Hacettepe Journal of Mathematics and Statistics
Volume 40 (5) (2011), 681 – 689

ON NEAR CONTINUITY FOR MINIMAL STRUCTURES

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Received 26 : 05 : 2010 : Accepted 21 : 02 : 2011

Abstract

The purpose of this paper is to introduce four kinds of near continuity for functions defined on minimal spaces. Basic properties and characterizations are established for such functions. We also define new minimal structures related to these near continuities. In this way, we obtain many well known results already in the literature, as special cases.

Keywords: Minimal structures, M-continuity, m-continuity, c-M-continuity, c-m-continuity, l-M-continuity, l-m-continuity, Co-m-compact, Co-m-Lindelöf.

2000 AMS Classification: 54A05, 54C08, 54D20.

1. Introduction

In the literature, there are a large number of papers including notions of near continuity, for example almost continuity [19, 23], H-continuity [15], c-continuity [9, 14, 16], almost c-continuity [20, 24], l-continuity [12], almost l-continuity [13], kc-continuity [10] and lc-continuity [11]. In each of these cases the definition of near continuity is equivalent to requiring continuity of the function when the range space is retopologised in a certain way. Some well known examples of these new topologies can be given as; cocompact [7], coLindelöf [8], almost coLindelöf [13], coKC [10] and coLC [11] topologies which define the continuity of a c-continuous, l-continuous, almost l-continuous, kc-continuous and lc-continuous function, respectively.

In this paper, we introduce some near continuities for functions between minimal spaces, namely c-M-continuity, l-M-continuity, c-m-continuity and l-m-continuity. We also define minimal structures related to these near continuities, called co-m-compact and co-m-Lindelöf structures, which are generalizations of cocompact and coLindelöf topologies, in the classical sense, respectively.

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2. Preliminaries

Let (X,τ) and (Y,τ') be topological spaces. Then a function $f:(X,\tau)\to(Y,\tau')$ is said to be c-continuous [9] (resp. l-continuous [12]) if for each point $x\in X$ and each open set V of Y containing $f(x)$ and having a compact (resp. Lindelöf) complement, there exists an open set U of X containing x such that $f(U)\subset V$. Furthermore, the cocompact topology and coLindelöf topology of τ on X (denoted by $c(\tau)$ and $l(\tau)$, respectively) are defined in [7] and [8] as; $c(\tau)=\{\emptyset\}\cup\{U\in\tau : X-U \text{ is compact in } (X,\tau)\}$ and $l(\tau)=\{\emptyset\}\cup\{U\in\tau : X-U \text{ is Lindelöf in } (X,\tau)\}$, respectively.

Now we recall some concepts and notations defined in [17]. Let X be a nonempty set and $m\subset \exp X$, then m is said to be a *minimal structure* (briefly, *m-structure*) on X if $\emptyset\in m$ and $X\in m$. Then, (X,m) is called a *minimal space*. The elements of m are called *m-open sets*, and their complements *m-closed sets*. If m is a minimal structure on X and $A\subset X$, the *m-interior of A* and the *m-closure of A* are defined as: $m\text{-Int}(A)=\bigcup\{U : U\subset A,\ U\in m\}$ and $m\text{-Cl}(A)=\bigcap\{F : A\subset F,\ X-F\in m\}$, respectively (we shall write briefly $\mathtt{I}_m A$ and $\mathtt{C}_m A$). Note that a m-structure is said to have property $(\mathcal{B})$ if the union of any family of subsets of m belongs to m.

2.1. Lemma. [17] *Let m be a minimal structure on X. Then*

(1) $\mathtt{I}_m A\subset A$ *and* $\mathtt{I}_m(\mathtt{I}_m A)=\mathtt{I}_m A$ *for each* $A\subset X$,
(2) $A\subset \mathtt{C}_m A$ *and* $\mathtt{C}_m(\mathtt{C}_m A)=\mathtt{C}_m A$ *for each* $A\subset X$,
(3) $\mathtt{C}_m A=X-\mathtt{I}_m(X-A)$ *for each* $A\subset X$,
(4) *If* $A\subset B\subset X$, *then* $\mathtt{I}_m A\subset \mathtt{I}_m B$ *and* $\mathtt{C}_m A\subset \mathtt{C}_m B$,
(5) *If A is m-open (resp. m-closed), then* $A=\mathtt{I}_m A$ *(resp.* $A=\mathtt{C}_m A$*)*,
(6) *If m has property* $(\mathcal{B})$, *then A is m-open (resp. m-closed) iff* $A=\mathtt{I}_m A$ *(resp.* $A=\mathtt{C}_m A$*).* □

