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ON SEMISYMMETRIC CUBIC GRAPHS OF ORDER 10p^3

Mehdi Alaeiyan*† and B. Naimeh Onagh‡

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Abstract

Connected cubic graphs of order $10p^3$ which admit an automorphism group acting semisymmetrically are investigated. We prove that every connected cubic edge-transitive graph of order $10p^3$ is vertex-transitive, where p is a prime.

Keywords: Automorphism group, Regular cover, Semisymmetric graph.

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1. Introduction

Throughout this paper, graphs are assumed to be finite, simple, undirected and connected. For the group-theoretic concepts and notation not defined here we refer the reader to [4, 8, 14]. Given a graph X, we let $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}X$ be the vertex set, the edge set, the arc set and the automorphism group of X, respectively.

If a subgroup G of $\mathrm{Aut}X$ acts transitively on $V(X)$, $E(X)$ and $A(X)$, then X is said to be *G-vertex-transitive, G-edge-transitive* and *G-arc-transitive*, respectively. It is easily seen that a graph X which is G-edge- but not G-vertex-transitive is necessarily bipartite, with the two parts of the bipartition coinciding with the orbits of G. In particular, if X is a regular, then these two parts have equal cardinalities, and such a graph is then referred to as being *G-semisymmetric*. In the case where $G = \mathrm{Aut}X$ the symbol G may be omitted from the definitions above, so that X is called semisymmetric if it is regular and $\mathrm{Aut}X$-edge-transitive but not $\mathrm{Aut}X$-vertex-transitive.

An *s-arc* in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$. A graph X is said to be *s-arc-transitive* if $\mathrm{Aut}(X)$ is transitive on the set of s-arcs of X. In particular, 0-arc-transitive means vertex-transitive, and 1-arc-transitive means *arc-transitive* or *symmetric*.

The study of semisymmetric graphs was initiated by Folkman [7].

*Department of Mathematics, Iran University of Science and Technology, Narmak, Tehran 16844, Iran. E-mail: alaeiyan@iust.ac.ir

†Corresponding Author.

‡Department of Mathematics, Golestan University, Gorgan, Iran. E-mail: b_onagh@yahoo.com

Semisymmetric graphs of order $2pq$, and semisymmetric cubic graphs of orders $2p^3$, $6p^2$ and $2p^2q$ are classified in [5, 10, 9, 13], where p and q are primes. Also in [1] it is proved that every edge-transitive cubic graph of order $8p^2$ is vertex-transitive, where p is a prime. In [3] an overview of known families of cubic semisymmetric graphs is given.

Semisymmetric cubic graphs of orders $10p$ and $10p^2$ are special cases of the semisymmetric graphs in [5, 13]. The objective of this paper is to investigate all connected cubic semisymmetric graphs of order $10p^3$. In particular, we have shown that there is no semisymmetric cubic graphs of order $10p^3$. The main result of this paper is as follows:

1.1. Theorem. *Let p be a prime. Then, every connected edge-transitive cubic graph of order $10p^3$ is vertex-transitive.*

As a result, we can conclude that every connected edge-transitive cubic graph of order $10p^3$ is symmetric.

2. Preliminaries

Given a finite group G, consider pairs of groups (H, Z), where $Z \subseteq Z(H)$ and $H/Z \cong G$. In this situation, we say H is a *central extension* of G. The largest possible second component of a pair (H, Z) associated with a given group G is called the *Schur multiplier* of G.

Let X be a graph, N a subgroup of $\mathrm{Aut}(X)$, and K a finite group. For u, $v \in V(\Gamma)$, denote by $\{u, v\}$ the edge incident to u and v in X, and by $N_X(u)$ the set of vertices adjacent to u in X. The *quotient graph* X/N or X_N induced by N is defined as the graph such that the set Σ of N-orbits in $V(X)$ is the vertex set of X/N and $B, C \in \Sigma$ are adjacent if and only if there exist $u \in B$ and $v \in C$ such that $\{u, v\} \in E(X)$.

A graph $\widetilde{X}$ is called a *covering* of a graph X with projection $\wp : \widetilde{X} \to X$ if there is a surjection $\wp : V(\widetilde{X}) \to V(X)$ such that $\wp_{N_{\widetilde{X}}(\tilde{v})} : N_{\widetilde{X}}(\tilde{v}) \to N_X(v)$ is a bijection for any vertex $v \in V(X)$ and $\tilde{v} \in \wp^{-1}(v)$.

