. Noor [21] has shown the equivalence between the implicit iterative method and the extragradient method for solving variational inequalities. This equivalence has been used to show that the convergence of the extragradient method requires only the pseudomonotoncity of the operator. Noor's result is a significant improvement of the result of Korpelevich [12]. Moudafi [15] suggested and analyzed the so-called viscosity method for solving variational inequalities, which was modified by Xu [31]. Related to variational inequalities, we have the problem of finding the fixed points of nonexpansive mappings, which is of current interest in functional analysis. It is natural to consider a unified approach to these different problems, see, for example, [1, 5-6, 8-10, 12-14, 18].

Inspired and motivated by the research going on in these fields, we suggest and analyze a new extragradient method for finding the common element of a solution set of variational inequalities and the set of fixed-points of nonexpansive mappings, by combining the extragradient method of Noor and the modified viscosity method of Xu [31]. We consider the convergence analysis of this new method under some suitable conditions. Several special cases are discussed. Results proved in this paper may be viewed as an improvement and refinement of the previously known results.

To be more precise, let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H and T a mapping from C into H. A classical variational inequality problem, denoted by $\mathrm{VI}(T, C)$, is to find a vector $u^* \in C$ such that

$$(1.1) \qquad \langle u - u^*, T(u^*)\rangle \geq 0, \quad \forall\, u \in C,$$

which was introduced by Stampacchia [28] in 1964. For recent applications, numerical techniques and a physical formulation, see [1-32].

We now recall the following well-known result.

1.1. Lemma. *Let C be a closed and convex set in the real Hilbert space H. For a given $z \in H$, $u \in C$ satisfies the inequality*

$$\langle u - z, v - u\rangle \geq 0, \ \ v \in C,$$

if and only if

$$u = P_C z,$$

where P_C is the projection of H onto the closed convex set C. □

Using Lemma 1.1, one can easily show that the variational inequality (1.1) is equivalent to a fixed-point problem:

1.2. Lemma. *$u \in C$ is a solution of the variational inequality (1.1) if and only if $u \in C$ satisfies the relation*

$$u = P_C[u - \rho T u],$$

where $\rho > 0$ is a constant. □

Lemma 1.2 implies that the variational inequality (1.1) is equivalent to a fixed-point problem. This equivalent formulation has played a fundamental and significant part in the study of variational inequalities and related optimization problems. This equivalent formulation also enables one to study sensitivity analysis as well as the dynamic system associated with a variational inequality.

Using Lemma 1.2, one can suggest and analyze the following iterative methods for solving the variational inequality (1.1), the origin of which can be traced back to Lions and Stampacchia [13].

1.3. Algorithm. *For a given $u^0 \in C$, compute the approximate solution u^{k+1} by the iterative scheme*

$$u^{k+1} = P_C[u^k - \rho T u^k], \; k = 0, 1, 2 \dots.$$

It is well-known [2,7,20,25,27] that the convergence of Algorithm 1.3 requires that the operator T must be strongly monotone and Lipschitz continuous.

These strict conditions rule out its application to many important problems which arise in the pure and applied sciences.

We again use Lemma 1.2 to suggest and analyze the following iterative method for solving the variational inequality (1.1).

1.4. Algorithm. *For a given $u^0 \in C$, compute the approximate solution u^{k+1} by the iterative scheme*

$$u^{k+1} = P_C[u^k - \rho T u^{k+1}], \; k = 0, 1, 2 \dots.$$

Algorithm **??** is known as the implicit iterative method, which is itself difficult to implement. Noor [20] has used the predictor-corrector technique to modify the implicit iterative method by using Algorithm 1.3 as the predictor and Algorithm 1.4 as the corrector. Consequently, we have the following iterative method.

1.5. Algorithm. *For a given $u^0 \in C$, compute the approximate solution u^{k+1} by the iterative schemes*

$$w^k = P_C[u^k - \rho T u^k]$$

$$u^{k+1} = P_C[u^k - \rho T w^k], \; k = 0, 1, 2 \dots,$$

which is known as the extragradient method *and is due to Korpelevich* [12].

We note that the implicit iterative method and extragradient method are equivalent. This equivalence has been used by Noor [21, 20] to show that the convergence of the extragradient method requires only pseudomonotonicity. The original result of Korpelevich requires that the operator must be monotone Lipschitz continuous. Clearly, Noor's result represents a significant improvement of the previously known result.

We also recall the following well-known concepts, which play a crucial part in our analysis.

Let T be α-inverse-strongly-monotone from C into H, that is, there exists an $\alpha > 0$ such that

$$\langle T(u) - T(v), u - v \rangle \geq \alpha \|T(u) - T(v)\|^2, \; \forall\, u, v \in C. \tag{1.2}$$

An operator S is said to be a *nonexpansive mapping* from C into itself, if and only if,

$$\|S(u) - S(v)\| \leq \|u - v\|, \ \forall\, u, v \in C.$$

We denote by S^* and $F(S)$ the solution set of Problem (1.1) and the set of fixed points of S, respectively.

For finding an element of $F(S) \cap S^*$ under the assumption that the set $C \subset H$ is nonempty, closed and convex, the mapping S of C into itself is nonexpansive, and the mapping T of C into H is α-inverse-strongly-monotone, Takahashi and Toyoda [30] introduced the following iterative scheme:

$$u^{k+1} = \alpha_k u^k + (1 - \alpha_k) S(P_C[u^k - \rho_k T(u^k)]), \ k \geq 0, \tag{1.3}$$

where $u^0 = u \in C$, $\{\alpha_k\}$ is a sequence in $(0, 1)$ and $\{\rho_k\}$ a sequence in $(0, 2\alpha)$. They proved that if $F(S) \cap S^* \neq \emptyset$, then the sequence $\{u^k\}$ generated by (1.3) converges weakly to some $z \in F(S) \cap S^*$.

Recently, Iiduka and Takahashi [10] presented another iterative scheme for finding an element of $F(S) \cap S^*$:

$$u^{k+1} = \alpha_k u + (1 - \alpha_k) S(P_C[u^k - \rho_k T(u^k)]), \ k \geq 1,$$

where $u^1 = u \in C$, $\{\alpha_k\}$ is a sequence in $[0, 1)$ and $\{\rho_k\}$ a sequence in $[0, 2\alpha]$. If $\{\alpha_k\}$ and $\{\rho_k\}$ are chosen that $\rho_k \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$,

$$\lim_{k\to\infty} \alpha_k = 0, \quad \sum_{k=1}^{\infty} \alpha_k = \infty, \quad \sum_{k=1}^{\infty} |\alpha_{k+1} - \alpha_k| < \infty, \text{ and } \sum_{k=1}^{\infty} |\rho_{k+1} - \rho_k| < \infty.$$

They showed that $\{u^k\}$ converges strongly to $P_{F(S)\cap S^*}[u]$.

