SUMS OF PRODUCTS OF THE TERMS OF THE GENERALIZED LUCAS SEQUENCE $\{V_{kn}\}$

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Abstract

In this study we consider the generalized Lucas sequence $\{V_n\}$ with indices in arithmetic progression. We also compute the sums of products of the terms of the Lucas sequence $\{V_{kn}\}$ for positive odd integers k.

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1. Introduction

The binary linear recurrence $W_n = W_n (a, b; p, q)$ is defined as follows for $n > 1$,

$$W_n = pW_{n-1} + qW_{n-2},$$

where $W_0 = a, W_1 = b$.

The Binet formula for $\{W_n\}$ is

(1.1) $W_n = A\alpha^n + B\beta^n,$

where $A = \frac{b-a\beta}{\alpha-\beta}$, $B = \frac{a\alpha-b}{\alpha-\beta}$ and $\alpha, \beta = \left(p \pm \sqrt{p^2 + 4q}\right)/2$.

For $n > 1$ and a fixed positive integer k, the terms of $\{W_{kn}\}$ satisfy the recursion [6, 7]:

$$W_{kn} = V_k W_{k(n-1)} - (-q)^k W_{k(n-2)},$$

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where $V_k = \alpha^k + \beta^k$. Specifically, define the generalized Fibonacci $\{U_n\}$ and Lucas $\{V_n\}$ sequences as $U_n = W_n(0, 1; p, 1)$, $V_n = W_n(2, p; p, 1)$, respectively. Thus:

$$(1.2) \qquad U_{kn} = V_k U_{k(n-1)} - (-1)^k U_{k(n-2)},$$

$$(1.3) \qquad V_{kn} = V_k V_{k(n-1)} - (-1)^k V_{k(n-2)}.$$

The Fibonomial coefficients $\begin{bmatrix} n \\ m \end{bmatrix}_F$ are defined by the relation

$$\begin{bmatrix} n \\ m \end{bmatrix}_F = \frac{F_1 F_2 \cdots F_n}{(F_1 F_2 \cdots F_{n-m})(F_1 F_2 \cdots F_m)},$$

for $n \geq m \geq 1$, with $\begin{bmatrix} n \\ 0 \end{bmatrix}_F = \begin{bmatrix} n \\ n \end{bmatrix}_F = 1$, where F_n is the nth Fibonacci number. These coefficients satisfy the relation:

$$\begin{bmatrix} n \\ m \end{bmatrix}_F = F_{m+1} \begin{bmatrix} n-1 \\ m \end{bmatrix}_F + F_{n-m-1} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_F.$$

Hoggatt [4] defined the following generalization by taking F_{kn} instead of F_n for a fixed positive integer k:

$$\begin{bmatrix} n \\ m \end{bmatrix}_{F_k} = \frac{F_k F_{2k} \cdots F_{kn}}{\left(F_k F_{2k} \cdots F_{k(n-m)}\right)\left(F_k F_{2k} \cdots F_{km}\right)}.$$

Jarden and Motzkin were the first to study the generalized Fibonomial coefficients formed by terms of the sequence $\{U_n\}$ as follows: for $n \geq m \geq 1$,

$$\begin{bmatrix} n \\ m \end{bmatrix}_U = \frac{U_1 U_2 \cdots U_n}{(U_1 U_2 \cdots U_{n-m})(U_1 U_2 \cdots U_m)},$$

with $\begin{bmatrix} n \\ 0 \end{bmatrix}_U = \begin{bmatrix} n \\ n \end{bmatrix}_U = 1$.

When $p = 1$, the generalized Fibonomial coefficients $\begin{bmatrix} n \\ m \end{bmatrix}_U$ are reduced to the Fibonomial coefficients $\begin{bmatrix} n \\ m \end{bmatrix}_F$.

