Hacettepe Journal of Mathematics and Statistics
Volume 40 (4) (2011), 515 – 522

FUNCTIONAL ALEXANDROFF SPACES

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Received 21 : 01 : 2010 : Accepted 22 : 11 : 2010

Abstract

In the following text a proper subclass of Alexandroff topological spaces, namely functional Alexandroff topological spaces, is introduced. We discuss relation between Alexandroff spaces and functional Alexandroff spaces, functional Alexandroff spaces as dynamical systems, and other related topics.

Keywords: Alexandroff topology, Dynamical system, Finite topology, Functional Alexandroff topology.

2000 AMS Classification: 54 C50, 54 F 05, 54 F 65.

1. Introduction

As it has been mentioned in [2] and several other papers such as [4] and [3], Alexandroff spaces have been introduced in [1] (under the name Diskrete Räume) for the first time. Diskrete Räume (e.g. in [1]), A-space (e.g. in [7]), principal topology (e.g. in [6]), and Alexandroff space (e.g. in [2]) are just several of the names for a topological space for which the intersection of every nonempty collection of its open sets is open, in this paper we will name such a space an *Alexandroff space*. Without any doubt one of the best and most important examples of Alexandroff spaces are the finite topological spaces, in this area several authors have worked (e.g. [5] as one of the oldest related papers).

Using the definition in Alexandroff space X, for every $a \in X$, $V(a) := \bigcap\{U \subseteq X : U$ is an open neighborhood of $a\}$ is the smallest open neighborhood of a in X. It is evident that X is Alexandroff if and only if for each $a \in X$ there exists smallest open neighborhood of a in X. Therefore the set of all maps $f : X \to \mathcal{P}(X)$ such that:

$\mathcal{A}_1$- $\forall a \in X \quad a \in f(a)$
$\mathcal{A}_2$- $\forall a \in X\, \forall b \in f(a) \quad f(b) \subseteq f(a)$

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coincides with all Alexandroff topologies on X (for such an f consider the topological basis $\{f(a) : a \in X\}$); some authors prefer to use this view (e.g. in [6], $f(x)$ is the orbit of x under S, where S is a semigroup with identity and acts on X).

Moreover if $(X, \leq)$ is a poset, then $f : X \to \mathcal{P}(X)$ with $f(a) = \{x \in X : x \leq a\} =: (-\infty, a]_{\leq}$ satisfies the above conditions and generates an Alexandroff topology on X with topological basis $\{(-\infty, a]_{\leq} : a \in X\}$.

On the other hand if X is an Alexandroff T_0 space, for each $a, b \in X$ define "$a \leq b$ if and only if $a \in V(b)$", in this way $(X, \leq)$ is a poset and the topology generated by $\{(-\infty, a]_{\leq} : a \in X\}$ is the same topology of X. Therefore there exists a bijection between Alexandroff T_0 topologies on X and partial orders on X, some authors also prefer to use this view (e.g. [1]).

In this paper we will introduce the class of functional Alexandroff spaces (a proper subclass of Alexandroff spaces) which arises by considering self-maps.

Let $\alpha : X \to X$ be an arbitrary map. Then $\alpha^* : X \to \mathcal{P}(X)$ with $\alpha^*(x) = \{x\} \cup \alpha^{-1}(x) \cup \alpha^{-2}(x) \cup \cdots$ (for $x \in X$) satisfies $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, so corresponds to an Alexandroff topology on X, we name this Alexandroff space the *functional Alexandroff space (induced by α)*. Moreover if α has no periodic points other than fixed points then this Alexandroff space is T_0 too, and the corresponding partial order relation on X is defined as "$a \leq b$ if and only if there exists $n \geq 0$ with $\alpha^n(a) = b$".

Consider $X = \{1, 2, 3\}$ under the topology $\{\emptyset, \{1, 2\}, X\}$. This Alexandroff space is not functional Alexandroff. Thus for $|X| \geq 3$, the set of all functional Alexandroff topologies on X is a proper subset of the set of all Alexandroff topologies on X. In this text by $A \subset B$ we mean $A \subseteq B$ and $A \neq B$.