The fundamental separation axioms (T_0, T_1, T_2, R_0) for minimal structures (denoted by m-T_0, m-T_1, m-T_2, m-R_0) are formulated in [2] and [22] by replacing open sets by m-open ones. Furthermore, m-compactness [22] and m-connectedness [22] are defined in the same manner. Similarly we shall say that a subset K of X is m-Lindelöf relative to (X,m) if any cover of K by m-open sets has a countable subcover.

2.2. Definition. Let m_X and m_Y be m-structures on X and Y, respectively and let $f:X\to Y$ be a function. Then f is said to be

(1) M-continuous [22] if for each $x\in X$ and each m_Y-open set V of Y containing $f(x)$, there exists $U\in m_X$ containing x such that $f(U)\subset V$.
(2) m-continuous [1] if $f^{-1}(V)\in m_X$ for each $V\in m_Y$.

On the other hand, the notions of generalized topology were introduced by Császár in [3]. Let X be a nonempty set and $g\subset\exp X$. Then g is said to be a *generalized topology* (briefly GT) on X if $\emptyset\in g$ and $G_j\in g$ for $j\in J\neq\emptyset$ implies $G=\bigcup_{j\in J}G_j\in g$. If $X\in g$, then g is said to be a *strongly generalized topology* [4] on X. The elements of g are called *g-open sets* and their complements *g-closed sets*. For $A\subset X$, the g-interior of A (denoted by i_gA) is the union of all $G\subset A$, $G\in g$ and the g-closure of A (denoted by c_gA) is the intersection of all g-closed sets containing A.

More generally, Császár [5] introduced $i_\lambda A=\bigcup\{L\in\lambda : L\subset A\}$ for an arbitrary $\lambda\subset\exp X$ (in particular, $i_\lambda A=\emptyset$ if no $L\in\lambda$ satisfies $L\subset A$), and by taking $\mu=\{X-L : L\in\lambda\}$, $c_\lambda A=\bigcap\{M\in\mu : M\supset A\}$ (in particular, $c_\lambda A=X$ if no $M\in\mu$ satisfies $M\supset A$). Also it is shown that $i_\lambda A\subset A$, $i_\lambda i_\lambda A=i_\lambda A$, $A\subset c_\lambda A$, $c_\lambda c_\lambda A=c_\lambda A$, $c_\lambda A=X-i_\lambda(X-A)$, and if $A\subset B\subset X$, then $i_\lambda A\subset i_\lambda B$ and $c_\lambda A\subset c_\lambda B$.

Note that if we give the role of the arbitrary family λ to a minimal structure m (resp. GT g), we get $\mathtt{I}_m$ and $\mathtt{C}_m$ (resp. i_g and c_g) instead of i_λ and c_λ. However it is observed that there exists a uniquely determined GT $g_{i_\lambda} = \{L \subset X : L = i_\lambda L\}$ satisfying $i_\lambda = i_{g_{i_\lambda}}$ for an arbitrary class $\lambda \subset \exp X$ by the following Lemma.

2.3. Lemma. [5] *If $\iota : \exp X \to \exp X$ satisfies $\iota A \subset \iota B$ when $A \subset B \subset X$, $\iota A \subset A$ and $\iota\iota A = \iota A$ for $A \subset X$, then there is a GT $\mu \subset \exp X$ such that $\iota = i_\mu$.* □

3. Near continuities for minimal structures

In this section we introduce new classes of functions between minimal spaces, namely c-M-continuous, l-M-continuous, c-m-continuous and l-m-continuous functions.

In the sequel, let m_X and m_Y be m-structures on X and Y, respectively.

3.1. Definition. Let $f : (X, m_X) \to (Y, m_Y)$ be a function. Then f is

(1) c-M-continuous (resp. l-M-continuous) if for each $x \in X$ and each m_Y-open set V containing $f(x)$ and having a m-compact (resp. m-Lindelöf) complement relative to Y, there exists a m_X-open set $U \subset X$ such that $f(U) \subset V$.
(2) c-m-continuous (resp. l-m-continuous) if whenever V is a m_Y-open set having a m-compact (resp. m-Lindelöf) complement relative to Y, then $f^{-1}(V) \in m_X$.