By taking U_{kn} instead of U_n for a fixed positive integer k, one can get

$$\begin{bmatrix} n \\ m \end{bmatrix}_{U_k} = \frac{U_k U_{2k} \cdots U_{kn}}{\left(U_k U_{2k} \cdots U_{k(n-m)}\right)\left(U_k U_{2k} \cdots U_{km}\right)}.$$

These coefficients satisfy the relations

$$\begin{bmatrix} n \\ m \end{bmatrix}_{U_k} = U_{km+1} \begin{bmatrix} n-1 \\ m \end{bmatrix}_{U_k} + U_{k(n-m)-1} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_{U_k}$$

and

$$\begin{bmatrix} n \\ m \end{bmatrix}_{U_k} = U_{km-1} \begin{bmatrix} n-1 \\ m \end{bmatrix}_{U_k} + U_{k(n-m)+1} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_{U_k}.$$

Golomb [3] found the generating function for the numbers F_n^2, and this result started the effort to find a recurrences or closed form for the generating function

$$f_m(x) = \sum_{n=0}^{\infty} F_n^m x^n$$

of the mth powers of the Fibonacci numbers.

In [8], Riordan found the general recurrence relation for $f_m(x)$ (see also [2]). Carlitz [1] and Horadam [5] generalized the result of Riordan and found similar recurrences for the generating functions of different types of generalized Fibonacci numbers. They also

found a closed form for the polynomial $N_m(x)$ in the numerator, and the polynomial $D_m(x)$ in the denominator of the generating functions. As a special case of Horadam's result, it is possible to get the following formula for the generating function of odd integer powers of the Fibonacci numbers:

$$(1.4) \qquad f_m(x) = \frac{\sum_{i=0}^{m} \sum_{j=0}^{i} (-1)^{\frac{j(j+1)}{2}} \left[\begin{matrix} m+1 \\ j \end{matrix} \right]_F F_{i-j}^m x^i}{\sum_{i=0}^{m+1} (-1)^{\frac{i(i+1)}{2}} \left[\begin{matrix} m+1 \\ i \end{matrix} \right]_F x^i}.$$

In [13], applying Carlitz's approach, Shannon obtained some special results for the numerator and the denominator in the expression of the generating function $f_m(x)$. Using the q-analogue of the terminating binomial theorem, he obtained the relation

$$\prod_{i=0}^{m} \left(1 - q^i x \right) = \sum_{i=0}^{m+1} (-1)^i q^{\frac{i}{2}(i-1)} \left\{ \begin{matrix} m+1 \\ i \end{matrix} \right\} x^i,$$

where $\left\{ \begin{matrix} m \\ i \end{matrix} \right\} = \frac{(1-q^m)(1-q^{m-1})\cdots(1-q^{m-i+1})}{(1-q)(1-q^2)\cdots(1-q^i)}$ is the Gaussian q-binomial coefficient for $i \geq 1$, any complex numbers q, x and any positive integer m with $\left\{ \begin{matrix} m \\ 0 \end{matrix} \right\} = 1$. Replacing q by β/α and x by $\alpha^m x$, one can get

$$\prod_{i=0}^{m} \left(1 - \alpha^{m-i} \beta^i x \right) = \sum_{i=0}^{m+1} (-1)^{\frac{i}{2}(i+1)} \left[\begin{matrix} m+1 \\ i \end{matrix} \right]_F x^i.$$

It is easy to obtain for any odd integer m that

$$(1.5) \qquad f_m(x) = 5^{-\frac{m-1}{2}} \sum_{i=0}^{\frac{m-1}{2}} \binom{m}{j} \frac{F_{m-2j} x}{1 - (-1)^j L_{m-2j} x - x^2},$$

after simplifications of one of Shannon's results. Seibert and Trojovsky [11] gave certain generalizations of the well-known formulas for the Fibonacci and Lucas numbers. For example,

$$\sum_{i=0}^{n} (-1)^i L_{n-2i} = 2F_{n+1}.$$

For odd positive integer m, the authors concentrated on the sums

$$\sum_{i_n=0}^{\frac{m-1}{2}} \sum_{i_{n-1}=i_n+1}^{\frac{m-1}{2}} \cdots \sum_{i_1=i_2+1}^{\frac{m-1}{2}} (-1)^{i_1+i_2+\cdots+i_n} \prod_{j=1}^{n} L_{m-2i_j}.$$

Combining (1.4) and (1.5), they gave some new results about these sums with the help of the Fibonomial coefficients.