2. Basic properties of functional Alexandroff spaces

From now on for $\alpha : X \to X$, let τ_α be the topology on X generated by the topological basis $\{\downarrow^\alpha x : x \in X\}$, where for each $x \in X$, $\downarrow^\alpha x := \bigcup\{\alpha^{-n}(x) : n \geq 0\}$. We have the following facts on the topological space (X, τ_α):

- for each $x, y \in X$, $\downarrow^\alpha x \subseteq \downarrow^\alpha y$ or $\downarrow^\alpha y \subseteq \downarrow^\alpha x$ or $\downarrow^\alpha y \cap \downarrow^\alpha x = \emptyset$;
- for each $x \in X$, $\downarrow^\alpha x$ is the smallest open neighborhood of x;
- $\alpha : X \to X$ is continuous.

We recall that under the topology τ_α, X is called the functional Alexandroff space (induced by α).

2.1. Theorem. [Elementary Properties] *Let $\alpha : X \to X$ be a map. In the topological space (X, τ_α), we have the following facts:*

(1) *$X \neq \emptyset$ is compact if and only if there exist $a_1, \ldots, a_n \in X$, such that $X = \bigcup_{1 \leq i \leq n} \downarrow^\alpha a_i$ (one may suppose the $\downarrow^\alpha a_i$s are disjoint too).*
(2) *For $D \subseteq X$ with $\alpha(D) \subseteq D$, we have $\tau_{\alpha|_D}$ is the same as the induced topology on D from X.*
(3) *For each $n \in \boldsymbol{N}$, $\tau_\alpha \subseteq \tau_{\alpha^n}$.*
(4) *For each $Y \subseteq X$, Y is closed if and only if Y is invariant under α (i.e., $\alpha(Y) \subseteq Y$).*
(5) *For each $Y \subseteq X$, $\overline{Y} = \bigcup\{\alpha^n(Y) : n \geq 0\}$.*
(6) *$\alpha : X \to X$ is closed.*
(7) *Each connected component of X is open (hence clopen).*

(8) *For each $a, b \in X$, let $a \sim b$ if and only if there exist $n, m \in \boldsymbol{N}$ with $\alpha^n(a) = \alpha^m(b)$, then $\sim$ is an equivalence relation on X and $X/\sim$ is the set of all connected components of X.*

Proof. (2) Since for each $x \in D$, $\bigcup\{\alpha^{-n}(x) : n \geq 0\} \cap D = \bigcup\{(\alpha|_D)^{-n}(x) : n \geq 0\}$.

(4) Let Y be a closed subset of X. For each $x \in X - Y$, $\alpha^{-1}(x) \subseteq \downarrow^\alpha x \subseteq X - Y$ (since $X - Y$ is open and $\downarrow^\alpha x$ is the smallest open neighborhood of x), so $\alpha^{-1}(x) \cap Y = \emptyset$ and $x \notin \alpha(Y)$. By $x \notin \alpha(Y)$ $(\forall x \in X - Y)$, we have $\alpha(Y) \subseteq Y$.

On the other hand if $\alpha(Y) \subseteq Y$, then for each $x \in X - Y$, $\downarrow^\alpha x \subseteq X - Y$ and Y is closed.

(5) Use (4) and the fact that $\bigcup\{\alpha^n(Y) : n \geq 0\}$ is the smallest $\alpha-$invariant subset of X containing Y.

(6) Use (5).

(7) It is enough to show $\downarrow^\alpha a$ is connected for each $a \in X$. But this is evident since $\downarrow^\alpha a$ is the smallest open subset of $\downarrow^\alpha a$ containing a.

(8) Let $a \in X$ and C denotes the connected component of a. By (7), $C \supseteq U := \bigcup\{\downarrow^\alpha c : a \in \downarrow^\alpha c\}$, moreover U is a nonempty clopen subset of C, since if $b \in C - U$ and $x \in \downarrow^\alpha b \cap U$, then there exists $c \in U$ such that $a, x \in \downarrow^\alpha c$, hence $\downarrow^\alpha b \subseteq \downarrow^\alpha c$ or $\downarrow^\alpha c \subseteq \downarrow^\alpha b$, which are both contradictions. So $\downarrow^\alpha b \subseteq C - U$ and U is a closed subset of C. Hence $C = U$ and therefore $b \in C$ if and only if there exists $c \in X$ such that $a, b \in \downarrow^\alpha c$, i.e., there exist $n, m \in \mathbf{N}$ and $c \in X$ such that $\alpha^n(a) = \alpha^m(b) = c$. □

If $\alpha : X \to X$ is a map, then as mentioned in Theorem 2.1. when X is considered under topology τ_α, $\alpha : X \to X$ is closed. The following theorem indicates the conditions under which $\alpha : X \to X$ is open.

