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Research Article

Interpolation between weighted Lorentz spaces with respect to a vector measure

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Abstract

In this paper, we consider weighted Lorentz spaces with respect to a vector measure and derive some of their properties. We describe the interpolation with a parameter function of these spaces. As an application, we get a type of the generalization of Steffensen's inequality for $L^p(\|m\|)$ and interpolation spaces for couples of Lorentz-Zygmund spaces.

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1. Introduction

We begin our work by recalling the classical Lorentz spaces. Let (Ω, Σ, μ) be a measure space. For $0 < p < \infty$ and $0 < q \leq \infty$ the Lorentz space $L^{p,q}(\mu)$ is the collection of all measurable functions f on Ω such that the quantity

$$\|f\|_{L^{p,q}(\mu)} := \begin{cases} \left(\int_0^\infty (t^{\frac{1}{p}} f_*(t))^q \frac{dt}{t}\right)^{\frac{1}{q}} & (0 < q < \infty), \\ \sup_{t>0} t^{\frac{1}{p}} f_*(t) & (q = \infty) \end{cases}$$

is finite, where f_* denotes the decreasing rearrangement of $|f|$. Note that $L^{p,p}(\mu)$ is just the Lebesgue space $L^p(\mu)$ and $L^{p,\infty}(\mu)$ is the weak-L^p space. The $L^{p,q}(\mu)$ spaces arise in the Lions-Peetre K-method of interpolation: in particular,

$$L^{p,q}(\mu) = (L^{p_0}(\mu), L^{p_1}(\mu))_{\theta,q},$$

where, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. For standard facts concerning Lorentz spaces and K-method, we refer the reader to [2, 4].

Integration of scalar functions with respect to a countably additive vector measure $m : \Sigma \to X$ with values in a Banach space X was introduced by Bartle-Dunford-schwartz [1] and studied by Klvanek-Knowles [18], and Lewis [19, 20]. Recently, several papers have analysed the properties of the spaces of (weakly) p–integrable functions $(L^p_w(m))$ $L^P(m)$, these may be found in, for example, [8, 14–17, 26].

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The Calderón complex interpolation $[X_0, X_1]_\theta$ and $[X_0, X_1]^\theta$, with $0 < \theta < 1$ of the couples (X_0, X_1) where X_0 and X_1 are spaces $L^p(m)$ or $L^p_w(m)$, with $1 \leq p < \infty$, were obtained in [16] and in [12] the Complex interpolation of Orlicz spaces with respect to a vector measure was identified. Moreover, the real interpolation spaces $(X_0, X_1)_{\theta,q}$, where $0 < \theta < 1 \leq q \leq \infty$, and X_0 and X_1 are, as above, $L^p(m)$ or $L^p_w(m)$, with $1 \leq p \leq \infty$, for vector measures on σ–algebras were studied in [14]. More precisely, Let $0 < \theta < 1 \leq q \leq \infty, 1 \leq p_0 < p_1 \leq \infty$, and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ we have

$$\begin{aligned} (L^{p_0}(m), L^{p_1}(m))_{\theta,q} &= (L^{p_0}_w(m), L^{p_1}(m))_{\theta,q} \\ &= (L^{p_0}_w(m), L^{p_1}_w(m))_{\theta,q} \\ &= L^{p,q}(\|m\|). \end{aligned} \tag{1.1}$$

The real interpolation spaces of these spaces for vector measures on δ-ring described in [11]. We recall that L^p spaces of vector measure on σ–algebras are as the finite measure scalar case, we always have that $L^p(m) \cap L^\infty(m) = L^\infty(m)$ and $L^p(m) \cap L^1(m) = L^p(m)$ and the same happens with the corresponding spaces of the semivariation $\|m\|$, and the case of δ–ring corresponds to the case of infinite scalar measures.

The aim of the present paper is to study several structure properties of the weighted Lorentz spaces $\Lambda^p_{\|m\|}(\varphi)$ and we describe interpolation with a parameter function between these spaces. Indeed, in this paper, by replacing t^θ by a more general (parameter) function $\varrho = \varrho(t)$ in (1.1), as $p_0 = 1, p_1 = \infty$, we prove that $(L^1(m), L^\infty(m))_{\varrho,q} = \Lambda^q_{\|m\|}(\frac{t}{\varrho(t)})$.

2. Weakly p-integrable and p-integrable functions

Let us recall that some basic facts and introduce some notations to a vector measure. Let $m : \Sigma \to X$ be a vector measure defined on a σ-algebra of subsets of a nonempty set Ω, this will means that m is countably additive on Σ with range in Banach space X. We denote by X^* its dual space and by X^{**} its bidual. Also $B(X)$ denotes the unit ball of X. The semivariation of m is the set function $\|m\|(A) = \sup\{|\langle m, x^*\rangle|(A) : x^* \in B(X^*)\}$, for each $A \in \Sigma$, where $|\langle m, x^*\rangle|$ is the total variation of the scalar measure $\langle m, x^*\rangle$ given by $\langle m, x^*\rangle(A) = \langle m(A), x^*\rangle$.