Note that if m_X has property ($\mathcal{B}$), then clearly c-M-continuity (resp. l-M-continuity) coincides with c-m-continuity (resp. l-m-continuity).

Now let us examine the particular case when m_X and m_Y are topologies on X and Y, respectively. Then it is clear that c-M-continuous and also c-m-continuous (resp. l-M-continuous and also l-m-continuous) functions coincide with c-continuous (resp. l-continuous) functions. Moreover, in the particular case when m_X is a minimal structure on X and m_Y is a topology on Y, the definition of c-M-continuous functions coincides with Definition 3.5 of Noiri and Popa [21].

3.2. Remark. (1) Every l-M-continuous (resp.l-m-continuous) function is c-M-continuous (resp. c-m-continuous).

(2) Every c-m-continuous (resp. l-m-continuous) function is c-M-continuous (resp. l-M-continuous).

By Remark 3.2 we have the following diagram, in which the converse of each implication fails to be true by the examples stated below.

Diagram

$$\begin{tikzcd}
l\text{-}m\text{-continuous} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & l\text{-}M\text{-continuous} \arrow[d, Rightarrow] \\
c\text{-}m\text{-continuous} \arrow[r, Rightarrow] & c\text{-}M\text{-continuous}
\end{tikzcd}$$

3.3. Example. [12] Let X and Y denote the real line equipped with the usual and discrete topologies, respectively, instead of m_X and m_Y. It is clear that the identity function $id : X \to Y$ is c-continuous but not l-continuous.

3.4. Example. Let $X = \{a, b, c, d\}$ and consider two minimal structures on X defined as $m_1 = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{b, c\}, \{b, d\}\}$ and $m_2 = \{\emptyset, X, \{a\}, \{b\}, \{b, c, d\}\}$. Then it is clear that the identity function $id : (X, m_1) \to (X, m_2)$ is c-M-continuous (resp. l-M-continuous) but it is not c-m-continuous (resp. l-m-continuous).

We shall denote the family of all unions of the elements of m_X (resp. m_Y) with g_{m_X} (resp. g_{m_Y}).

3.5. Theorem. *Let $f : (X, m_X) \to (Y, m_Y)$ be a function. Then the following conditions are equivalent.*

(1) *f is c-M-continuous (resp. l-M-continuous),*
(2) *If V is a m_Y-open subset having a m-compact (resp. m-Lindelöf) complement relative to Y, then $f^{-1}(V) \in g_{m_X}$,*
(3) *If F is m-compact (resp. m-Lindelöf) relative to Y and m_Y-closed, then $f^{-1}(F)$ is g_{m_X}-closed.*

Proof. (1) $\Longrightarrow$ (2) Let $V \in m_Y$ have a m-compact complement relative to Y. Then for each $x \in f^{-1}(V)$ there exists a m_X-open set U_x containing x such that $f(U_x) \subset V$. Hence $f^{-1}(V) \in g_{m_X}$.

(2) $\Longrightarrow$ (3) Let F be m-compact relative to Y and m_Y-closed. Then $Y - F$ is a m_Y-open subset having a m-compact complement relative to Y. Therefore $f^{-1}(Y-F) \in g_{m_X}$. Hence $f^{-1}(F)$ is g_{m_X}-closed.

(3) $\Longrightarrow$ (1) Let $x \in X$ and V a m_Y-open set containing $f(x)$ and having a m-compact complement relative to Y. Then for $F = Y - V$, $f^{-1}(F)$ is g_{m_X}-closed. Thus $f^{-1}(V) \in g_{m_X}$. Hence there exists $U \in m_X$ containing x such that $U \subset f^{-1}(V)$. This completes the proof.