Using the extragradient method (Algorithm 1.3) of Korpelevich [12], Nadezhkina and Takahasaki [16] introduced an iterative process for finding an element of $F(S) \cap S^*$ and they proved that the sequence converges weakly to a common element of the two sets, while Zeng and Yao [32] presented another iterative scheme for finding an element of $F(S) \cap S^*$ and they proved that the two sequences generated by method converge strongly to a common element of the two sets under the following condition

$$\lim_{k\to\infty} \|u^{k+1} - u^k\| = 0.$$

Recall that a self-mapping $f : C \longrightarrow C$ is said to be a c_1-contraction if $c_1 \in [0, 1]$ and

$$\|f(u) - f(v)\| \leq c_1 \|u - v\|, \ \forall u, v \in C.$$

Xu [31] proposed a viscosity approximating method for nonexpansive mappings which can be viewed as an improved extension of Moudafi's method [15]. The method of Xu is as follows:

$$u^{k+1} = \alpha_k f(u^k) + (1 - \alpha_k) S(u^k), \ k \geq 0,$$

where $u^0 \in C$, $\{\alpha_k\}$ is a sequence in $[0, 1)$. If $\{\alpha_k\}$ satisfies

$$\lim_{k\to\infty} \alpha_k = 0, \ \sum_{k=1}^{\infty} \alpha_k = \infty, \ \sum_{k=1}^{\infty} |\alpha_{k+1} - \alpha_k| < \infty \text{ or } \lim_{k\to\infty} \frac{\alpha_{k+1}}{\alpha_k} = 1,$$

then $\{u^k\}$ converges strongly to u, where u is the unique solution in $F(S)$ of the following variational inequality:

$$\langle f(u) - u, v - u \rangle \leq 0, \ \forall\, v \in F(S).$$

Chen *et al.*[3] proposed another iterative scheme for finding an element of $F(S) \cap S^*$:

$$u^{k+1} = \alpha_k f(u^k) + (1 - \alpha_k) S(P_C[u^k - \rho_k T(u^k)]), \ k \geq 0,$$

where $u^0 \in C$, $\{\alpha_k\}$ is a sequence in $[0,1)$ and $\{\rho_k\}$ a sequence in $[0,2\alpha]$. If $\{\alpha_k\}$ and $\{\rho_k\}$ are chosen such that $\rho_k \in [a,b]$ for some a,b with $0 < a < b < 2\alpha$,

$$\lim_{k\to\infty} \alpha_k = 0,\ \sum_{k=1}^{\infty} \alpha_k = \infty,\ \sum_{k=1}^{\infty} |\alpha_{k+1} - \alpha_k| < \infty \text{ and } \sum_{k=1}^{\infty} |\rho_{k+1} - \rho_k| < \infty,$$

then $\{u^k\}$ converges strongly to u, where u is the unique solution in $F(S) \cap S^*$ of the following variational inequality:

$$\langle f(u) - u, v - u\rangle \le 0, \quad \forall v \in F(S).$$

On the other hand, while finding a common fixed point of a countable family of nonexpansive mappings in a Banach space, Aoyama *et al.* [1] introduced a Halpern type iteration for such family as follows:

$$u^1 = u \in C, \quad u^{k+1} = \alpha_k u + (1-\alpha_k) S_k(u^k),\ k \ge 2,$$

where C is a nonempty closed convex subset of a Banach space, $\{\alpha_k\}$ is a sequence in $[0,1)$ and $\{S_k\}$ a sequence of nonexpansive mappings of C into itself with some appropriate conditions, and then they proved that the sequence $\{u^k\}$ converges strongly to a common fixed point of $\{S_k\}$.

Inspired and motivated by the above research, we suggest and analyze a new iterative scheme for finding a common element of the set of fixed points of a countable family of nonexpansive mappings and the solution set of a variational inequality for an inverse strongly monotone mapping in a Hilbert space. Under mild assumptions, we obtain a strong convergence theorem for two sequences generated by the proposed method. We would like to mention that our proposed method is quite general and flexible and includes the iterative methods considered by Takahashi and Toyoda [30], Iiduka and Takahashi [10], Nadezhkina and Takahasaki [16], Zeng and Yao [32], Chen *et al.* [3], Aoyama *et al.*[1], and several other iterative methods as special cases. The results proved in this paper continue to hold for these problems.

2. Preliminaries

In this section, we list some fundamental lemmas that are useful in the consequent analysis. The first lemma provides some basic properties of projections onto C.

2.1. Lemma. *Let P_C denote the projection of H onto C. Then, we have the following inequalities.*

$$\langle z - P_C[z], P_C[z] - v\rangle \ge 0,\ \forall\, z \in H,\ v \in C; \tag{2.1}$$

$$\|P_C[u] - P_C[v]\| \le \|u - v\|,\ \forall\, u,v \in H; \tag{2.2}$$

$$\|u - P_C[z]\|^2 \le \|z - u\|^2 - \|z - P_C[z]\|^2,\ \forall z \in H,\ u \in C. \tag{2.3}$$

□

2.2. Lemma. *If T is α-inverse-strongly monotone in H and $0 < \lambda \le 2\alpha$, then $I - \lambda T$ is a nonexpansive mapping in H.*

Proof. For all $u, v \in H$, we have

$$\begin{aligned}\|(I-\lambda T)(u) - (I-\lambda T)(v)\|^2 &= \|(u-v) - \lambda(T(u) - T(v))\|^2 \\ &= \|u-v\|^2 - 2\lambda\langle u-v, T(u)-T(v)\rangle \\ &\qquad + \lambda^2\|T(u)-T(v)\|^2 \\ &\le \|u-v\|^2 + \lambda(\lambda - 2\alpha)\|T(u)-T(v)\|^2 \\ &\le \|u-v\|^2.\end{aligned}$$

□

2.3. Lemma. [29] *Let* $\{x_k\}$ *and* $\{y_k\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_k\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{k\to\infty} \beta_k \le \limsup_{k\to\infty} \beta_k < 1$. *Suppose that* $x_{k+1} = (1-\beta_k)y_k + \beta_k x_k$ *for all integers* $k \ge 0$ *and* $\limsup_{k\to\infty}(\|y_{k+1} - y_k\| - \|x_{k+1} - x_k\|) \le 0$. *Then,* $\lim_{k\to\infty} \|y_k - x_k\| = 0$. □

2.4. Lemma. [14] *Let* H *be a real Banach space. Then the following inequality holds:*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \ \forall\, x, y \in H.$$ □

2.5. Lemma. [31] *Assume* $\{a_k\}$ *is a sequence of nonnegative real numbers such that*

$$a_{k+1} \le (1-\sigma_k)a_k + \delta_k,$$

where $\{\sigma_k\}$ *is a sequence in* $(0,1)$ *and* δ_k *is a sequence such that*

(1) $\sum_{k=1}^{\infty} \sigma_k = \infty$;

(2) $\limsup_{k\to\infty} \delta_k/\sigma_k \le 0$ *or* $\sum_{k=1}^{\infty} |\delta_k| < \infty$.