A measurable function $f : \Omega \to \mathbb{R}$ is called weakly integrable (with respect to m) if $f \in L^1(|\langle m, x^*\rangle|)$ for any $x^* \in X^*$ and for each $A \in \Sigma$ there exists an element $\int_A f dm \in X^{**}$ such that $\langle \int_A f dm, x^*\rangle = \int_A f d\langle m, x^*\rangle$ for $x^* \in X^*$. The space $L^1_w(m)$ of all (equivalence classes of) weakly integrable functions becomes a Banach lattice when it is endowed with the norm

$$\|f\|_1 := \sup\left\{\int_\Omega |f| d|\langle m, x^*\rangle| : x^* \in B(X^*)\right\}.$$

We say that a weakly integrable function f is integrable (with respect to m) if the vector $\int_A f dm \in X$ for all $A \in \Sigma$. It is clear from the definition that $L^1(m) \subseteq L^1_w(m)$ and in general, this inclusion is strict. In [27] Stefansson obtains conditions under which the equality $L^1(m) = L^1_w(m)$ holds. Properties of the space of integrable functions $L^1(m)$ have already been studied in [6–8, 17, 24, 27].

Let $1 < p < \infty$. The spaces of p-integrable functions was introduced by Sánchez-Pérez and the corresponding spaces $L^p(m)$ and $L^p_w(m)$ have been studied in depth by many authors being their behavior well understood, (see [9, 15, 26]). We say that a measurable function f is weakly p-integrable with respect to m, if $|f|^p \in L^1_w(m)$ and p-integrable with respect to m, if $|f|^p \in L^1(m)$. We denote by $(L^p_w(m))$ $L^p(m)$ the corresponding spaces of (weakly) p-integrable functions with respect to m, which is a Banach space when equipped with the norm

$$\|f\|_p := \sup\left\{\left(\int_\Omega |f|^p d|\langle m, x^*\rangle|\right)^{\frac{1}{p}} : x^* \in B(X^*)\right\}.$$

Clearly $L^p(m) \subseteq L^p_w(m)$. In particular in [15] the authors studied the case equality $L^p(m) = L^p_w(m)$ holds. For the general theory of vector measures we refer the reader to [10].

3. Weighted Lorentz spaces with respect to a vector measure

For the measurable function f on a measure space (Ω, m) where m is a vector measure, we define its distribution function by $\|m\|_f(t) := \|m\|(\{w \in \Omega : |f(w)| > t\})$, where $\|m\|$ is the semivariation of the measure m. This distribution function $\|m\|_f$ has similar properties that in the scalar case [2, 14]. Also, the decreasing rearrangement of f, defined by

$$f_*(s) := \inf\{t > 0 : \|m\|_f(t) \leq s\}$$

for all $s > 0$. Note that

$$\begin{aligned} \inf\{t > 0 : \|m\|_f(t) \leq s\} &= \sup\{t > 0 : \|m\|_f(t) > s\} \\ &= \lambda\{t > 0 : \|m\|_f(t) > s\} = \lambda_{\|m\|_f}(s), \end{aligned}$$

where $\lambda_{\|m\|_f}$ is the distribution function of $\|m\|_f$, with respect to the Lebesgue measure λ on the interval $[0, \infty)$.
In [14] Fernandez et al. introduced Lorentz spaces with respect to a vector measure and given some of their fundamental properties. For $1 \leq p, q \leq \infty$ the Lorentz space $L^{p,q}(\|m\|)$, is the space of all measurable functions f such that the quantity

$$\|f\|_{L^{p,q}(\|m\|)} := \begin{cases} \left(\int_0^\infty (t^{\frac{1}{p}} f_*(t))^q \frac{dt}{t}\right)^{\frac{1}{q}} & (1 \leq q < \infty), \\ \sup_{t>0} t^{\frac{1}{p}} f_*(t) & (q = \infty) \end{cases}$$

is finite. In the special case in which $1 \leq p = q \leq \infty$, we denote the space $L^{p,p}(\|m\|)$ simply by $L^p(\|m\|)$. The next result gives alternative descriptions of the $\|.\|_{L^p(\|m\|)}$ in term of distribution function and the decreasing rearrangement.

Remark 3.1. Let f be a measurable function. If $1 \leq p < \infty$, then by definition of norm in $L^p(\|m\|)$ and [14, Proposition 2], we have

$$\|f\|^p_{L^p(\|m\|)} = \int_0^\infty f_*(s)^p ds = p \int_0^\infty t^{p-1} \|m\|_f(t) dt, \tag{3.1}$$

Furthermore, in the case $p = \infty$, $\|f\|_{L^\infty(\|m\|)} = \sup_{s>0} f_*(s) = f_*(0)$. It follows from (3.1) that $L^p(\|m\|)$ are rearrangement-invariant function spaces as $1 < p < \infty$. Aspects related to rearrangement-invariant spaces can be seen in [2].