The proof is similar for l-M-continuity. □

3.6. Corollary. *If $f : (X, m_X) \to (Y, m_Y)$ is a function and m_Y has the property $(\mathcal{B})$, then the following statements are equivalent.*

(1) *f is c-M-continuous (resp. l-M-continuous),*
(2) *$\mathtt{C}_{m_X} f^{-1}(B) \subset f^{-1}(\mathtt{C}_{m_Y} B)$ for every subset B of Y such that $\mathtt{C}_{m_Y} B$ is m-compact (resp. m-Lindelöf) relative to Y,*
(3) *$f^{-1}(\mathtt{I}_{m_Y} B) \subset \mathtt{I}_{m_X} f^{-1}(B)$ for every subset B of Y such that $Y - \mathtt{I}_{m_Y} B$ is m-compact (resp. m-Lindelöf) relative to Y.*

These statements are implied by:

(4) *f is c-m-continuous (resp. l-m-continuous).*

Moreover, if m_X has property $(\mathcal{B})$, all the statements are equivalent.

Proof. (1) $\Longrightarrow$ (2) Let B be an arbitrary subset of Y such that $\mathtt{C}_{m_Y} B$ is m-compact relative to Y. Then $f^{-1}(\mathtt{C}_{m_Y} B)$ is g_{m_X}-closed. On the other hand, for the m_X-interior operator $\mathtt{I}_{m_X}$, we have a uniquely determined strongly GT,

$$g_{I_{m_X}} = \{L \subset X : L = \mathtt{I}_{m_X} L\}$$

satisfying $\mathtt{I}_{m_X} = i_{g_{I_{m_X}}}$ and $\mathtt{C}_{m_X} = c_{g_{I_{m_X}}}$ by Lemma 2.3. Also, $g_{I_{m_X}} = g_{m_X}$ by [6, Proposition 2.8]. Thus

$$\mathtt{C}_{m_X} f^{-1}(\mathtt{C}_{m_Y} B) = \mathtt{C}_{g_{m_X}} f^{-1}(\mathtt{C}_{m_Y} B) = f^{-1}(\mathtt{C}_{m_Y} B).$$

Hence $\mathtt{C}_{m_X} f^{-1}(B) \subset f^{-1}(\mathtt{C}_{m_Y} B)$.

(2) $\Longrightarrow$ (3) Let B be an arbitrary subset of Y such that $Y - \mathtt{I}_{m_Y} B$ is m-compact relative to Y. Then we have $\mathtt{C}_{m_X} f^{-1}(Y - \mathtt{I}_{m_Y} B) \subset f^{-1}(\mathtt{C}_{m_Y}(Y - \mathtt{I}_{m_Y} B))$. Hence, $f^{-1}(\mathtt{I}_{m_Y} B) \subset \mathtt{I}_{m_X} f^{-1}(B)$ since $\mathtt{C}_{m_X} f^{-1}(Y - \mathtt{I}_{m_Y} B) = X - \mathtt{I}_{m_X} f^{-1}(\mathtt{I}_{m_Y} B)$ and $f^{-1}(\mathtt{C}_{m_Y}(Y - \mathtt{I}_{m_Y} B)) = X - f^{-1}(\mathtt{I}_{m_Y} B)$.

(3) $\Longrightarrow$ (1) Let V be a m_Y-open subset having a m-compact complement relative to Y. Then $f^{-1}(V) = f^{-1}(\mathtt{I}_{m_Y} V) \subset \mathtt{I}_{m_X} f^{-1}(V)$, since $Y - \mathtt{I}_{m_Y} V$ is m-compact relative to Y. Hence, $f^{-1}(V) \in g_{m_X}$.

For the rest of the proof see Remark 3.2 (2) and consider the fact that c-M-continuity coincides with c-m-continuity when m_X has the property $(\mathcal{B})$.

The proof is similar for l-M-continuity. $\square$

3.7. Remark. In the particular case when m_X and m_Y are topologies on X and Y, respectively, we obtain the results given in Long and Hendrix [14, Theorem 1], and a part of Kohli [12, Theorem 2.1], as a corollary by Theorem 3.5. Also, Noiri and Popa [21, Corollary 3.2] follows from Corollary 3.6.

4. Minimal structures related to near continuities

In this section, we introduce minimal structures related to the near continuous functions given in the previous section and investigate some basic properties of these structures.

4.1. Definition. Let m be a minimal structure on X. The collection

(1) $c(m) = \{\emptyset\} \cup \{A \in m : X - A$ is m-compact relative to $X\}$ is a minimal structure with $c(m) \subset m$, called the *co-m-compact structure on X*,
(2) $l(m) = \{\emptyset\} \cup \{A \in m : X - A$ is m-Lindelöf relative to $X\}$ is a minimal structure with $l(m) \subset m$, called the *co-m-Lindelöf structure on X*.