Then $\lim_{k\to\infty} a_k = 0$. □

2.6. Lemma. [1] *Let* C *be a nonempty closed convex subset of a Banach space* E. *Let* $S_1, S_2, \dots$ *be a sequence of mappings of* C *into itself. Suppose that* $\sum_{k=1}^{\infty} \sup\{\|S_{k+1}(x) - S_k(x)\|\| : x \in C\} < \infty$. *Then for each* $y \in C$, $\{S_k(y)\}$ *converges strongly to some point of* C. *Moreover, let* S *be the mapping of* C *into itself defined by* $Sy = \lim_{k\to\infty} S_k(y)\ \forall\, y \in C$. *Then* $\limsup_{k\to\infty}\{\|S(x) - S_k(x)\|\| : x \in C\} = 0$. □

A set-valued mapping $T : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $u \in T(x)$ and $v \in T(y)$ imply $\langle x - y, u - v\rangle \ge 0$. A monotone mapping $T : H \to 2^H$ is *maximal* if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if, for $(x, u) \in H \times H$, $\langle x - y, u - v\rangle \ge 0$ for every $(y, v) \in G(T)$ implies $u \in T(x)$.

Let T be a monotone mapping of C into H and let $N_C(\,\cdot\,)$ be the normal cone operator to C defined by

$$N_C(v) := \{w \in H : \langle w, v - u\rangle \ge 0, \ \forall\, u \in C\}.$$

Define

$$A(v) = \begin{cases} T(v) + N_C(v), & \forall\, v \in C, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then A is maximal monotone and $0 \in A(v)$ if and only if $v \in S^*$ (see [27]).

It is also known that H satisfies Opial's condition [24], i.e., for any sequence $\{u^k\}$ with $u^k \rightharpoonup u$, the inequality

$$\liminf_{k\to\infty} \|\tilde{u}^k - u\| < \liminf_{k\to\infty} \|\tilde{u}^k - v\|$$

holds for every $v \in H$ with $u \ne v$.

3. New extragradient method

In this Section, we suggest and analyze a modified extragradient method for finding the common element of the set of fixed points of a countable family of nonexpansive mappings $\{S_k\}$ and the solution set of the corresponding variational inequalities. Let T be α-inverse-strongly-monotone from C into H and $\{S_k\}$ a sequence of nonexpansive mapping from C into itself. Suppose that $\sum_{k=1}^{\infty} \sup\{\|S_{k+1}(x) - S_k(x)\|\| : x \in C\} < \infty$.

Let S be the mapping of C into itself defined by $Sy = \lim_{k\to\infty} S_k(y),\ \forall y \in C$, such that $F(S) = \bigcap_{k=1}^{\infty} F(S_k)$ and $F(S) \cap S^* \neq \emptyset$.

3.1. Algorithm. *For a given $u^0 \in C$, find the approximate solution u^{k+1} by the iterative schemes;*

$$\tilde{u}^k = P_C[u^k - \rho_k T(u^k)], \tag{3.1}$$

$$u^{k+1} = \alpha_k f(u^k) + \beta_k u^k + \gamma_k S_k \left(P_C[u^k - \rho_k T(\tilde{u}^k)] \right), \tag{3.2}$$

where $f : C \longrightarrow C$ is a c_1-contraction and the sequences $\{\rho_k\}$, $\{\alpha_k\}$, $\{\beta_k\}$ in $(0,1)$ satisfy the following conditions:

(i) $\lim_{k\to\infty} \alpha_k = 0,\ \sum_{k=1}^{\infty} \alpha_k = \infty$;
(ii) $\{\rho_k/\alpha\} \subset (\tau, 1-\delta)$ *for some* $\tau, \delta \in (0,1),\ \lim_{k\to\infty} \rho_k = 0$;
(iii) $0 < \liminf_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \beta_k < 1$.
(iv) $\alpha_k + \beta_k + \gamma_k = 1$.

Clearly for $\alpha_k = \beta_k = \gamma_k = 0$, Algorithm 3.1 is exactly Algorithm 1.3, and for $\alpha_k = \beta_k = 0,\ \gamma_k = 1$ and $S_k \equiv I$, the identity operator, Algorithm 3.1 is just the extragradient method (Algorithm 1.5). In a similar way for different choices of the parameters, one can obtain several iterative methods for solving variational inequalities and nonexpansive mappings. This show that Algorithm 3.1 is quite general and unifying.

3.2. Lemma. *Let $\bar{u}^k = P_C[u^k - \rho_k T(\tilde{u}^k)]$ and $u^* \in F(S) \cap S^*$. Then we have*

$$\|\bar{u}^k - u^*\|^2 \leq \|u^k - u^*\|^2 + (\frac{\rho_k^2}{\alpha^2} - 1)\|u^k - \tilde{u}^k\|^2 \tag{3.3}$$

$$\leq \|u^k - u^*\|^2, \tag{3.4}$$

and $\{u^k\}$ is bounded.