Now we define the weighted Lorentz spaces with respect to a vector measure m which are generalization of the Lorentz spaces $L^{p,q}(\|m\|)$ and derive some of their elementary properties. Let $1 \leq p < \infty$ and $\varphi(t)$ be a given weight, nonnegative measurable function on $(0, \infty)$. The weighted Lorentz space $\Lambda^p_{\|m\|}(\varphi)$ with respect to a vector measure m, is defined to be the collection of all functions f for which the quantity

$$\|f\|_{\Lambda^p_{\|m\|}(\varphi)} := \left(\int_0^\infty (f_*(t)\varphi(t))^p \frac{dt}{t}\right)^{\frac{1}{p}} \qquad 1 \leq p < \infty,$$

is finite.
Moreover, integration by parts yields

$$\int_0^\infty (f_*(t)\varphi(t))^p \frac{dt}{t} = p \int_0^\infty y^{p-1} \left\{\int_0^{\|m\|_f(y)} \varphi^p(t) \frac{dt}{t}\right\} dy \qquad 1 \leq p < \infty,$$

and hence

$$\int_0^\infty (f_*(t)\varphi(t))^p \frac{dt}{t} = p \int_0^\infty y^{p-1} w^p (\|m\|_f(y)) \, dy,$$

where $w(t) = \{\int_0^t \varphi^p(s)\frac{ds}{s}\}^{\frac{1}{p}}$ is a positive, nondecreasing weight (see [5]). From now on, we delete the subscript $\|m\|$. For $p = \infty$ we define

$$\|f\|_{\Lambda^\infty(\varphi)} = \|f\|_{\Lambda^\infty(w)} := \sup_s f_*(s)w(s) = \sup_y yw(\|m\|_f(y)) < \infty.$$

Note that, if $\varphi(t) = t^{\frac{1}{q}}$, then $\Lambda^p(\varphi) = L^{q,p}(\|m\|)$ and $\Lambda^\infty(\varphi)$ coincides with $L^{q,\infty}(\|m\|)$. Recall that for $1 \le p \le \infty$, $\|.\|_{\Lambda^p(\varphi)}$ is a quasi-norm if its "*fundamental function*" $w(t) = \{\int_0^t \varphi^p(s)\frac{ds}{s}\}^{1/p}$ satisfies the Δ_2-condition, $w(2t) \le cw(t)$, for some $c > 0$, in fact, since w is a nondecreasing function one has that $w(x+y) \le c(w(x) + w(y))$ and hence,

$$\begin{aligned}
\|f+g\|^p_{\Lambda^p(\varphi)} &= p\int_0^\infty y^{p-1}w^p\left(\|m\|_{f+g}(y)\right)dy \\
&\le p\int_0^\infty y^{p-1}w^p\left(\|m\|_f(\frac{y}{2}) + \|m\|_g(\frac{y}{2})\right)dy \\
&\le c\int_0^\infty y^{p-1}\left(w^p(\|m\|_f(\frac{y}{2})) + w^p(\|m\|_g(\frac{y}{2})\right)dy \\
&\le c\left(\|f\|^p_{\Lambda^p(\varphi)} + \|g\|^p_{\Lambda^p(\varphi)}\right).
\end{aligned}$$

Example 3.2. For $\varphi(t) = t^{\frac{1}{q}}(1 + |\log t|)^\alpha$ with $1 \le p, q \le +\infty$ and $-\infty < \alpha < +\infty$, $\Lambda^p(\varphi)$ is the Lorentz–Zygmund space $L^{q,p}_{\|m\|}(\log L)^\alpha$. This is the Lorentz space $L^{q,p}(\|m\|)$ if $\alpha = 0$.

The next proposition contains elementary property of weighted Lorentz spaces.

Proposition 3.3. *If $w_1(t) < cw_0(t)$, for all $t > 0$, then*

(1) $\Lambda^p(\varphi_0) \subset \Lambda^p(\varphi_1)$ *for* $1 \le p \le \infty$,
(2) $\Lambda^p(\varphi_0) \subset \Lambda^\infty(\varphi_1)$ *for* $1 \le p < \infty$.