A covering $\widetilde{X}$ of X with a projection $\wp$ is said to be *regular* (or a *K-covering*) if there is a semiregular subgroup K of the automorphism group $\mathrm{Aut}(\widetilde{X})$ such that X is isomorphic to the quotient graph $\widetilde{X}/K$, say by h, and the quotient map $\widetilde{X} \to \widetilde{X}/K$ is the composition $\wp h$ of $\wp$ and h; to emphasize this we sometimes write $\wp_N$ instead of just $\wp$.

Let X be a graph and K a finite group. By a^{-1} we mean the reverse arc to an arc a. A *voltage assignment* (or, *K-voltage assignment*) of X is a function $\xi : A(X) \to K$ with the property that $\xi(a^{-1}) = (\xi(a))^{-1}$ for each arc $a \in A(X)$. The values of ξ are called *voltages*, and K is the *voltage group*. The graph $X \times_\xi K$ derived from a voltage assignment $\xi : A(X) \to K$ has vertex set $V(X) \times K$ and edge set $E(X) \times K$, so that an edge (e, g) of $X \times K$ joins a vertex (u, g) to $(v, g\xi(a))$ for $a = (u, v) \in A(X)$ and $g \in K$, where $e = \{u, v\}$. If $\xi(a) = 1$ for each arc $a \in A(X)$, then the covering $X \times_\xi \mathbb{Z}_2$ is called the *canonical double covering* of X.

Let $\wp : \widetilde{X} \to X$ be a covering projection. The vertices in $\wp^{-1}(v)$ form the *fibre* over the vertex v; we similarly define the fibre over an edge $e \in E(X)$. If X is connected, as we assume in this paper, then any two vertex or edge fibres are of the same cardinality n. This number is called the *fold-number* of the covering, and we say that $\wp$ is an *n-fold covering.* We remark that any covering of a bipartite graph is bipartite, but:

2.1. Proposition. *[2] If $\tilde{X}$ is a bipartite covering of a non-bipartite graph X, then the fold number is even.* □

The next proposition is a special case of [9, Lemma 3.2].

2.2. Proposition. *Let X be a connected G-semisymmetric cubic graph with bipartition sets $U(X)$ and $W(X)$, where $G \leq \mathrm{Aut}(X)$. Moreover, suppose that N is a normal subgroup of G. If N is intransitive on bipartition sets, then N acts semiregularly on both $U(X)$ and $W(X)$, and X is an N-regular covering of a G/N-semisymmetric cubic graph.*

We quote the following propositions.

2.3. Proposition. [10] *The vertex stabilizers of a connected G-semisymmetric cubic graph X have order $2^r \cdot 3$, where $r \geq 0$. Moreover, if u and v are two adjacent vertices, then $G = \langle G_u, G_v \rangle$, and the edge stabilizer $G_u \cap G_v$ is a common Sylow 2-subgroup of G_u and G_v.* □

2.4. Proposition. [10] *Let X be a connected bipartite graph admitting an abelian subgroup $G \leq \mathrm{Aut}X$ acting regularly on each of the bipartition sets. Then, X is vertex-transitive.* □

2.5. Proposition. [12] *Every both edge-transitive and vertex-transitive cubic graph is symmetric.* □

2.6. Proposition. [6] *Let p be a prime and X a connected cubic symmetric graph of order $10p$ or $10p^2$. Then, X is 2-, 3- or 5-regular. Furthermore,*

(1) *X is 2-regular if and only if X is isomorphic to the Dodecahedron D_{20} of order 20, the Cayley graph C_{50} or C_{250} of orders 50 and 250, respectively.*
(2) *X is 3-regular if and only if X is isomorphic to the canonical double covering $O_3^{(2)}$ of the Petersen graph O_3, the canonical double covering $D_{20}^{(2)}$ of the Dodecahedron D_{20}, or the Coxeter-Frucht graph CF_{110}.*
(3) *X is 5-regular if and only if X is isomorphic to the Levi graph L_{30} of order 30, or the Biggs-Smith graph BS_{90}.* □

3. Proof of Theorem 1.1

Let X be a connected semisymmetric cubic graph of order $10p^3$, where p is a prime. Note that by [3], for $p = 2$ and $p = 3$ there is no semisymmetric cubic graph of order $10p^3$. So, we may assume that $p \geq 5$. Therefore, we divide our proof into the following two cases. First, we consider the case $p = 5$.