Proof. Since $u^* \in F(S) \cap S^*$, then $u^* = P_C[u^* - \rho_k T(u^*)]$. It follows from (2.3) that

$$\begin{aligned}
\|\bar{u}^k - u^*\|^2 &\leq \|u^k - \rho_k T(\tilde{u}^k) - u^*\|^2 - \|u^k - \rho_k T(\tilde{u}^k) - \bar{u}^k\|^2 \\
&= \|u^k - u^*\|^2 - \|u^k - \bar{u}^k\|^2 + 2\rho_k \langle T(\tilde{u}^k), u^* - \bar{u}^k \rangle \\
&= \|u^k - u^*\|^2 - \|u^k - \bar{u}^k\|^2 + 2\rho_k \langle T(\tilde{u}^k) - T(u^*), u^* - \tilde{u}^k \rangle \\
&\quad + 2\rho_k \langle T(u^*), u^* - \tilde{u}^k \rangle + 2\rho_k \langle T(\tilde{u}^k), \tilde{u}^k - \bar{u}^k \rangle \\
&\leq \|u^k - u^*\|^2 - \|u^k - \bar{u}^k\|^2 + 2\rho_k \langle T(\tilde{u}^k), \tilde{u}^k - \bar{u}^k \rangle \\
&= \|u^k - u^*\|^2 - \|u^k - \tilde{u}^k\|^2 - 2\langle u^k - \tilde{u}^k, \tilde{u}^k - \bar{u}^k \rangle - \|\tilde{u}^k - \bar{u}^k\|^2 \\
&\quad + 2\rho_k \langle T(\tilde{u}^k), \tilde{u}^k - \bar{u}^k \rangle \\
&= \|u^k - u^*\|^2 - \|u^k - \tilde{u}^k\|^2 - \|\tilde{u}^k - \bar{u}^k\|^2 + 2\langle u^k - \rho_k T(\tilde{u}^k) - \tilde{u}^k, \\
&\qquad \bar{u}^k - \tilde{u}^k \rangle,
\end{aligned}$$

where the second inequality follows because T is α-inverse-strongly-monotone and u^* is a solution of (1.1).

By using (2.1), we have

$$\begin{aligned}\langle u^k - \rho_k T(\tilde{u}^k) - \tilde{u}^k, \bar{u}^k - \tilde{u}^k\rangle &= \langle u^k - \rho_k T(u^k) - \tilde{u}^k, \bar{u}^k - \tilde{u}^k\rangle + \langle \rho_k T(u^k) \\ &\qquad - \rho_k T(\tilde{u}^k), \bar{u}^k - \tilde{u}^k\rangle \\ &\le \langle \rho_k T(u^k) - \rho_k T(\tilde{u}^k), \bar{u}^k - \tilde{u}^k\rangle \\ &\le \frac{\rho_k}{\alpha}\|u^k - \tilde{u}^k\|\|\bar{u}^k - \tilde{u}^k\|.\end{aligned}$$

Then, we obtain

$$\begin{aligned}\|\bar{u}^k - u^*\|^2 &\le \|u^k - u^*\|^2 - \|u^k - \tilde{u}^k\|^2 - \|\tilde{u}^k - \bar{u}^k\|^2 + 2\frac{\rho_k}{\alpha}\|u^k - \tilde{u}^k\|\|\bar{u}^k - \tilde{u}^k\| \\ &\le \|u^k - u^*\|^2 - \|u^k - \tilde{u}^k\|^2 - \|\tilde{u}^k - \bar{u}^k\|^2 + \frac{\rho_k^2}{\alpha^2}\|u^k - \tilde{u}^k\|^2 \\ &\qquad + \|\bar{u}^k - \tilde{u}^k\|^2 \\ &\le \|u^k - u^*\|^2 + (\frac{\rho_k^2}{\alpha^2} - 1)\|u^k - \tilde{u}^k\|^2 \\ &\le \|u^k - u^*\|^2. \qquad (3.5)\end{aligned}$$

Now, we prove the second part of the lemma. Since S_k is a nonexpansive mapping from C into itself and, from (3.2) and (3.5), we get

$$\begin{aligned}\|u^{k+1} - u^*\| &= \|\alpha_k(f(u^k) - u^*) + \beta_k(u^k - u^*) + \gamma_k(S_k(\bar{u}^k) - u^*)\| \\ &\le \alpha_k\|f(u^k) - f(u^*)\| + \alpha_k\|f(u^*) - u^*\| + \beta_k\|u^k - u^*\| \\ &\qquad + \gamma_k\|\bar{u}^k - u^*\| \\ &\le \alpha_k c_1\|u^k - u^*\| + \alpha_k\|f(u^*) - u^*\| + \beta_k\|u^k - u^*\| \\ &\qquad + (1 - \beta_k - \alpha_k)\|u^k - u^*\| \\ &\le (1 - (1 - c_1)\alpha_k)\|u^k - u^*\| + \alpha_k\|f(u^*) - u^*\| \\ &\le \max\{\|u^k - u^*\|, \frac{1}{1 - c_1}\|f(u^*) - u^*\|\}.\end{aligned}$$

It follows from induction that

$$\|u^{k+1} - u^*\| \le \max\{\|u^0 - u^*\|, \frac{1}{1 - c_1}\|f(u^*) - u^*\|\}.$$

Therefore, $\{u^k\}$ is bounded. Hence $\{\bar{u}^k\}$, $\{T(u^k)\}$ and $\{T(\tilde{u}^k)\}$ are also bounded. □

3.3. Lemma. *The sequence $\{u^k\}$ generated by Algorithm 3.1 satisfies the following condition:*

$$\lim_{k\to\infty}\|u^{k+1} - u^k\| = 0. \qquad (3.6)$$

Proof. From (2.2) and Lemma 2.2, we have

$$\begin{aligned}\|\bar{u}^{k+1}-\bar{u}^{k}\| &\le \|P_C[u^{k+1}-\rho_{k+1}T(\tilde{u}^{k+1})]-P_C[u^{k}-\rho_k T(\tilde{u}^{k})]\| \\ &\le \|(u^{k+1}-\rho_{k+1}T(\tilde{u}^{k+1}))-(u^{k}-\rho_k T(\tilde{u}^{k}))\| \\ &\le \|(u^{k+1}-\rho_{k+1}T(u^{k+1}))-(u^{k}-\rho_{k+1}T(u^{k}))+\rho_{k+1}\{T(u^{k+1}) \\ &\qquad -T(u^{k})-T(\tilde{u}^{k+1})\}+\rho_k T(\tilde{u}^{k})\| \\ &\le \|(u^{k+1}-\rho_{k+1}T(u^{k+1}))-(u^{k}-\rho_{k+1}T(u^{k}))\| \\ &\qquad +\rho_{k+1}\{\|T(u^{k+1})\|+\|T(u^{k})\|+\|T(\tilde{u}^{k+1})\|\}+\rho_k\|T(\tilde{u}^{k})\| \\ &\le \|u^{k+1}-u^{k}\|+\rho_{k+1}\{\|T(u^{k+1})\|+\|T(u^{k})\|+\|T(\tilde{u}^{k+1})\|\} \\ &\qquad +\rho_k\|T(\tilde{u}^{k})\|. \end{aligned} \tag{3.7}$$