2.2. Theorem. *Let $\alpha : X \to X$ be a map. In the topological space (X, τ_α), we have:*

(1) *$\alpha : X \to X$ is open if and only if for all $x \in X$ we have $\alpha(\downarrow^\alpha x) = \downarrow^\alpha \alpha(x)$.*
(2) *If $\alpha : X \to X$ is open, then it is onto.*
(3) *$\alpha : X \to X$ is a homeomorphism if and only if it is one to one and open.*
(4) *$\alpha : X \to X$ is a homeomorphism if and only if it is a bijection.*

Proof. (1) Let $\alpha : X \to X$ be open and $x \in X$. It is clear that $\alpha(\downarrow^\alpha x) \subseteq \downarrow^\alpha \alpha(x)$. Since $\alpha : X \to X$ is open so $\alpha(\downarrow^\alpha x)$ is an open neighborhood of $\alpha(x)$, therefore $\downarrow^\alpha \alpha(x) \subseteq \alpha(\downarrow^\alpha x)$.

On the other hand if for all $x \in X$ we have $\alpha(\downarrow^\alpha x) = \downarrow^\alpha \alpha(x)$, since $\{\downarrow^\alpha x : x \in X\}$ is a basis for τ_α, thus $\alpha : X \to X$ is open.

(2) Let $x \in X$, by (1) we have $x \in \downarrow^\alpha \alpha(x) = \alpha(\downarrow^\alpha x) \subseteq \alpha(X)$. □

2.3. Note. Let $\alpha : X \to X$ be an arbitrary map, then $\phi : (X, \tau_\alpha) \to (X \times X, \tau_{\alpha \times \alpha})$ with $\phi(x) = (x, x)$ an embedding.

Proof. Using Theorem 2.1 (2), $\{(x, x) : x \in X\}$ is an invariant subset of $X \times X$ under $\alpha \times \alpha$. □

2.4. Note. Define $\lambda_i : A_i \to A_i$ by setting $A_0 = \mathbf{N}$, $A_1 = \mathbf{Z}$, $A_{n+1} = \{1, \ldots, n\}$ (for $n \in \mathbf{N}$), and also let $\lambda_0(k) = k + 1$, $\lambda_1(k) = k + 1$, and $\lambda_{n+1}(k) \equiv k + 1 (\mathrm{mod}\, n)$. Thus $\tau_{\lambda_i} = \{A_i, \emptyset\}$ for $i > 1$ and $\tau_{\lambda_i} = \{\{n \in A_i : n < k\} : k \in A_i\} \cup \{A_i\}$ for $i = 0, 1$.

If $\lambda : A \to A$ is one-to-one, then the topological space A is a disjoint union of copies of the A_i s.

If $\lambda : A \to A$ is one-to-one without any periodic point, then the topological space A is a disjoint union of copies of the A_i s for $i = 0, 1, 2$.

If $\lambda : A \to A$ is a bijection without any periodic point, then the topological space A is a disjoint union of copies A_1.

2.5. Remark. In an Alexandroff space X, the smallest open neighborhood of $x \in X$ is denoted by $V(x)$ [2]. By [2, Theorem 2.3] the Alexandroff space X is T_0 if and only if

$$\forall\, x, y \in X \quad (V(x) = V(y) \Rightarrow x = y).$$

The following theorem deals with conditions under which a functional Alexandroff space is T_0 or T_1.

2.6. Lemma (T_0 and T_1 spaces)**.** *Let $\alpha : X \to X$ be an arbitrary map. Then (X, τ_α) is:*

(1) T_0 *if and only if α does not have any periodic point except fixed points;*
(2) T_1 *if and only if $\alpha = \mathrm{id}_X$.*

Proof. Since for $x, y \in X$ with $x \neq y$ we have $\downarrow^\alpha x = \downarrow^\alpha y$ if and only if there exists $n \geq 2$ such that $y \in \{x, \alpha(x), \ldots, \alpha^{n-1}(x), \alpha^n(x) = x\}$, using Remark 2.5, (1) is established. Moreover X is T_1 if and only if for each $x \in X$, $\{x\}$ is a closed subset of X, which is equivalent to $\alpha(x) \in \{x\}$ (by Theorem 2.1 (4)), i.e., for each $x \in X$ we have $\alpha(x) = x$ and $\alpha = \mathrm{id}_X$. □

3. Functional Alexandroff spaces as a proper subclass of Alexandroff spaces

The main aim of this section is to find some topological properties which characterize functional Alexandroff spaces amongst all Alexandraff spaces (Main Theorem).