Proof. Let us start with the first one. For every measurable function f we have $w_1(\|m\|_f(t)) < cw_0(\|m\|_f(t))$, if $w_1(t) < cw_0(t)$, for all $t > 0$, and it follows that

$$\int_0^\infty y^{p-1}w_1^p\left(\|m\|_f(y)\right)dy < c\int_0^\infty y^{p-1}w_0^p\left(\|m\|_f(y)\right)dy,$$

therefore $\Lambda^p(\varphi_0) \subset \Lambda^p(\varphi_1)$. Next we are going to prove the second one. Consider a function $f \in \Lambda^p(\varphi_0)$. Since f_* is a decreasing function, so for each $t > 0$ we have

$$\begin{aligned}
f_*(t)w_1(t) < cf_*(t)w_0(t) &= cf_*(t)\left(\int_0^t \varphi_0(s)^p\frac{ds}{s}\right)^{\frac{1}{p}} \\
&\le c\left(\int_0^t (\varphi_0(s)f_*(s))^p\,\frac{ds}{s}\right)^{\frac{1}{p}} \\
&\le c\left(\int_0^\infty (\varphi_0(s)f_*(s))^p\,\frac{ds}{s}\right)^{\frac{1}{p}} = c\|f\|_{\Lambda^p(\varphi_0)}.
\end{aligned}$$

Now, taking supremum over all $t > 0$, it follows that $f \in \Lambda^\infty(\varphi_1)$, that is, $\Lambda^p(\varphi_0) \subset \Lambda^\infty(\varphi_1)$. ☐

4. Estimates of K-functional with respect to a vector measure

We let (A_0, A_1) denote a compatible couple of quasi-Banach pair (i.e. A_0 and A_1 are quasi-Banach spaces, which both are continuously embedded in some Hausdorff topological vector space). For every $f \in A_0+A_1$ and any $0 < t < \infty$, the so-called Peetre K-functional is defined by

$$K(t, f, A_0, A_1) = K(t, f) := \inf_{f_0+f_1=f}(\|f_0\|_{A_0} + t\|f_1\|_{A_1}),$$

where $f_i \in A_i$, $i = 0, 1$.
For $1 \le q \le \infty$ and each measurable function ϱ, the real interpolation space $(A_0, A_1)_{\varrho,q}$ consists of all elements of $f \in A_0 + A_1$ such that the quantity

$$\|f\|_{(A_0,A_1)_{\varrho,q}} := \begin{cases} \left(\int_0^\infty \left(\frac{K(t,f)}{\varrho(t)}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} & (1 \le q < \infty), \\ \sup_{t>0} \frac{K(t,f)}{\varrho(t)} & (q = \infty) \end{cases}$$

is finite. By replacing measurable function $\varrho = \varrho(t)$ by t^θ we obtain $(A_0, A_1)_{\theta,q}$.

We shall write $A \preceq B$ if $A \le cB$, where c is some positive constant independent of appropriate quantities involved in A, B. If both $A \preceq B$ and $B \preceq A$ are satisfied (with possibly different constants), we write $A \approx B$. In order to estimate the K–functional we can see that $K(t, f) \approx K(t, |f|)$ for a general function f and for every $t > 0$.So in the sequel, we will suppose that $f \ge 0$ when we want to estimate the K–functional $K(t, f)$.

Theorem 4.1. *Let f be a function in $L^p(m), 1 \le p < \infty$. Then*

$$K(t, f, L^p(m), L^\infty(m)) \preceq \left(\int_0^{t^p} f_*(s)^p ds\right)^{\frac{1}{p}}, \qquad t > 0.$$

Proof. Take $s_* = f_*(t^p)$ and consider the nonnegative function

$$f_0(w) = \begin{cases} 0, & f(w) \le s_* \\ f(w) - s_*, & f(w) > s_* \end{cases} \tag{4.1}$$

and $f_1 = f - f_0$. Since $f_1(w) \le s_*$, so $f_1 \in L^\infty(m)$ and $\|f_1\|_{L^\infty(m)} \le s_*$. On the other hand $f_0 \in L^p(m)$ since $0 \le f_0 \le f - s_*$ and $f \in L^p(m)$. Moreover, for all $s > 0$ we have

$$\begin{aligned} \|m\|_{f_0}(s) &= \|m\|\{w : f_0(w) > s\} \\ &= \|m\|\{w : f(w) - s_* > s\} \\ &= \|m\|\{w : f(w) > s + s_*\} \\ &= \|m\|_f(s + s_*). \end{aligned}$$

Now from definition of norm in $L^p(m)$, we obtain

$$\begin{aligned} \| f_0 \|^p_{L^p(m)} &= \sup\left\{\int_\Omega f_0^p d|\langle m, x^*\rangle| : x^* \in B(X^*)\right\} \\ &= \sup\left\{\int_0^\infty |\langle m, x^*\rangle|_{f_0^p}(s)ds : x^* \in B(X^*)\right\} \\ &\le \int_0^\infty \|m\|_{f_0^p}(s)ds = \int_0^\infty \|m\|_{f_0}(s^{\frac{1}{p}})ds = \int_0^\infty \|m\|_f(s^{\frac{1}{p}} + s_*)ds \\ &= \int_0^{f_*^p(t^p)} \|m\|_f(s^{\frac{1}{p}} + s_*)ds + \int_{f_*^p(t^p)}^\infty \|m\|_f(s^{\frac{1}{p}} + s_*)ds \\ &\le \int_0^{f_*^p(t^p)} \|m\|_f(s_*)ds + \int_{f_*^p(t^p)}^\infty \|m\|_f(s^{\frac{1}{p}})ds \\ &\le \int_0^{f_*^p(t^p)} t^p ds + \int_{f_*^p(t^p)}^\infty \|m\|_{f^p}(s)ds \\ &= \int_0^{f_*^p(t^p)} t^p ds + \int_{f_*^p(t^p)}^\infty \lambda_{f_*^p}(s)ds. \end{aligned}$$