Define a sequence $\{v^k\}$ by $v^k=\frac{u^{k+1}-\beta_k u^k}{1-\beta_k}$, $\forall\, k\ge 1$. Then we have

$$\begin{aligned}\|v^{k+1}-v^{k}\| &= \left\|\frac{u^{k+2}-\beta_{k+1}u^{k+1}}{1-\beta_{k+1}}-\frac{u^{k+1}-\beta_k u^k}{1-\beta_k}\right\| \\ &= \left\|\frac{\alpha_{k+1}f(u^{k+1})+(1-\alpha_{k+1}-\beta_{k+1})S_{k+1}(\bar{u}^{k+1})}{1-\beta_{k+1}}\right. \\ &\qquad\qquad \left. -\frac{\alpha_k f(u^k)+(1-\alpha_k-\beta_k)S_k(\bar{u}^k)}{1-\beta_k}\right\| \\ &\le \frac{\alpha_{k+1}}{1-\beta_{k+1}}\big(\|f(u^{k+1})\|+\|S_{k+1}(\bar{u}^{k+1})\|\big)+\frac{\alpha_k}{1-\beta_k}\big(\|f(u^k)\| \\ &\qquad +\|S_k(\bar{u}^k)\|\big)+\|S_{k+1}(\bar{u}^{k+1})-S_k(\bar{u}^{k+1})\| \end{aligned} \tag{3.8}$$

and from (3.7), we obtain

$$\begin{aligned}\|S_{k+1}(\bar{u}^{k+1})-S_k(\bar{u}^{k})\| &= \|S_{k+1}(\bar{u}^{k+1})-S_{k+1}(\bar{u}^{k})\|+\|S_{k+1}(\bar{u}^{k})-S_k(\bar{u}^{k})\| \\ &\le \|\bar{u}^{k+1}-\bar{u}^{k}\|+\|S_{k+1}(\bar{u}^{k})-S_k(\bar{u}^{k})\| \\ &\le \|u^{k+1}-u^{k}\|+\rho_{k+1}\big\{\|T(u^{k+1})\|+\|T(u^{k})\| \\ &\qquad +\|T(\tilde{u}^{k+1})\|\big\}+\rho_k\|T(\tilde{u}^{k})\|+\|S_{k+1}(\bar{u}^{k})-S_k(\bar{u}^{k})\| \\ &\le \|u^{k+1}-u^{k}\|+(\rho_{k+1}+\rho_k)M+\|S_{k+1}(\bar{u}^{k})-S_k(\bar{u}^{k})\|, \end{aligned} \tag{3.9}$$

where $M=\sup_{k\ge 1}\{\|T(u^{k+1})\|+\|T(u^k)\|+\|T(\tilde{u}^{k+1})\|+\|T(\tilde{u}^k)\|\}$. Combining (3.8) and (3.9), we get

$$\begin{aligned}&\|v^{k+1}-v^{k}\|-\|u^{k+1}-u^{k}\| \\ &\quad\le \frac{\alpha_{k+1}}{1-\beta_{k+1}}\big(\|f(u^{k+1})\|+\|S_{k+1}(\bar{u}^{k+1})\|\big)+\frac{\alpha_k}{1-\beta_k}(\|f(u^k)\|+\|S_k(\bar{u}^k)\|) \\ &\qquad\quad +(\rho_{k+1}+\rho_k)M+\|S_{k+1}(\bar{u}^{k})-S_k(\bar{u}^{k})\| \\ &\quad\le \frac{\alpha_{k+1}}{1-\beta_{k+1}}\big(\|f(u^{k+1})\|+\|S_{k+1}(\bar{u}^{k+1})\|\big)+\frac{\alpha_k}{1-\beta_k}\big(\|f(u^k)\|+\|S_k(\bar{u}^k)\|\big) \\ &\qquad\quad +(\rho_{k+1}+\rho_k)M+\sup\big\{\|S_{k+1}(x)-S_k(x)\|: x\in\{\bar{u}^k\}\big\},\end{aligned}$$

which implies that

$$\limsup_{k\to\infty}(\|v^{k+1}-v^k\|-\|u^{k+1}-u^k\|)\le 0.$$

And by Lemma 2.3, we obtain $\|v^k-u^k\|\to 0$ as $k\to\infty$. Therefore,

$$\lim_{k\to\infty}\|u^{k+1}-u^k\|=\lim_{k\to\infty}(1-\beta_k)\|v^k-u^k\|=0. \qquad \square$$

4. Convergence analysis

In this section, convergence of the new method is proved under mild assumptions. The following theorem plays an important role in the convergence analysis.

4.1. Theorem. *Let $\tilde{u}^k$ be defined by (3.1). Then we have*

$$\lim_{k\to\infty} \|S(\tilde{u}^k) - \tilde{u}^k\| = 0.$$

Proof. Since S_k is a nonexpansive mapping from C into itself, from Lemma 3.2 and (3.2), we obtain

$$\begin{aligned}
\|u^{k+1} - u^*\|^2 &= \|\alpha_k(f(u^k) - u^*) + \beta_k(u^k - u^*) + \gamma_k(S(\bar{u}^k) - u^*)\|^2 \\
&\le \alpha_k\|f(u^k) - u^*\|^2 + \beta_k\|u^k - u^*\|^2 + \gamma_k\|\bar{u}^k - u^*\|^2 \\
&\le \alpha_k\|f(u^k) - u^*\|^2 + \beta_k\|u^k - u^*\|^2 + \gamma_k\|u^k - u^*\|^2 \\
&\qquad + \gamma_k\left(\frac{\rho_k^2}{\alpha^2} - 1\right)\|u^k - \tilde{u}^k\|^2 \\
&= \alpha_k\|f(u^k) - u^*\|^2 + (1-\alpha_k)\|u^k - u^*\|^2 \\
&\qquad + \gamma_k\left(\frac{\rho_k^2}{\alpha^2} - 1\right)\|u^k - \tilde{u}^k\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
\gamma_k\delta\|u^k - \tilde{u}^k\|^2 &\le \gamma_k\left(1 - \frac{\rho_k^2}{\alpha^2}\right)\|u^k - \tilde{u}^k\|^2 \\
&\le \alpha_k\big(\|f(u^k) - u^*\|^2 - \|u^k - u^*\|^2\big) + \|u^k - u^*\|^2 - \|u^{k+1} - u^*\|^2 \\
&= \alpha_k\big(\|f(u^k) - u^*\|^2 - \|u^k - u^*\|^2\big) + \big(\|u^k - u^*\| + \|u^{k+1} - u^*\|\big) \\
&\qquad \times \big(\|u^k - u^*\| - \|u^{k+1} - u^*\|\big) \\
&\le \alpha_k\big(\|f(u^k) - u^*\|^2 - \|u^k - u^*\|^2\big) + \big(\|u^k - u^*\| + \|u^{k+1} - u^*\|\big) \\
&\qquad \times \|u^k - u^{k+1}\|.
\end{aligned}$$