It is clear that $c(m) \subset l(m) \subset m$.

Let us examine the particular case when m is a topology on X. Clearly the co-m-compact (resp. co-m-Lindelöf) structure on X coincides with the cocompact (resp. coLindelöf) topology on X. Also, it is evident that $c(m)$ $(l(m))$ need not to be a topology on X, except in this particular case.

4.2. Theorem. *Let m be a minimal structure on X. Then $(X, c(m))$ (resp. $(X, l(m))$) is m-compact (resp. m-Lindelöf).*

Proof. Let $\mathcal{A} = (A_j)_{j \in J} \subset c(m)$ be a cover of X. Then for an arbitrary $A_j \in \mathcal{A}$, finitely many members of $\mathcal{A}$ cover $X - A_j$. Hence, $(X, c(m))$ is m-compact since $X = A_j \cup (X - A_j)$.

The proof is similar for $(X, l(m))$. $\square$

4.3. Lemma. *Let m be a minimal structure on X and $A \subset X$.*

(1) *If X is m-compact and $\mathtt{C}_m A = A$ then A is m-compact relative to X [18].*
(2) *If X is m-Lindelöf and $\mathtt{C}_m A = A$ then A is m-Lindelöf relative to X.*

Proof. (1) Given in [18].

(2) Let $A \subset X$ satisfying $A = C_m A$. Then $X - A = \mathtt{I}_m(X - A) = \bigcup_{\imath \in I}\{U_\imath : U_\imath \subset X - A,\ U_\imath \in m\}$. So, for an arbitrary m-open cover $(M_j)_{j \in J}$ of A, we have $B \subset (\bigcup_{j \in J} M_j) \cup (\bigcup_{\imath \in I} U_\imath)$. Therefore, m-Lindelöfness of B implies the existence of countable subsets F, K of J, I, respectively, such that $B \subset (\bigcup_{j \in F} M_j) \cup (\bigcup_{\imath \in K} U_\imath)$. Thus, $A \subset \cup_{j \in F} M_j$. Hence A is m-Lindelöf relative to X. $\square$

So we can give the following result by Theorem 4.2.

4.4. Corollary. *Let m be a minimal structure on X. Then*

(1) *(X, m) is m-compact iff $c(m) = m$.*
(2) *(X, m) is m-Lindelöf iff $l(m) = m$.* $\square$

4.5. Remark. In the particular case when m is a topology on X, we obtain the results given in Gauld [7, Theorem 2] and of Gauld *et. al.* [8, Theorem 2] as a corollary by Theorem 4.2. Also, Gauld [7, Corollary 3] and Gauld *et. al.* [8, Corollary 1] follow from Corollary 4.4.

4.6. Proposition. *Let m be a minimal structure on X. Then we have*

(1) $l(c(m)) = c(c(m)) = c(m) \subset c(l(m))$,
(2) $l(l(m)) = l(m)$,
(3) *If m has property $(\mathcal{B})$, then $c(m) = c(l(m))$.*

Proof. (1) It is clear that $l(c(m)) = c(m)$ and $c(c(m)) = c(m)$ by Theorem 4.2 and Corollary 4.4. Also, $c(m) \subset c(l(m))$ follows from $c(m) \subset l(m) \subset m$, since m-Lindelöfness is implied by m-compactness.

(2) Follows from Theorem 4.2 and Corollary 4.4 (2).

(3) Let $A \in c(l(m))$ and $\mathcal{A} = (A_j)_{j\in J}$ be a m-open cover of $X - A$. Since m has the property $(\mathcal{B})$, we have $G_j = A \cup A_j \in m$ for each $j \in J$. Then $X - G_j = (X-A) \cap (X-A_j)$ is m-Lindelöf relative to X since $A \in l(m)$. Therefore, $(G_j)_{j\in J}$ is a $l(m)$-open cover of $X - A$. Thus, there exists a finite set $F \subset J$ such that $X - A \subset \bigcup_{j\in F} G_j$, and then $(A_j)_{j\in F}$ covers $X - A$. Hence, $A \in c(m)$. □

4.7. Remark. If $m = \tau$ is a topology on X, then we obtain Gauld *et. al.* [8, Proposition 1] as a corollary to Proposition 4.6.