Applying the equality $\lambda_{\chi_{[0,t^p)}f_*^p} = t^p\chi_{[0,f_*^p(t^p))} + \lambda_{f_*^p}\chi_{[f_*^p(t^p),\infty)}$ we can conclude

$$\begin{aligned}
\| f_0 \|^p_{L^p(m)} &\leq \int_0^{f_*^p(t^p)} t^p ds + \int_{f_*^p(t^p)}^{\infty} \lambda_{f_*^p}(s)ds \\
&= \int_0^{\infty} \lambda_{\chi_{[0,t^p)}f_*^p}(s)ds = \int_0^{\infty} \lambda_{\chi_{[0,t^p)}f_*}(s^{\frac{1}{p}})ds \\
&= p\int_0^{\infty} s^{p-1}\lambda_{\chi_{[0,t^p)}f_*}(s)ds, \quad \text{(by Proposition 2.1.8 in [2])} \\
&= \int_0^{t^p} f_*(s)^p ds.
\end{aligned}$$

For for a fixed $t > 0$ we get

$$\begin{aligned}
K(t, f, L^p(m), L^\infty(m)) &\leq \|f_0\|_{L^p(m)} + t\|f_1\|_{L^\infty(m)} \\
&\leq \left(\int_0^{t^p} f_*(s)^p ds\right)^{\frac{1}{p}} + tf_*(t^p) \\
&= \left(\int_0^{t^p} f_*(s)^p ds\right)^{\frac{1}{p}} + \left(\int_0^{t^p} f_*(t^p)^p ds\right)^{\frac{1}{p}} \\
&\preceq \left(\int_0^{t^p} f_*(s)^p ds\right)^{\frac{1}{p}}.
\end{aligned}$$

The proof is complete. □

Proposition 4.2. *Let f be a function in $L^1(m)$. Then*

$$tf_*(t) \preceq k(t, f, L^1(m), L^\infty(m)), \quad t > 0.$$

For the proof of above proposition you can see [14].
In the sequel, we prove the generalization of Steffensen's inequality for $L^p(\|m\|)$ spaces. To this end, we need the next theorem.

Theorem 4.3. *Let f be a function in $L^p(\|m\|), 1 \leq p < \infty$. Then*

$$K(t, f, L^p(\|m\|), L^\infty(\|m\|)) \approx \left(\int_0^{t^p} f_*^p(s)ds\right)^{\frac{1}{p}}, \qquad t > 0. \tag{4.2}$$

Proof. First we prove " $\leq$ " of (4.2). Choose the nonnegative functions f_0 as it is considered in Theorem 4.1 and $f_1 = f - f_0$. Let $A = \{w : f_0(w) > 0\}$. Then

$$\begin{aligned}
\|m\|(A) &= \|m\|\{w : f_0(w) > 0\} = \|m\|\{w : f(w) - s_* > 0\} \\
&= \|m\|\{w : f(w) > s_*\} = \|m\|_f(s_*) = \|m\|_f(f_*(t^p)) \leq t^p.
\end{aligned}$$

Since $f_*(s)$ is decreasing and constant on $[\|m\|(A), t^p]$, so we have

$$\begin{aligned}
K(t, f, L^p(\|m\|), L^\infty(\|m\|)) &\leq \|f_0\|_{L^p(\|m\|)} + t\|f_1\|_{L^\infty(\|m\|)} \\
&\leq \left(\int_0^{\infty} f_{0*}(s)^p ds\right)^{\frac{1}{p}} + tf_*(t^p) \\
&= \left(\int_0^{t^p} f_{0*}(s)^p ds\right)^{\frac{1}{p}} + \left(\int_0^{t^p} f_*(t^p)^p ds\right)^{\frac{1}{p}} \\
&\leq 2\left(\int_0^{t^p} f_*(s)^p ds\right)^{\frac{1}{p}}.
\end{aligned} \tag{4.3}$$

To obtain the converse inequality, assume that $f = f_0 + f_1, f_0 \in L^p(\|m\|)$ and $f_1 \in L^\infty(\|m\|)$. Taking into account the inequality

$$f_*(s) \leq f_{0_*}(s/2) + f_{1_*}(s/2) \leq f_{0_*}(s/2) + \|f_1\|_{L^\infty(\|m\|)},$$

we observe that

$$\begin{aligned}\left(\int_0^{t^p} f_*(s)^p ds\right)^{\frac{1}{p}} &\leq \left(\int_0^{t^p} \left(f_{0_*}(s/2) + \|f_1\|_{L^\infty(\|m\|)}\right)^p ds\right)^{\frac{1}{p}} \\ &\leq c\left\{\left(\int_0^\infty (f_{0_*}(s))^p\, ds\right)^{\frac{1}{p}} + t\|f_1\|_{L^\infty(\|m\|)}\right\} \\ &= c\left\{\|f_0\|_{L^p(\|m\|)} + t\|f_1\|_{L^\infty(\|m\|)}\right\}.\end{aligned}$$