Since $\lim_{k\to\infty}\|u^{k+1} - u^k\| = 0$, $\{u^k\}$ is bounded and $\lim_{k\to\infty}\alpha_k = 0$, we have

$$\lim_{k\to\infty}\|\tilde{u}^k - u^k\| = 0. \tag{4.1}$$

Note that

$$\begin{aligned}
\|\bar{u}^k - \tilde{u}^k\| &= \|P_C[u^k - \rho_k T(\tilde{u}^k)] - P_C[u^k - \rho_k T(u^k)]\| \\
&\le \rho_k\|T(\tilde{u}^k) - T(u^k)\| \\
&\le \frac{\rho_k}{\alpha}\|\tilde{u}^k - u^k\|.
\end{aligned}$$

Then, we get

$$\lim_{k\to\infty}\|\bar{u}^k - \tilde{u}^k\| = 0. \tag{4.2}$$

It follows from (3.2) that

$$\begin{aligned}
\|u^k - S_k(\bar{u}^k)\| &\le \|u^{k+1} - u^k\| + \|u^{k+1} - S_k(\bar{u}^k)\| \\
&\le \|u^{k+1} - u^k\| + \alpha_k\|f(u^k) - S_k(\bar{u}^k)\| + \beta_k\|u^k - S_k(\bar{u}^k)\|,
\end{aligned}$$

which implies that

$$\|u^k - S_k(\bar{u}^k)\| \le \frac{1}{1-\beta_k}\big(\|u^{k+1} - u^k\| + \alpha_k\|f(u^k) - S_k(\bar{u}^k)\|\big).$$

Since $\lim_{k\to\infty}\|u^{k+1}-u^k\|=0$ and $\lim_{k\to\infty}\alpha_k=0$, we have

$$\lim_{k\to\infty}\|u^k-S_k(\bar{u}^k)\|=0. \tag{4.3}$$

Also

$$\begin{aligned}\|S_k(\tilde{u}^k)-\tilde{u}^k\| &\le \|S_k(\tilde{u}^k)-S_k(\bar{u}^k)\|+\|S_k(\bar{u}^k)-u^k\|+\|u^k-\tilde{u}^k\|\\ &\le \|\tilde{u}^k-\bar{u}^k\|+\|S_k(\bar{u}^k)-u^k\|+\|u^k-\tilde{u}^k\|,\end{aligned}$$

and we can conclude that $\|S_k(\tilde{u}^k)-\tilde{u}^k\|\to 0$ as $k\to\infty$. Using Lemma 2.6 and (4.3), we obtain

$$\begin{aligned}\|S(\tilde{u}^k)-\tilde{u}^k\| &\le \|S(\tilde{u}^k)-S_k(\tilde{u}^k)\|+\|S_k(\tilde{u}^k)-\tilde{u}^k\|\\ &\le \sup\{\|S(x)-S_k(x)\| : x\in\{\tilde{u}^k\}\}+\|S_k(\tilde{u}^k)-\tilde{u}^k\|,\end{aligned}$$

which implies that $\|S(\tilde{u}^k)-\tilde{u}^k\|\to 0$ as $k\to\infty$. □

4.2. Theorem. *The sequences $\{u^k\}$ and $\{\tilde{u}^k\}$ generated by the proposed method converge strongly to the same point $u=P_{F(S)\cap S^*}[f(u)]$.*

Proof. First we show that

$$\limsup_{k\to\infty}\langle f(u)-u,\tilde{u}^k-u\rangle\le 0.$$

In order to show the above result, we choose a subsequence $\{\tilde{u}^{k_i}\}$ of $\{\tilde{u}^k\}$ such that

$$\limsup_{k\to\infty}\langle f(u)-u,S(\tilde{u}^k)-u\rangle=\lim_{i\to\infty}\langle f(u)-u,S(\tilde{u}^{k_i})-u\rangle.$$

As $\tilde{u}^{k_i}$ is bounded, we have that a subsequence $\{\tilde{u}^{k_{i_j}}\}$ of $\{\tilde{u}^{k_i}\}$ converges weakly to $\hat{u}$. We may assume without loss of generality that $\tilde{u}^{k_i}\rightharpoonup\hat{u}$. Since $\|S(\tilde{u}^k)-\tilde{u}^k\|\to 0$, we obtain $S(\tilde{u}^{k_i})\rightharpoonup\hat{u}$ as $i\to\infty$.

Now, we show that $\hat{u}\in F(S)\cap S^*$. First we prove that $\hat{u}\in S^*$.

Let

$$A(v)=\begin{cases}T(v)+N_C(v), & \forall\, v\in C,\\ \emptyset, & \text{otherwise.}\end{cases}$$

Then A is maximal monotone. Let $(v,w)\in G(A)$. Since $w-T(v)\in N_C(v)$ and $\tilde{u}^k\in C$, we have $\langle v-\tilde{u}^k, w-T(v)\rangle\ge 0$. On the other hand, from $\tilde{u}^k=P_C[u^k-\rho_kT(u^k)]$, we have $\langle v-\tilde{u}^k,\tilde{u}^k-(u^k-\rho_kT(u^k))\rangle\ge 0$, then

$$\left\langle v-\tilde{u}^k,\frac{\tilde{u}^k-u^k}{\rho_k}+T(u^k)\right\rangle\ge 0.$$

Therefore, we have

$$\begin{aligned}\langle v-\tilde{u}^{k_i}, w\rangle &\geq \langle v-\tilde{u}^{k_i}, T(v)\rangle\\ &\geq \langle v-\tilde{u}^{k_i}, T(v)\rangle - \left\langle v-\tilde{u}^{k_i}, \frac{\tilde{u}^{k_i}-u^{k_i}}{\rho_{k_i}} + T(u^{k_i})\right\rangle\\ &= \left\langle v-\tilde{u}^{k_i}, T(v)-T(u^{k_i}) - \frac{\tilde{u}^{k_i}-u^{k_i}}{\rho_{k_i}}\right\rangle\\ &= \langle v-\tilde{u}^{k_i}, T(v)-T(\tilde{u}^{k_i})\rangle + \langle v-\tilde{u}^{k_i}, T(\tilde{u}^{k_i})-T(u^{k_i})\rangle\\ &\qquad - \left\langle v-\tilde{u}^{k_i}, \frac{\tilde{u}^{k_i}-u^{k_i}}{\rho_{k_i}}\right\rangle\\ &\geq \langle v-\tilde{u}^{k_i}, T(\tilde{u}^{k_i})-T(u^{k_i})\rangle - \left\langle v-\tilde{u}^{k_i}, \frac{\tilde{u}^{k_i}-u^{k_i}}{\rho_{k_i}}\right\rangle.\end{aligned}$$