3.1. Lemma. *Let $\alpha : X \to X$ be a map and $\{x_n\}_{n \in \boldsymbol{N}}$ a sequence in X.*

(1) *If $n, m \in \boldsymbol{N}$ are such that $\downarrow^\alpha x_1 \subset \downarrow^\alpha x_2 \subset \downarrow^\alpha x_3$, $\alpha^n(x_1) = x_2$ and $\alpha^m(x_1) = x_3$, then $n < m$.*
(2) *If $\downarrow^\alpha x_1 \supset \downarrow^\alpha x_2 \supset \cdots$, then $\bigcap_{n \in \boldsymbol{N}} \downarrow^\alpha x_n = \emptyset$.*
(3) *If $\downarrow^\alpha x_1 \subset \downarrow^\alpha x_2 \subset \cdots$, then for all $y \in X$, $\bigcup_{n \in \boldsymbol{N}} \downarrow^\alpha x_n \not\subseteq \downarrow^\alpha y$.*
(4) *For all $x \in X$, $\{z \in X : \downarrow^\alpha z = \downarrow^\alpha x\}$ is finite.*
(5) *Let $x, y \in X$. If $\downarrow^\alpha x \subset \downarrow^\alpha y$, then for all $z \in X - \{x\}$ we have $\downarrow^\alpha x \neq \downarrow^\alpha z$.*
(6) *If $x \in X$ is such that for all $y \in X - \{x\}$, $\downarrow^\alpha x \neq \downarrow^\alpha y$, then $\downarrow^\alpha x$ is a T_0 subspace of X.*

Proof. (1) If $n \geq m$, then $x_2 = \alpha^n(x_1) = \alpha^{n-m}(x_3)$ and $x_3 \in \downarrow^\alpha x_2$, so $\downarrow^\alpha x_3 \subseteq \downarrow^\alpha x_2$, which is a contradiction.

(2) If $x \in \bigcap_{n \in \mathbf{N}} \downarrow^\alpha x_n$, then $\downarrow^\alpha x \subseteq \cdots \subset \downarrow^\alpha x_2 \subset \downarrow^\alpha x_1$, so for each $n \in \mathbf{N}$ there exists $m_n \in \mathbf{N}$ such that $\alpha^{m_n}(x) = x_n$. By (1) we have $\cdots < m_2 < m_1$, which is a contradiction.

(3) If $y \in X$ is such that $\bigcup_{n \in \mathbf{N}} \downarrow^\alpha x_n \subseteq \downarrow^\alpha y$, then there exists $k \in \mathbf{N}$ such that $\alpha^k(x_1) = y$ and for all $n \geq 2$ there exists $m_n \in \mathbf{N}$ with $\alpha^{m_n}(x_1) = x_n$. By (1) we have $m_2 < m_3 < \cdots < k$, which is a contradiction.

(4) If $z \neq x$ is such that $\downarrow^\alpha x = \downarrow^\alpha z$, then there exist $n, m \in \mathbf{N}$ such that $\alpha^n(x) = z$ and $\alpha^m(z) = x$. It is easy to show $\{w \in X : \downarrow^\alpha w = \downarrow^\alpha x\} = \{x, \alpha(x), \ldots, \alpha^{n+m-1}(x)\}$, and $\alpha^{n+m}(x) = x$.

(5) Let $\downarrow^\alpha x \subset \downarrow^\alpha y$ and $z \in X - \{x\}$ be such that $\downarrow^\alpha x = \downarrow^\alpha z$. Using the same method as in (4), there exists $p \in \mathbf{N}$ such that $\alpha^p(x) = x$ and $z \in \{\alpha(x), \ldots, \alpha^p(x) = x\}$,

moreover it is evident that $\downarrow^\alpha x = \downarrow^\alpha \alpha^i(x)$ for all $i \in \{1,\dots,p\}$. Since $x \in \downarrow^\alpha y$, thus there exists $q \geq 0$ such that $\alpha^q(x) = y$, by $\alpha^p(x) = x$ we may choose $q \in \{1,\dots,p\}$, which leads to the contradiction $\downarrow^\alpha x = \downarrow^\alpha y$.