Taking the infimum over all decompositions $f = f_0 + f_1 \in L^p(\|m\|) + L^\infty(\|m\|)$, we reach desire inequality. □

We then deduce immediately the following that is a type of the generalization of Steffensen's inequality, see [3]. Recall that if Y be a Banach function space, we will denote by Y' the Banach function space consisting of all measurable functions g on $(0, \infty)$ such that

$$\|g\|_{Y'} = \sup_{\|f\|_Y \leq 1} \left|\int_0^\infty f(s)g(s)ds\right|$$

is finite. We will need the following representation of the norm of Y given by Lorentz and Luxemburg [21]

$$\|f\|_Y = \sup_{\|g\|_{Y'} \leq 1} \left|\int_0^\infty f(s)g(s)ds\right|.$$

This gives that $Y'' = Y$. Moreover if Y is a rearrangement-invariant space then we have $\|f_*\|_Y = \|f\|_Y$; in fact,

$$\|f\|_Y = \sup_{\|g\|_{Y'} \leq 1} \int_0^\infty f_*(s)g_*(s)ds.$$

Corollary 4.4. *Let f and g be positive functions on $(0, \infty)$, f decreasing and g measurable. Assume that, for some $p > 1$, $f \in L^p(\|m\|) + L^\infty(\|m\|)$ and $g \in (L^p(\|m\|))' \cap L^1(\|m\|)$, with*

$$\|g\|_{(L^p(\|m\|))'} = 1, \quad \|g\|_{L^1(\|m\|)} = t.$$

Then

$$\int_0^\infty f(x)g(x)dx \leq 2\left(\int_0^{t^p} (f_*(x))^p dx\right)^{\frac{1}{p}}.$$

Proof. Let $f = f_0 + f_1, f_0 \in L^p(\|m\|), f_1 \in L^\infty(\|m\|)$. Then from above descriptions we obtain

$$\begin{aligned}\int_0^\infty f(x)g(x)dx &= \int_0^\infty f_0(x)g(x)dx + \int_0^\infty f_1(x)g(x)dx \\ &\leq \|f_0\|_{L^p(\|m\|)} + \|f_1\|_{L^p(\|m\|)} \\ &= \|f_0\|_{L^p(\|m\|)} + \sup_{\|g\|_{(L^p(\|m\|))'} \leq 1} \int_0^\infty f_{1_*}(s)g_*(s)ds \\ &\leq \|f_0\|_{L^p(\|m\|)} + t\|f_1\|_{L^\infty(\|m\|)}.\end{aligned}$$

Finally from (4.3) in Theorem 4.3 follows that

$$\int_0^\infty f(x)g(x)dx \leq K(t, f, L^p(\|m\|), L^\infty(\|m\|)) \leq 2\left(\int_0^{t^p} (f_*(x))^p dx\right)^{\frac{1}{p}}.$$

□

5. Interpolation of weighted Lorentz spaces

Let a and b be two real numbers such that $a < b$. The notation $\varphi(t) \in Q[a,b]$ means that $\varphi(t)t^{-a}$ is nondecreasing and $\varphi(t)t^{-b}$ is nonincreasing for all $t > 0$. Moreover, we say that $\varphi(t) \in Q(a,b)$, wherever $\varphi(t) \in Q[a+\epsilon, b-\epsilon]$ for some $\epsilon > 0$. The notation $\varphi(t) \in Q(a,-)$ means that $\varphi(t) \in Q(a,b)$ for some real number b. In this paper we shall consider the interpolation spaces $(A_0, A_1)_{\varrho,q}$ with a parameter function $\varrho = \varrho(t) \in Q(0,1)$, which means that, for some $\epsilon > 0$, $\varrho(t)t^{-\epsilon}$ is increasing and $\varrho(t)t^{-1+\epsilon}$ is decreasing. To prove the main result of this section, we need the following lemma which is proved by Persson [25].

Lemma 5.1. *Let $0 < q \leq \infty, 0 < p < \infty$ and $\psi(t) \in Q(-,-)$. Let $h(t)$ be a positive and nonincreasing function. If $\varphi(t) \in Q(-,0)$, then*

$$\left(\int_0^\infty (\varphi(t))^q \left(\int_0^t (h(u)\psi(u))^p \frac{du}{u}\right)^{\frac{q}{p}} \frac{dt}{t}\right)^{\frac{1}{q}} \leq C \left(\int_0^\infty (\varphi(t)h(t)\psi(t))^q \frac{dt}{t}\right)^{\frac{1}{q}}.$$

Now we have the following fundamental interpolation theorem for couples of weighted Lorentz spaces with respect to a vector measure.