Hence, we obtain $\langle v-\hat{u}, w\rangle \geq 0$ as $i \to \infty$. Since A is maximal monotone, we have $\hat{u} \in A^{-1}(0)$ and hence $\hat{u} \in S^*$. Let us now show that $\hat{u} \in F(S)$. Assume $\hat{u} \notin F(S)$. From Opial's condition, we have

$$\begin{aligned}\liminf_{i\to\infty} \|\tilde{u}^{k_i}-\hat{u}\| &< \liminf_{i\to\infty} \|\tilde{u}^{k_i}-S(\hat{u})\|\\ &= \liminf_{i\to\infty} \|\tilde{u}^{k_i}-S(\tilde{u}^{k_i})+S(\tilde{u}^{k_i})-S(\hat{u})\|\\ &\leq \liminf_{i\to\infty} \|S(\tilde{u}^{k_i})-S(\hat{u})\|\\ &\leq \liminf_{i\to\infty} \|\tilde{u}^{k_i}-\hat{u}\|.\end{aligned}$$

This is a contradiction. Thus, we obtain $\hat{u} \in F(S)$. Then, we have

$$\begin{aligned}\limsup_{k\to\infty}\langle f(u)-u, \tilde{u}^k-u\rangle &= \limsup_{k\to\infty} \langle f(u)-u, S(\tilde{u}^k)-u\rangle\\ &= \lim_{i\to\infty} \langle f(u)-u, S(\tilde{u}^{k_i})-u\rangle\\ &= \langle f(u)-u, \hat{u}-u\rangle\\ &\leq 0.\end{aligned}$$

Therefore, from (4.1) we have

$$\limsup_{k\to\infty}\langle f(u)-u, u^k-u\rangle \leq 0. \tag{4.4}$$

It follows from Lemma 2.4 and Lemma 3.2 that

$$\begin{aligned}\|u^{k+1}-u\|^2 &= \|\alpha_k(f(u^k)-u)+\beta_k(u^k-u)+\gamma_k(S(\bar{u}^k)-u)\|^2\\ &\leq \|\beta_k(u^k-u)+\gamma_k(S(\bar{u}^k)-u)\|^2 + 2\alpha_k\langle f(u^k)-u, u^{k+1}-u\rangle\\ &\leq [\beta_k\|u^k-u\|+\gamma_k\|\bar{u}^k-u\|]^2 + 2\alpha_k\langle f(u^k)-u, u^{k+1}-u\rangle\\ &\leq [\beta_k\|u^k-u\|+\gamma_k\|u^k-u\|]^2 + 2\alpha_k\langle f(u^k)-f(u), u^{k+1}-u\rangle\\ &\qquad + 2\alpha_k\langle f(u)-u, u^{k+1}-u\rangle\\ &\leq (1-\alpha_k)^2\|u^k-u\|^2 + 2\alpha_k c_1\|u^k-u\|\|u^{k+1}-u\|\\ &\qquad + 2\alpha_k\langle f(u)-u, u^{k+1}-u\rangle\\ &\leq (1-\alpha_k)^2\|u^k-u\|^2 + \alpha_k c_1(\|u^k-u\|^2+\|u^{k+1}-u\|^2)\\ &\qquad + 2\alpha_k\langle f(u)-u, u^{k+1}-u\rangle\end{aligned}$$

$$\begin{aligned}\|u^{k+1}-u\|^2 &\le \frac{(1-\alpha_k)^2}{1-\alpha_k c_1}\|u^k-u\|^2+\frac{\alpha_k c_1}{1-\alpha_k c_1}\|u^k-u\|^2\\ &\qquad +\frac{2\alpha_k}{1-\alpha_k c_1}\langle f(u)-u,u^{k+1}-u\rangle\\ &\le (1-\frac{2\alpha_k(1-c_1)}{1-\alpha_k c_1})\|u^k-u\|^2+\frac{2\alpha_k(1-c_1)}{1-\alpha_k c_1}\{\frac{\alpha_k}{2(1-c_1)}\|u^k-u\|^2\\ &\qquad +\frac{1}{1-c_1}\langle f(u)-u,u^{k+1}-u\rangle\}\\ &\le (1-\sigma_k)\|u^k-u\|^2+\delta_k,\end{aligned}$$

where $\sigma_k=\frac{2\alpha_k(1-c_1)}{1-\alpha_k c_1}$ and $\delta_k=\frac{2\alpha_k(1-c_1)}{1-\alpha_k c_1}\big\{\frac{\alpha_k}{2(1-c_1)}\|u^k-u\|^2+\frac{1}{1-c_1}\big\langle f(u)-u,u^{k+1}-u\big\rangle\big\}$. It easily seen that $\sum\limits_{k=1}^{\infty}\sigma_k=\infty$ and from (4.4) we get

$$\limsup_{k\to\infty}\delta_k/\sigma_k\le 0.$$

Then by Lemma 2.5, we have $\|u^k-u^0\|\to 0$ as $k\to\infty$. Since $\lim_{k\to\infty}\|u^k-\tilde{u}^k\|=0$ (see (4.1)), we obtain $\tilde{u}^k\to u^0$. □

5. Applications

In this section, we obtain three results by using special cases of the proposed method.

5.1. Theorem. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be α-inverse-strongly-monotone from C into itself such that $S^*\neq\emptyset$. For given $u^0\in C$ arbitrarily. Let $\{u^k\}$ be the sequence generated by*

$$\begin{cases}\tilde{u}^k=P_C[u^k-\rho_k T(u^k)],\\ u^{k+1}=\alpha_k f(u^k)+\beta_k u^k+\gamma_k P_C(u^k-\rho_k T(\tilde{u}^k)),\quad \forall\, k\ge 0,\end{cases}$$

where $f:C\longrightarrow C$ is a c_1-contraction and the sequences $\{\rho_k\}$, $\{\alpha_k\}$, $\{\beta_k\}$, $\{\gamma_k\}$ in $(0,1)$ satisfy the following conditions:

(i) $\lim_{k\to\infty}\alpha_k=0$, $\sum\limits_{k=1}^{\infty}\alpha_k=\infty$;
(ii) $\{\rho_k/\alpha\}\subset(\tau,1-\delta)$ *for some* $\tau,\delta\in(0,1),\lim_{k\to\infty}\rho_k=0$;
(iii) $0<\liminf_{k\to\infty}\beta_k\le\limsup_{k\to\infty}\beta_k<1$.
(iv) $\alpha_k+\beta_k+\gamma_k=1$.