(6) Use (5) and [2, Theorem 2.3]. ☐

3.2. Lemma. *Let X be an Alexandroff space such that for all $x, y \in X$ we have*

$$V(x) \subseteq V(y) \vee V(y) \subseteq V(x) \vee V(x) \cap V(y) = \emptyset.$$

Then the following statements are equivalent:

(1) *For all $x, y \in X$, $\{V(z) : z \in X \wedge V(x) \subset V(z) \subset V(y)\}$ is finite.*
(2) *For all sequences $\{x_n\}_{n\in\boldsymbol{N}}$ in X we have:*
   i. *if $V(x_1) \supset V(x_2) \supset \cdots$, then $\bigcap\limits_{n\in\boldsymbol{N}} V(x_n) = \emptyset$;*
   ii. *if $V(x_1) \subset V(x_2) \subset \cdots$, then for all $y \in X$, $\bigcup_{n\in\boldsymbol{N}} V(x_n) \not\subseteq V(y)$.*

Proof. Let for all $x, y \in X$, $\{V(z) : z \in X \wedge V(x) \subset V(z) \subset V(y)\}$ is finite. If $V(x_1) \supset V(x_2) \supset \cdots$ is such that $\bigcap_{n\in\mathbf{N}} V(x_n) \neq \emptyset$, then for $w \in \bigcap\limits_{n\in\mathbf{N}} V(x_n)$ we have $V(w) \subseteq \bigcap\limits_{n\in\mathbf{N}} V(x_n)$ and $\{V(x_i) : i \geq 2\} \subseteq \{V(z) : V(w) \subset V(z) \subset V(x_1)\}$, which is a contradiction since $\{V(x_i) : i \geq 2\}$ is infinite. Moreover if $V(x_1) \subset V(x_2) \subset \cdots$ and $\bigcup_{n\in\mathbf{N}} V(x_n) \subseteq V(y)$, then $\{V(x_i) : i \geq 2\} \subseteq \{V(z) : V(x_1) \subset V(z) \subset V(y)\}$, which is again a contradiction since $\{V(x_i) : i \geq 2\}$ is infinite. Thus (i) and (ii) are valid.

Conversely suppose (i) and (ii) are valid. If $V(x) \subset V(y)$, then $(\{V(z) : V(x) \subset V(z) \subset V(y)\}, \subseteq)$ is a chain, using (i) and (ii) it is a finite chain, which leads to (1). ☐

3.3. Corollary. *In a functional Alexandroff space X, for any $x, y \in X$, $\{z \in X : V(x) \subseteq V(z) \subseteq V(y)\}$ is finite.*

Proof. By Lemma 3.1, any functional Alexandroff space X satisfies Lemma 3.2 (2), therefore it satisfies Lemma 3.2 (1), in particular for any $x, y \in X$, $\{z \in X : V(x) \subseteq V(z) \subseteq V(y)\}$ is finite, again by using Lemma 3.1. ☐

3.4. Lemma. *Disjoint unions of functional Alexandroff spaces are functional Alexandroff.*

Proof. Let $\{X_t : t \in \Gamma\}$ be a collection of disjoint functional Alexandroff spaces such that for each $t \in \Gamma$, the topology on X_t is that induced by $f_t : X_t \to X_t$. Then the disjoint topology on $X := \bigcup_{t\in\Gamma} X_t$ (i.e. $\{U \subseteq X : \forall t \in \Gamma \quad U \cap X_t$ is an open subset of $X_t\}$) is the functional Alexandroff topology induced by $\bigcup_{t\in\Gamma} f_t$. ☐

3.5. Theorem. [Main Theorem] *Let X be an Alexandroff space. Then, X is a functional Alexandroff space if and only if the following statements hold:*

($\mathcal{C}_1$) $\forall x, y \in X \quad V(x) \subseteq V(y) \vee V(y) \subseteq V(x) \vee V(x) \cap V(y) = \emptyset$;
($\mathcal{C}_2$) *For $x \in X$, if there exists $y \in X$ with $V(x) \subset V(y)$ then for all $z \in X - \{x\}$ we have $V(x) \neq V(z)$;*
($\mathcal{C}_3$) *For all $x, y \in X$, $\{z \in X : V(x) \subseteq V(z) \subseteq V(y)\}$ is finite.*

Proof. If X is a functional Alexandroff space, then by Lemma 3.1 and Lemma 3.2, X satisfies ($\mathcal{C}_1$), ($\mathcal{C}_2$) and ($\mathcal{C}_3$).