Theorem 5.2. *Let $\varphi_i(t) \in Q(0,-), i = 0,1$ be two weights, $\varphi_0(t)/\varphi_1(t) \in Q(0,1)$ or $\varphi_0(t)/\varphi_1(t) \in Q(1,0)$, and $\varrho \in Q(0,1)$ be a parameter function. If $1 \leq p_0, p_1, q \leq \infty$, then*

$$(\Lambda^{p_0}(\varphi_0), \Lambda^{p_1}(\varphi_1))_{\varrho,q} = \Lambda^q(\varphi), \tag{5.1}$$

where $\varphi(t) = \varphi_0(t)/\varrho(\varphi_0(t)/\varphi_1(t))$.

Proof. First we show that if $1 \leq q \leq \infty$ and $\varrho \in Q(0,1)$, then

$$(L^1(m), L^\infty(m))_{\varrho,q} = \Lambda^q\left(\frac{t}{\varrho(t)}\right). \tag{5.2}$$

Let f be a function in $L^1(m)$, $t > 0$ and $1 \leq q < \infty$. From Proposition 4.2 we have

$$\int_0^\infty \left(\frac{tf_*(t)}{\varrho(t)}\right)^q \frac{dt}{t} \leq \int_0^\infty \left(\frac{k(t,f)}{\varrho(t)}\right)^q \frac{dt}{t}. \tag{5.3}$$

Now, if $f \in (L^1(m), L^\infty(m))_{\varrho,q}$, then the right-hand side in (5.3) is finite and so $f \in \Lambda^q(\frac{t}{\varrho(t)})$. Thus we have proved that

$$(L^1(m), L^\infty(m))_{\varrho,q} \subseteq \Lambda^q(\frac{t}{\varrho(t)}).$$

To obtain the opposite inclusion in (5.2), since $\frac{1}{\varrho(t)} \in Q(-1,0)$ by Lemma 1.1 in [25], so we apply Proposition 4.1 and Lemma 5.1 for $p = 1$, to the nonnegative decreasing function $f_*(s)$, therefor

$$\begin{aligned}\int_0^\infty \left(\frac{k(t,f,L^1(m),L^\infty(m)}{\varrho(t)}\right)^q \frac{dt}{t} &\leq \int_0^\infty (\frac{1}{\varrho(t)})^q \left(\int_0^t f_*(s)ds\right)^q \frac{dt}{t} \\ &\leq \int_0^\infty \left(\frac{tf_*(t)}{\varrho(t)}\right)^q \frac{dt}{t}.\end{aligned}$$

Now, from the definition of weighted Lorentz spaces, we deduce that

$$\Lambda^q\left(\frac{t}{\varrho(t)}\right) \subseteq \left(L^1(m), L^\infty(m)\right)_{\varrho,q}.$$

When $q=\infty$, by Proposition 4.2, we get

$$\begin{aligned}\|f\|_{\Lambda^\infty(\frac{t}{\varrho(t)})} &= \sup_{t>0}\frac{tf_*(t)}{\varrho(t)}\\ &\leq C\sup_{t>0}\frac{k(t,f,L^1(m),L^\infty(m)}{\varrho(t)}\\ &= C\|f\|_{(L^1(m),L^\infty(m))_{\varrho,\infty}}.\end{aligned}$$

Hence, $(L^1(m),L^\infty(m))_{\varrho,\infty}\subseteq\Lambda^\infty(\frac{t}{\varrho(t)})$. For the converse, since $\varrho(t)\in Q(0,1)$, then there exist a constant $\epsilon>0$ such that $\varrho(t)t^{-\epsilon}$ is nondecreasing on $(0,\infty)$. So we have

$$\begin{aligned}\|f\|_{(L^1(m),L^\infty(m))_{\varrho,\infty}} &= C\sup_{t>0}\frac{k(t,f,L^1(m),L^\infty(m))}{\varrho(t)}\\ &\leq C\sup_{t>0}\frac{\int_0^t f_*(s)ds}{\varrho(t)}\\ &\leq C\sup_{s>0}\frac{sf_*(s)}{\varrho(s)}.\sup_{t>0}\frac{\varrho(t)t^{-\epsilon}\int_0^t s^{\epsilon-1}ds}{\varrho(t)}\\ &\leq C\|f\|_{\Lambda^\infty(\frac{t}{\varrho(t)})}.\end{aligned}$$