Then the sequences $\{u^k\}$ and $\{\tilde{u}^k\}$ generated by the proposed method converge strongly to the same point $u=P_{S^*}[f(u)]$.

Proof. Set $S_k=I$ in Algorithm 3.1. By using Theorems 4.1 and 4.2, we obtain the desired result. □

5.2. Theorem. *Let H be a real Hilbert space. Let $T:H\longrightarrow H$ be α-inverse-strongly-monotone and $\{S_k\}$ a sequence of nonexpansive mapping from H into itself. Suppose that*

$$\sum_{k=1}^{\infty}\sup\{\|S_{k+1}(x)-S_k(x)\|\|:x\in C\}<\infty.$$

Let S be the mapping of H into itself defined by $Sy=\lim_{k\to\infty}S_k(y)$, $\forall\, y\in C$ such that $F(S)=\bigcap\limits_{k=1}^{\infty}F(S_k)$ and $F(S)\cap T^{-1}(0)\neq\emptyset$. For given $u^0\in H$, let $\{u^k\}$ be a sequence *generated by*

$$\begin{cases} \tilde{u}^k = u^k - \rho_k T(u^k), \\ u^{k+1} = \alpha_k f(u^k) + \beta_k u^k + \gamma_k S_k(u^k - \rho_k T(\tilde{u}^k)), \quad \forall\, k \geq 0, \end{cases}$$

where $f : H \longrightarrow H$ *is a* c_1*-contraction and the sequences* $\{\rho_k\}$, $\{\alpha_k\}$, $\{\beta_k\}$, $\{\gamma_k\}$ *in* $(0,1)$ *satisfy to the following conditions:*

(i) $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=1}^{\infty} \alpha_k = \infty$;
(ii) $\{\rho_k/\alpha\} \subset (\tau, 1-\delta)$ *for some* $\tau, \delta \in (0,1)$, $\lim_{k\to\infty} \rho_k = 0$;
(iii) $0 < \liminf_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \beta_k < 1$.
(iv) $\alpha_k + \beta_k + \gamma_k = 1$.

Then the sequences $\{u^k\}$ *and* $\{\tilde{u}^k\}$ *generated by the proposed method converge strongly to the same point* $u = P_{F(S)\cap T^{-1}(0)}[f(u)]$.

Proof. We have $T^{-1}(0) = S^*$ and $P_H = I$. By using Theorems 4.1 and 4.2, we obtain the desired result. □

5.3. Definition. Let H be a Hilbert space and C a nonempty closed convex subset of H. A mapping $B : C \longrightarrow C$ is called *strictly* L*-pseudocontractive* if there exists $L \in [0, 1[$ such that

$$\|B(u) - B(v)\|^2 \leq \|u - v\|^2 + L\|(I - B)(u) - (I - B)(v)\|^2, \ \forall\, u, v \in C.$$

Note that if $B : C \longrightarrow C$ is strictly L-pseudocontractive, then the mapping $T := I - B$ is $\frac{1-L}{2}$-inverse-strongly-monotone. Moreover, we have

$$\langle u - v, T(u) - T(v)\rangle \geq \frac{1-L}{2}\|T(u) - T(v)\|^2, \ \forall\, u, v \in C. \tag{5.1}$$

Then from Theorems 4.1 and 4.2, we can obtain now a strong convergence theorem for the common fixed point of a nonexpansive mapping and strictly pseudocontractive mapping.

5.4. Theorem. *Let* H *be a real Hilbert space and* C *a nonempty closed convex subset of* H*. Let* $B : C \longrightarrow C$ *be strictly* L*-pseudocontractive and* $\{S_k\}$ *a sequence of nonexpansive mapping from* C *into itself.*

Suppose that $\sum_{k=1}^{\infty} \sup\{\|S_{k+1}(x) - S_k(x)\|\| : x \in C\} < \infty$*. Let* S *be the mapping of* C *into itself defined by* $Sy = \lim_{k\to\infty} S_k(y)$, $\forall\, y \in C$ *such that* $F(S) = \bigcap_{k=1}^{\infty} F(S_k)$ *and* $F(S) \cap F(B) \neq \emptyset$*. For given* $u^0 \in C$ *let* $\{u^k\}$ *be a sequence generated by*

$$\begin{cases} \tilde{u}^k = (1-\rho_k)u^k + \rho_k B(u^k), \\ u^{k+1} = \alpha_k f(u^k) + \beta_k u^k + \gamma_k S_k(u^k - \rho_k(\tilde{u}^k - B(\tilde{u}^k)) \quad \forall\, k \geq 0, \end{cases}$$

where $f : C \longrightarrow C$ *is a* c_1*-contraction and the sequences* $\{\rho_k\}$, $\{\alpha_k\}$, $\{\beta_k\}$, $\{\gamma_k\}$ *in* $(0,1)$ *satisfy to the following conditions:*

(i) $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=1}^{\infty} \alpha_k = \infty$;
(ii) $\{2\rho_k/(1-L)\} \subset (\tau, 1-\delta)$ *for some* $\tau, \delta \in (0,1)$, $\lim_{k\to\infty} \rho_k = 0$;
(iii) $0 < \liminf_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \beta_k < 1$.
(iv) $\alpha_k + \beta_k + \gamma_k = 1$.

Then the sequences $\{u^k\}$ *and* $\{\tilde{u}^k\}$ *generated by the proposed method converge strongly to the same point* $u = P_{F(S)\cap F(B)}[f(u)]$.

Proof. Setting $T = I - B$, from (5.1) we have that T is $\frac{1-L}{2}$-inverse-strongly-monotone. Since $F(B) = S^*$ we have $P_C[u^k - \rho_k T(u^k)] = u^k - \rho_k T(u^k) = (I - \rho_k)u^k + \rho_k B(u^k)$ and $P_C[u^k - \rho_k T(\tilde{u}^k)] = u^k - \rho_k(\tilde{u}^k - B(\tilde{u}^k))$. By using Theorems 4.1 and 4.2, we obtain the desired result. □

6. Conclusions

This paper contributes a new extragradient iterative method obtained by combining a modified extragradient method with the viscosity approximation method iterative scheme for finding a common element of the set of the fixed points of a countable family of nonexpansive mappings and the solution set of the variational inequality for an inverse strongly monotone mapping on a Hilbert space. The proposed method is quite general and flexible and includes some existing methods (e.g. [1, 32, 10, 16, 25, 27]) as special cases. Therefore, the proposed method is expected to be widely applicable.

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