Now suppose X is an Alexandroff space that satisfies ($\mathcal{C}_1$), ($\mathcal{C}_2$) and ($\mathcal{C}_3$). We distinguish the following cases:

First case: $(\{V(x) : x \in X\}, \subseteq)$ has a maximum element, (i.e.,there exists $a \in X$ such that $V(a) = X$).

By $(\mathcal{C}_3)$ there exist distinct elements $a_1, \dots, a_n \in X$ such that $\{a_1, \dots, a_n\} = \{x \in X : V(x) = X\} =: A$. For all $x \in X \backslash A$, $\{z \in X : V(x) \subset V(z)\} \neq \emptyset$. By $(\mathcal{C}_3)$ and $(\mathcal{C}_1)$ there exist $b_1, \dots, b_p \in X$ such that $V(x) \subset V(b_1) \subset \cdots \subset V(b_p)$ and $\{V(z) : V(x) \subset V(z)\} = \{V(b_i) : i \in \{1, \dots, p\}\}$ (note that if $b_1 \in X \backslash A$, then b_1 is determined uniquely by $(\mathcal{C}_1)$), define:

$$\hat{x} = \begin{cases} a_1 & \text{if } b_1 \in A, \\ b_1 & \text{if } b_1 \in X \backslash A. \end{cases}$$

Define $\alpha : X \to X$ by $\alpha(x) = \hat{x}$ for $x \in X \backslash A$, $\alpha(a_1) = a_2$, $\alpha(a_2) = a_3, \dots, \alpha(a_{n-1}) = a_n$, $\alpha(a_n) = a_1$. We claim that τ_α coincides with the topology on X.

It is enough to show for each $x \in X$ we have $V(x) = \downarrow^\alpha x$, with this aim we follow the steps below:

- By the definition of α, for all $x \in X \backslash A$ we have $V(x) \subset V(\alpha(x))$.
- For $x \in X \backslash A$ and $z \in \downarrow^\alpha x - \{x\}$ there exists $m \geq 1$ such that $\alpha^m(z) = x$, therefore $z \in V(z) \subset V(\alpha(z)) \subset \cdots \subset V(\alpha^m(z)) = V(x)$. Thus $\downarrow^\alpha x \subseteq V(x)$.
- For $x \in X$ and $z \in V(x) - \{x\}$ we have $V(z) \subseteq V(x)$ using $(\mathcal{C}_1)$, $(\mathcal{C}_3)$ and the definition of α, there exists $m \geq 1$ such that $\alpha^m(z) = x$ and $z \in \downarrow^\alpha x$. Therefore $V(x) \subseteq \downarrow^\alpha x$.

Second case: $(\{V(x) : x \in X\}, \subseteq)$ does not have a maximal element.

By $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$, X is T_0. Suppose $x \in X$. There exists $y \in X$ such that $V(x) \subset V(y)$. By $(\mathcal{C}_1)$ and $(\mathcal{C}_3)$ there exist $b_1, \dots, b_p \in X$ such that $V(x) \subset V(b_1) \subset \cdots \subset V(b_p) = V(y)$ and $\{V(z) : V(x) \subset V(z) \subseteq V(y)\} = \{V(b_i) : i \in \{1, \dots, p\}\}$. Set $\hat{x} := b_1$. Define $\alpha : X \to X$ by $\alpha(x) = \hat{x}$. Using a similar method to that described in the first case, we have $V(x) = \downarrow^\alpha x$.

Third case: The general case.