Furthermore, the basic relation between the cocompact (coLindelöf) minimal structures and c-M-continuous (resp. l-M-continuous), and also c-m-continuous (resp. l-m-continuous), functions is given by the following theorem.

4.8. Theorem. *Let m_X and m_Y be m-structures on X and Y, respectively. Then the following statements are valid.*

(1) *$f : X \to (Y, m_Y)$ is c-M-continuous iff $f : X \to (Y, c(m_Y))$ is M-continuous,*
(2) *$f : X \to (Y, m_Y)$ is c-m-continuous iff $f : X \to (Y, c(m_Y))$ is m-continuous,*
(3) *$f : X \to (Y, m_Y)$ is l-M-continuous iff $f : X \to (Y, l(m_Y))$ is M-continuous,*
(4) *$f : X \to (Y, m_Y)$ is l-m-continuous iff $f : X \to (Y, l(m_Y))$ is m-continuous.*

Proof. Follows from Definitions 3.1 and 4.1. □

4.9. Remark. In the particular case when m_X and m_Y are topologies on X and Y, respectively, we obtain Gauld [7, Theorem 1] and Gauld *et. al.* [8, Theorem 1] as a corollary to Theorem 4.8.

Now consider the transfer of the seperation properties (m-T_0, m-T_1 and m-T_2) from a minimal space (X, m) to its co-m-compact and co-m-Lindelöf structures. Clearly any property preserved by an enlargement of minimal structures will be transferred from $c(m)$ to $l(m)$, and from $l(m)$ to m, since $c(m) \subset l(m) \subset m$. On the other hand, by [8, Examples 1, 2, 3 and 4] it is shown that the opposite preservations do not occur in general for the m-T_0 and m-T_2 properties. In Proposition 4.11 below we show that if m has property $(\mathcal{B})$, then the opposite preservation occurs for the m-T_1 property.

However we give a new separation axiom, analogous to m-T_1 for minimal structures, which will be transferred from $c(m)$ to $l(m)$, from $l(m)$ to m and also from (X, m) to $(X, c(m))$ $((X, l(m)))$, without the assumption that m has property $(\mathcal{B})$.

4.10. Definition. Let m be a minimal structure on X. Then X is said to be *strongly m-T_1* if $\{x\}$ is m-closed for each $x \in X$.

Clearly every strongly m-T_1 space is m-T_1.

Recall that for a minimal structure m on X, we denote the family of all unions of the elements of m by g_m. Clearly, g_m is a strongly generalized topology on X.

4.11. Proposition. *Let m be a minimal structure on X. Then the following statements are true.*

(1) *(X, m) is m-T_1 iff $\{x\}$ is g_m-closed for each $x \in X$,*
(2) *If m has property $(\mathcal{B})$, then (X, m) is m-T_1 iff (X, m) is strongly m-T_1,*
(3) *(X, m) is strongly m-T_1 iff $(X, c(m))$ $((X, l(m)))$ is strongly m-T_1.*

Proof. (1) For an arbitrary $y \in X - \{x\}$ there exists $U_x \in m$ containing x such that $y \notin U_x$, and $U_y \in m$ containing y such that $x \notin U_y$, since (X, m) is m-T_1. Thus $y \in U_y \subset X - \{x\}$ implies that $y \in i_{g_m}(X - \{x\})$ since $m \subset g_m$. Hence $X - \{x\}$ is g_m-open.

Conversely, for any pair of distinct points x, y of X, $X - \{x\}$ and $X - \{y\}$ are g_m-open sets of X. On the other hand, for the m-interior operator $\mathtt{I}_m$, we have an uniquely determined strongly GT $g_{I_m} = \{L \subset X : L = \mathtt{I}_m L\}$ satisfying $\mathtt{I}_m = i_{g_{I_m}}$ and $\mathtt{C}_m = c_{g_{I_m}}$ by Lemma 2.3. Also $g_{I_m} = g_m$ by [6, Proposition 2.8]. Therefore $X - \{x\} = \mathtt{I}_m(X - \{x\})$ and $X - \{y\} = \mathtt{I}_m(X - \{y\})$, and this implies the existence of $G_x \in m_X$ such that $x \in G_x \subset X - \{y\}$, and of $G_y \in m_X$ such that $y \in G_y \subset X - \{x\}$, respectively. Hence (X, m) is m-T_1.