Hence, $\Lambda^\infty(\frac{t}{\varrho(t)})\subseteq(L^1(m),L^\infty(m))_{\varrho,\infty}$. Then the proof of the assertion is completed. Put $\varrho_i(t)=\frac{t}{\varphi_i(t)}$ so by Lemma 1.1(c) in [25] we see that $\varrho_i(t)\in Q(0,1)$. According to (5.2), we obtain

$$\Lambda^{p_i}(\varphi_i)=(L^1(m),L^\infty(m))_{\varrho_i,p_i}, i=0,1$$

It follows from [25, Corollary 4.4] that

$$\begin{aligned}(\Lambda^{p_0}(\varphi_0),\Lambda^{p_1}(\varphi_1))_{\varrho,q} &= \Big((L^1(m),L^\infty(m))_{\varrho_0,p_0},(L^1(m),L^\infty(m))_{\varrho_1,p_1}\Big)_{\varrho,q}\\ &= \Big(L^1(m),L^\infty(m)\Big)_{\kappa,q}=\Lambda^q(\frac{t}{\kappa(t)})=\Lambda^q(\varphi),\end{aligned}$$

where $\kappa(t)=\varrho_0(t)\varrho(\varrho_1(t)/\varrho_0(t))=\frac{t}{\varphi(t)}$. Note that $\kappa(t)\in Q(0,1)$ by [25, Lemma 3.3]. Thus (5.1) holds and the proof is complete. □

According to Theorem 5.2 we have the following corollary.

Corollary 5.3. *Let $1\leq q\leq\infty$ and $1\leq p_0<p_1\leq\infty$ and $\varrho\in Q(0,1)$. If $p_0\neq p_1$, then*

$$(L^{p_0,q_0}(\|m\|),L^{p_1,q_1}(\|m\|))_{\varrho,q}=\Lambda^q(t^{\frac{1}{p_0}}/\varrho(t^{\frac{1}{p_0}-\frac{1}{p_1}})).$$

Remark 5.4. Let $0<\theta<1$ and $1\leq q\leq\infty$. Putting $\varrho(t)=t^\theta$ in (5.2) we obtain

$$\Big(L^1(m),L^\infty(m)\Big)_{\theta,q}=\Lambda^q(t^{1-\theta})=L^{p,q}(\|m\|)$$

where $1<p<\infty$ and $\theta=1-\frac{1}{p}$.

The following result is a simple application of Theorem 5.2 by replacing parameter function $\varrho=\varrho(t)$ by t^θ.

Corollary 5.5. *Under the same hypothesis of Theorem 5.2, we have*

$$(\Lambda^{p_0}(\varphi_0),\Lambda^{p_1}(\varphi_1))_{\theta,q}=\Lambda^q(\varphi_0^{1-\theta}\varphi_1^\theta).$$

Remark 5.6. For $\varphi(t)=t^{\frac{1}{q}}(1+|\log t|)^\alpha$ with $1\leq p,q\leq+\infty$ and $-\infty<\alpha<+\infty$, $\Lambda^p(\varphi)$ is the Lorentz-Zygmund space $L^{q,p}_{\|m\|}(\log L)^\alpha$ (this is the Lorentz space $L^{q,p}(\|m\|)$ if $\alpha=0$). So, interpolation with a suitable parameter function ϱ can be used to describe the interpolation spaces for couples of these Lorentz-Zygmund with respect to a vector measure. For example if $\varrho(t) = t^\theta(1 + |\log t|)^\gamma$, $\varphi_0(t) = t^{\frac{1}{p}}(1 + |\log t|)^{\alpha_0}$ and $\varphi_1(t) = t^{\frac{1}{p}}(1 + |\log t|)^{\alpha_1}$, then

$$\begin{aligned}(L^{p,q}_{\|m\|}(\log L)^{\alpha_0}, L^{p,q}_{\|m\|}(logL)^{\alpha_1})_{\varrho,q} &= \Lambda^q(t^{\frac{1}{p}}(1 + |\log t|)^{\alpha_0(1-\theta)+\alpha_1\theta}(1 + |\log(1 + |\log t|)|)^\gamma) \\ &= L^{p,q}_{\|m\|}(\log L)^{\alpha_0(1-\theta)+\alpha_1\theta}(\log\log L)^\gamma.\end{aligned}$$

The above results are like those for the Lorentz-Zygmund space of a positive measure, described for example in [13, 22, 23]

Corollary 5.7. *Let* $0 < \theta < 1 \le q \le \infty$ *and* $1 \le p_0 < p_1 \le \infty$, *then*

$$\begin{aligned}(L^{p_0,q_0}(\|m\|), L^{p_1,q_1}(\|m\|))_{\theta,q} &= L^{p,q}(\|m\|) \\ &= (L^{p_0}(m), L^{p_1}(m))_{\theta,q} \\ &= (L^{p_0}_w(m), L^{p_1}(m))_{\theta,q} \\ &= (L^{p_0}_w(m), L^{p_1}_w(m))_{\theta,q}.\end{aligned}$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Proof. By [14, Corollary 17] it is enough to show that $(L^{p_0,q_0}(\|m\|), L^{p_1,q_1}(\|m\|))_{\theta,q} = L^{p,q}(\|m\|)$. To this end, we consider $\varphi_i(t) = t^{\frac{1}{p_i}}$ and $\varrho(t) = t^\theta$, then the equality follows from Theorem 5.2. □

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