Let

$$A := \{y \in X : V(y) \text{ be a maximal element of } (\{V(x) : x \in X\}, \subseteq)\}.$$

Set $Y := X \backslash \bigcup\{V(y) : y \in A\}$. By $(\mathcal{C}_1)$ and the definition of A, Y is an open subset of X. Clearly, $(\{V(x) : x \in Y\}, \subseteq)$ does not have a maximal element, so using the second case, Y is a functional Alexandroff space. For all $y \in A$, using the first case, $V(y)$ is an functional Alexandroff space. But X is a disjoint union of elements of $\{V(y) : y \in A\} \cup \{Y\}$. So X is a functional Alexandroff (use Lemma 3.4). □

3.6. Counterexamples. In Theorem 3.5:

- Consider $X = \{1, 2, 3\}$ under the topology $\tau = \{\emptyset, X, \{1\}, \{1, 2\}, \{1, 3\}\}$. Then, (X, τ) is a T_0 Alexandroff space, but is not a Functional Alexandroff space. Indeed, (X, τ) satisfies items $(\mathcal{C}_2)$ and $(\mathcal{C}_3)$, but does not satisfy $(\mathcal{C}_1)$.
- Consider $X = \{1, 2, 3\}$ under the topology $\tau = \{\emptyset, X, \{1, 2\}\}$. Then, (X, τ) is an Alexandroff space, but is not a Functional Alexandroff space. Indeed, (X, τ) satisfies items $(\mathcal{C}_1)$ and $(\mathcal{C}_3)$, but does not satisfy $(\mathcal{C}_2)$.
- Let $X = \{0, 1, 2, \dots\}$, $V(0) = \{0\}$, $V(n) = \{0, n, n+1, \dots\}(n \in \mathbf{N})$. Consider X under the topology τ generated by the topological basis $\{V(n) : n = 0, 1, 2, \dots\}$. Then (X, τ) is a T_0 Alexandroff space, but is not Functional Alexandroff space. Indeed, (X, τ) satisfies items $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$, but it does not satisfy $(\mathcal{C}_3)$.

3.7. Corollary. *In an infinite functional Alexandroff space X, $w(X) = |X|$ (where $w(X) = \min\{|\mathcal{B}| : \mathcal{B}$ is a basis for a topological space $X\}$ is the weight of X).*

Proof. In an Alexandroff space X, $w(X) = |\{V(x) : x \in X\}| \leq |X|$. If X is an infinite functional Alexandroff space, then by $(\mathcal{C}_3)$ in Theorem 3.5 (Main Theorem), for each $x \in X$, $\{z \in X : V(z) = V(x)\}$ is finite, which leads to $|\{V(x) : x \in X\}| = |X|$ and $w(X) = |X|$. □

4. A functional Alexandroff space as a dynamical system

For an arbitrary map $\alpha : X \to X$, when X is endowed with the topology τ_α, $\alpha : X \to X$ is continuous and one may be interested in the dynamical system (X, α). In this section we will study such dynamical systems.

Let (X, α) and (Y, β) be two dynamical systems. A continuous function $\varphi : (X, \alpha) \to (Y, \beta)$ is called a homomorphism if $\varphi\alpha = \beta\varphi$, moreover if it is a homeomorphism too, then it is called an isomorphism. The following theorem characterizes all of these homomorphisms and isomorphisms.

4.1. Theorem. *In dynamical systems (X, α) and (Y, β), when X is considered under the topology τ_α and Y under the topology τ_β, we have:*

(1) *$\varphi : (X, \alpha) \to (Y, \beta)$ is a homomorphism of dynamical systems if and only if $\varphi\alpha = \beta\varphi$.*
(2) *A bijection $\varphi : (X, \alpha) \to (Y, \beta)$ is an isomorphism of dynamical systems if and only if $\varphi\alpha = \beta\varphi$.*

Proof. Let $\varphi : X \to Y$ be such that $\varphi\alpha = \beta\varphi$. We claim that $\varphi : X \to Y$ is continuous. Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a net in X converging to x. Then there exists $\lambda_0 \in \Lambda$ such that for all $\lambda \geq \lambda_0$ we have $x_\lambda \in \downarrow^\alpha x$; therefore for all $\lambda \geq \lambda_0$ there exists $n_\lambda \geq 0$ such that $\alpha^{n_\lambda}(x_\lambda) = x$, so $\varphi(x) = \varphi(\alpha^{n_\lambda}(x_\lambda)) = \beta^{n_\lambda}(\varphi(x_\lambda))$ and $\varphi(x_\lambda) \in \downarrow^\beta (\varphi(x))$, hence $\{\varphi(x_\lambda)\}_{\lambda \in \Lambda}$ converges to $\varphi(x)$ and $\varphi : X \to Y$ is continuous, which completes the proof of (1).