(2) Follows from (1) and Definition 4.10, since $g_m = m$ by the assumption that m has property $(\mathcal{B})$.

(3) Take an arbitrary $x \in X$. Clearly $\{x\}$ is m-closed since (X, m) is strongly m-T_1. Thus the m-compactness of $\{x\}$ implies that $\{x\}$ is $c(m)$-closed. Hence, $(X, c(m))$ is strongly m-T_1.

The reverse implication is clear by the definition of $c(m)$.

The proof is similar for $(X, l(m))$. □

4.12. Lemma. *Let m be a minimal structure on X with property $(\mathcal{B})$. Then $c(m)$ and $l(m)$ are strongly generalized topologies on X.*

Proof. Clearly $\emptyset$ and $X \in c(m)$. Now let $J \neq \emptyset$ and $A = \bigcup_{j \in J} U_j$, $(U_j)_{j \in J} \subset c(m)$. Then $X - A = \bigcap_{j \in J}(X - U_j) \subset X - U_j$ for each $j \in J$ implies that $X - A$ is m-compact relative to X since $X - A$ is m-closed. Hence $A \in c(m)$.

The proof is similar for $l(m)$. □

A minimal space (X, m) is said to be m-R_0 [22] if for each m-open set U and each $x \in U$, $\mathtt{C}_m\{x\} \subset U$. By the following proposition we show that the m-R_0 property will be transferred from m to $l(m)$, and from $l(m)$ to $c(m)$, similar to the particular case when m is a topology.

4.13. Proposition. *Let m be a minimal structure on X. Then the following statements are true.*

(1) *If (X, m) is m-R_0, then $\mathtt{C}_m\{x\}$ is m-compact relative to X, for each $x \in X$,*
(2) *If m has property $(\mathcal{B})$ and (X, m) is m-R_0, then $(X, l(m))$ is m-R_0,*
(3) *If m has property $(\mathcal{B})$ and $(X, l(m))$ is m-R_0, then $(X, c(m))$ is m-R_0.*

Proof. (1) Let $x \in X$ and $\mathtt{C}_m\{x\} \subset \bigcup_{j \in J} U_j$, $(U_j)_{j \in J} \subset m$. Then there exists $U_{j_0} \in m$ such that $x \in U_{j_0}$. Thus $\mathtt{C}_m\{x\} \subset U_{j_0}$, since (X, m) is m-R_0. Hence $\mathtt{C}_m\{x\}$ is m-compact relative to X.

(2) Consider $U \in l(m)$ and $x \in U$. Then $\mathtt{C}_m\{x\} \subset U$ since (X, m) is m-R_0. On the other hand, $\mathtt{C}_m\{x\}$ is m-closed since m has property ($\mathcal{B}$) and is m-compact. Thus $\mathtt{C}_m\{x\}$ is $l(m)$-closed and so $\mathtt{C}_{l(m)}\{x\} \subset \mathtt{C}_m\{x\}$. Hence, $(X, l(m))$ is m-R_0.

(3) Consider $U \in c(m)$ and $x \in U$. Then $\mathtt{C}_{l(m)}\{x\} \subset U$ since $(X, l(m))$ is m-R_0. On the other hand, $\mathtt{C}_{l(m)}\{x\}$ is $l(m)$-closed by Lemma 4.12 and m-compact relative to $(X, l(m))$. Therefore, $\mathtt{C}_{l(m)}\{x\}$ is $c(l(m))$-closed. By Proposition 4.6(3), $\mathtt{C}_{l(m)}\{x\}$ is $c(m)$-closed. Thus, $\mathtt{C}_{c(m)}\{x\} \subset \mathtt{C}_{l(m)}\{x\}$. Hence $(X, c(m))$ is m-R_0. □

4.14. Remark. If $m = \tau$ is a topology on X, then we obtain Gauld *et. al.* [8, Proposition 2] as a corollary to Proposition 4.11 (2) and (3). Also, Gauld *et. al.* [8, Proposition 3] follows from Proposition 4.13 (2) and (3).

Acknowledgement.

We would like to thank the referees for their valuable comments on the paper.

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