3.1. Lemma. *Let X be a connected semisymmetric cubic graph of order $2 \cdot 5^4$. Then, X is vertex-transitive.*

Proof. Let X be a connected semisymmetric cubic graph of order $2 \cdot 5^4$. Denote by $U(X)$ and $W(X)$ the bipartition sets of X, where $|U(X)| = |W(X)| = 5^4$. Set $A := \mathrm{Aut}(X)$, and let $Q := O_5(A)$ be the maximal normal 5-subgroup of A. By Proposition 2.3, we have $|A| = 2^r \cdot 3 \cdot 5^4$, $r \geq 0$.

First suppose that $|Q| = 1$. Let N be a minimal normal subgroup of A. If N is not solvable, then $N \cong T^k$, where T is a non-abelian $\{2, 3, 5\}$-simple group. So, N is isomorphic to A_5. Since $2.5^4 \nmid |N|$, N is intransitive on the bipartition sets and then, by Proposition 2.2, N must be semiregular on $U(X)$ and $W(X)$, a contradiction. So, N is solvable. If N is transitive on $U(X)$ and $W(X)$, then $|N| = 5^4$. If N is intransitive, then, by Proposition 2.2 N acts semiregularly on both $U(X)$ and $W(X)$. So, $|N| = 5$, 5^2 or 5^3. In all cases, we get a contradiction to $|Q| = 1$. Therefore, $|Q| \neq 1$.

Now suppose that $|Q| = 5^i$, $(1 \leq i \leq 2)$. Let X_Q be the quotient graph of X relative to Q, where X_Q is A/Q-semisymmetric. We have $|U(X_Q)| = |W(X_Q)| = 5^{4-i}$. Suppose that N/Q is a minimal normal subgroup of A/N. One can see that N/Q is solvable and then $|N/Q| = 5, \ldots, 5^{4-i}$. Therefore, N is a normal subgroup of A of order $5^{i+1}, \ldots, 5^4$, a contradiction.

If $|Q| = 5^3$, then by Proposition 2.2, X is a Q-regular covering of the A/Q-semisymmetric graph X_Q, where X_Q is an edge-transitive cubic graph of order 10. Observe that the quotient graph X_Q must be vertex-transitive since the smallest semisymmetric cubic graph, the Gray graph, has order 54. Then, by Proposition 2.5, X_Q is a symmetric cubic graph of order 10. Therefore, X_Q is the Petersen graph O_3, the only symmetric cubic graph of order 10. Since X is bipartite and O_3 is non-bipartite, the fold number 5^3 must be even, a contradiction.

Now let $|Q| = 5^4$. Since Q and A/Q are solvable, then A is also an edge-transitive solvable group. By [11, Corollary 4.5], X is a $\mathbb{Z}_5^4$-cover of the 3-dipole Dip_3, a contradiction to [11, Proposition 3.1]. □

In the second case, we assume that $p \geq 7$. Then, we have:

3.2. Lemma. *Let X be a connected semisymmetric cubic graph of order $10p^3$, where $p \geq 7$ is a prime. Set $A := \mathrm{Aut}(X)$, and also let $Q := O_p(A)$ be the maximal normal p-subgroup of A. Then, $|Q| = p^3$.*

Proof. Let X be a cubic graph satisfying the above assumptions. Therefore X is a bipartite graph. Denote by $U(X)$ and $W(X)$, the bipartition sets of X, where $|U(X)| = |W(X)| = 5p^3$. The automorphism group A acts transitively on the set $U(X)$ (and also $W(X)$). So, by Proposition 2.3, $|A| = 2^r \cdot 3 \cdot 5 \cdot p^3$, $(r \geq 0)$.

Let N be a minimal normal subgroup of A. One can deduce that N is solvable. Because otherwise $N \cong T^k$, where T is a non-abelian $\{2, 3, p\}$- or $\{2, 3, 5, p\}$-simple group (see [8]). So, by Proposition 2.2, N is semiregular on $U(X)$ (and also $W(X)$). However this is impossible because $3 \mid |N|$. Thus, we can assume that N is elementary abelian.