Moreover, if $\varphi : (X, \alpha) \to (Y, \beta)$ is a bijection such that $\varphi\alpha = \beta\varphi$, then since $\alpha\varphi^{-1} = \varphi^{-1}\beta$ and using (1) we see that both $\varphi : (X, \alpha) \to (Y, \beta)$ and $\varphi^{-1} : (Y, \beta) \to (X, \alpha)$ are continuous homomorphisms, which leads to (2). □

4.2. Note. In the dynamical system (X, α), when X is endowed with the topology τ_α, calculation of the orbit closure of a point is easy by Theorem 2.1, moreover orbit closure and the orbit of a point are the same, i.e., for each $x \in X$,

$$\overline{\{x\}} = \overline{\{\alpha^n(x) : n \geq 0\}} = \{\alpha^n(x) : n \geq 0\}.$$

In the dynamical system (X, α), we call $(x, y) \in X \times X$ a *proximal pair* if there exists a net $\{n_\lambda\}_{\lambda \in \Lambda}$ in $\mathbf{N} \cup \{0\}$ and $z \in X$ such that $\{\alpha^{n_\lambda}(x)\}_{\lambda \in \Lambda}$ and $\{\alpha^{n_\lambda}(y)\}_{\lambda \in \Lambda}$ converge to z. The set of all proximal pairs is denoted by $\mathrm{P}(X)$. We call (X, α) *distal* (resp. *proximal*), if $\mathrm{P}(X) = \Delta_X$ (resp. $\mathrm{P}(X) = X \times X$).

4.3. Theorem. *In the dynamical system (X, α), when X is endowed with the topology τ_α, we have*

$$\begin{aligned}\mathrm{P}(X) &= \{(x, y) \in X \times X : \exists z \in X \quad x, y \in \downarrow^\alpha z\} \\ &= \{(x, y) \in X \times X : x, y \textit{ belong to a connected component of } X\}.\end{aligned}$$

Proof. Let $x, y, z \in X$ and suppose the net $\{n_\lambda\}_{\lambda \in \Lambda}$ in $\mathbf{N} \cup \{0\}$ is such that $\{\alpha^{n_\lambda}(x)\}_{\lambda \in \Lambda}$ and $\{\alpha^{n_\lambda}(y)\}_{\lambda \in \Lambda}$ converge to z. Then

$$z \in \overline{\{\alpha^n(x) : n \geq 0\}} \cap \overline{\{\alpha^n(y) : n \geq 0\}} = \{\alpha^n(x) : n \geq 0\} \cap \{\alpha^n(y) : n \geq 0\},$$

and there exist $n, m \geq 0$ such that $\alpha^n(x) = z = \alpha^m(y)$. Therefore

$$\mathrm{P}(X) \subseteq \{(x, y) \in X \times X : \exists z \in X \quad x, y \in \downarrow^\alpha z\}.$$

On the other hand suppose $x, y, z \in X$ are such that $x, y \in \downarrow^\alpha z$, thus there exist $n, m \geq 0$ such that $\alpha^n(x) = z = \alpha^m(y)$. Let $n \leq m$. The constant sequences $\{\alpha^n(x)\}_{k\in\mathbf{N}}$ and $\{\alpha^n(y)\}_{k\in\mathbf{N}}$ converge to z, therefore $(x, y) \in \mathrm{P}(X)$. □

4.4. Corollary. *A dynamical system (X, α), when X is endowed with the topology τ_α, is distal if and only if $\alpha = \mathrm{id}_X$.*

Proof. By Theorem 4.3, (X, α) is distal if and only if for each $z \in X$ we have $\downarrow^\alpha z = \{z\}$. So if (X, α) is distal, then for each $z \in X$ we have $z \in \downarrow^\alpha \alpha(z) = \{\alpha(z)\}$, which shows $\alpha(z) = z$. □

4.5. Corollary. *A dynamical system (X, α), when X is endowed with the topology τ_α, is proximal if and only if X is connected.*

Proof. Use Theorem 4.3. □

Acknowledgements

With thanks to Prof. D. Dikranjan for his helpful comments, ideas and encouragement.

With thanks to the Research Division of the University of Tehran for the grant which supported this research (ref. no. 6103027/5/04).

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