First, suppose that $|Q| = 1$. Clearly, N is intransitive on $U(X)$ (and also $W(X)$). Thus, by Proposition 2.2, $|N| = 5$. Now we consider the quotient graph X_N, where $|U(X_N)| = |W(X_N)| = p^3$. Suppose that M/N is a normal minimal subgroup of A/N. If M/N is not solvable, then M/N is isomorphic to a non-abelian $\{2, 3, p\}$-simple group. So, by Proposition 2.2, M/N is semiregular on $U(X_N)$ (and also $W(X_N)$), a contradiction. Therefore, M/N is solvable and so elementary abelian. We have that M/N is either transitive or intransitive on $U(X_N)$ (and also $W(X_N)$). So, $|M/N| = p, p^2$ or p^3. Thus M/N has a characteristic normal subgroup of order p, p^2 or p^3. We can deduce that A has a normal subgroup of order p, p^2 or p^3, which is a contradiction. Thus $|Q| \neq 1$.

Now suppose that $|Q| = p$. Since $5p^3 \nmid p$, by Proposition 2.2, Q is semiregular on $U(X)$ (and also $W(X)$). Let X_Q be the quotient graph, where $|U(X_Q)| = |W(X_Q)| = 5p^2$. Suppose that T/Q is a minimal normal subgroup of A/Q, where $|A/Q| = 2^r \cdot 3 \cdot 5 \cdot p^2$. If T/Q is not solvable, then T/Q is a non-abelian $\{2, 3, p\}$- or $\{2, 3, 5, p\}$-simple group. So, by Proposition 2.2, T/Q is semiregular on $U(X_Q)$, a contradiction. Therefore, T/Q is solvable and then elementary abelian. Since $5p^2 \nmid |T/Q|$, then, by Proposition 2.2, T/Q acts semiregularly on $U(X_Q)$ (and also $W(X_Q)$). So, $|T/Q| = 5$.

Now let X_T be the quotient graph, where $|U(X_T)| = |W(X_T)| = p^2$ and also suppose that K/T is a minimal normal subgroup of A/T. Note that K/T can be intransitive or transitive on $U(X_T)$. So, $|K| = 5p^2$ or $5p^3$, respectively. Therefore, A has a normal subgroup of order p^2 or p^3, a contradiction. Thus $|Q| \neq p$.

Finally, assume that $|Q| = p^2$. Since $5p^3 \nmid p^2$, by Proposition 2.2, Q acts intransitively on $U(X)$ (also $W(X)$) and X is a Q-regular covering of the A/Q-semisymmetric graph X_Q. The quotient graph X_Q is an edge-transitive graph of order $10p$.

Now suppose that $p = 11$ and let $\bar{R} \cong R/Q$ be the minimal normal subgroup of A/Q. If $\bar{R}$ is solvable, then $|\bar{R}| = 5$. Let X_R be the quotient graph, where $|U(X_R)| = |W(X_R)| = 11$. Let L/R be a minimal normal subgroup of A/R. It is obvious that $|L/R| = 11$, so A has a normal subgroup of order 11^3, a contradiction. On the other hand, if $\bar{R}$ is not solvable, then $\bar{R}$ is a non-abelian simple group and $|\bar{R}| = 2^s \cdot 3 \cdot 5 \cdot 11$. It is easy to see that $Z(R) \cong Q$. Then, the simple group $\bar{R}$ has Schur multiplier isomorphic to $Z(R)$, a contradiction to the order of $\bar{R}$.

If $p = 7$ or ≥ 13, then by [5] and by Proposition 2.6, there is no semisymmetric or symmetric cubic graph of order $10p$, which is a contradiction. The result now follows. □

Proof of Theorem 1.1 Now we complete the proof of the main theorem. Suppose to the contrary that X is a connected semisymmetric cubic graph of order $10p^3$, where p is a prime. We remark that there is no semisymmetric cubic graph of order $10p^3$ for $p = 2$ or 3. If $p = 5$, then, by Lemma 3.1, X is vertex-transitive. So, we suppose that $p \geq 7$. By Lemma 3.2, $|Q| = p^3$. Then, by Proposition 2.2, X is a Q-regular covering of the A/Q-semisymmetric graph X_Q. One can see that X_Q must be a symmetric cubic graph of order 10. So, X_Q is the Petersen graph O_3. Now, since X is bipartite and O_3 is non-bipartite, the fold number p^3 must be even, which is a contradiction. Thus Theorem 1.1 now follows. □

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