In [12], for arbitrary positive integer m, taking the sums

$$(1.6) \qquad \sum_{i_n=0}^{\lfloor \frac{m-1}{2} \rfloor} \sum_{i_{n-1}=i_n+1}^{\lfloor \frac{m-1}{2} \rfloor} \cdots \sum_{i_1=i_2+1}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^{i_1+i_2+\cdots+i_n} \prod_{j=1}^{n} L_{m-2i_j},$$

the authors gave analogous new results for an even integer m using the method from [11].

We consider the generalized Lucas sequence $\{V_n\}$ with indices in arithmetic progression, and then compute the sums of products of terms of the sequence $\{V_{kn}\}$ for a positive odd integer k.

2. Some identities including the terms of $\{U_{kn}\}$ and $\{V_{kn}\}$

We shall give some results for later use. Throughout this study, we will denote $W_n(a, b, p, 1)$ by H_n.

2.1. Lemma. *For $m, n > 0$,*

$$(2.1) \qquad V_{k(m+n)} + V_{k(m-n)} = \begin{cases} \frac{V_k^2+4}{U_k^2} U_{km} U_{kn} & \text{if } n \text{ is odd,} \\ V_{km} V_{kn} & \text{if } n \text{ is even,} \end{cases}$$

$$(2.2) \qquad V_{k(m+n)} - V_{k(m-n)} = \begin{cases} V_{km} V_{kn} & \text{if } n \text{ is odd,} \\ \frac{V_k^2+4}{U_k^2} U_{km} U_{kn} & \text{if } n \text{ is even,} \end{cases}$$

$$(2.3) \qquad U_{k(m+n)} + U_{k(m-n)} = \begin{cases} V_{km} U_{kn} & \text{if } n \text{ is odd,} \\ U_{km} V_{kn} & \text{if } n \text{ is even,} \end{cases}$$

$$(2.4) \qquad U_{k(m+n)} - U_{k(m-n)} = \begin{cases} U_{km} V_{kn} & \text{if } n \text{ is odd,} \\ V_{km} U_{kn} & \text{if } n \text{ is even,} \end{cases}$$

$$(2.5) \qquad V_{k(m-2n)} V_{k(2(m-2n)-3)} = \begin{cases} V_{3k(m-2n-1)} - V_{k(m-2n-3)} & \text{if } m \text{ is odd,} \\ V_{3k(m-2n-1)} + V_{k(m-2n-3)} & \text{if } m \text{ is even,} \end{cases}$$

$$(2.6) \qquad V_{k(m-2n)} U_{k(m-2n-1)} = \begin{cases} U_{k(2m-4n-1)} - U_k & \text{if } m \text{ is odd,} \\ U_{k(2m-4n-1)} + U_k & \text{if } m \text{ is even,} \end{cases}$$

$$(2.7) \qquad V_{3k(m+1)} + V_{3k(m-1)} = (V_k^2 + 4) U_{3km} U_{3k},$$

$$(2.8) \qquad V_{kn} U_{k(n-1)} - U_{k(2n-1)} = (-1)^{kn} U_k,$$

$$(2.9) \qquad U_{k(m+n)} = U_{km} V_{kn} + (-1)^{n+1} U_{k(m-n)},$$

$$(2.10) \qquad U_{k(m+n)} U_{k(m+t)} - U_{km} U_{k(m+t+n)} = (-1)^m U_{kn} U_{kt}.$$

Proof. The proof follows by the Binet formulas for $\{U_{kn}\}$ and $\{V_{kn}\}$.   □

2.2. Theorem. *For any integers r, c, d with $c \neq 0$ and $n \geq 0$,*

$$\text{i)} \quad (2.11) \quad \sum_{i=r}^{n} H_{k(ci+d)} = [H_{k(cr+d)} - H_{k(c(n+1)+d)} - (-1)^c H_{k(c(r-1)+d)}$$
$$+ (-1)^c H_{k(cn+d)}]/1 - V_{kc} + (-1)^c,$$

$$\text{ii)} \quad (2.12) \quad \sum_{i=r}^{n} (-1)^i H_{k(ci+d)} = [(-1)^r H_{k(cr+d)} + (-1)^n H_{k(c(n+1)+d)}$$
$$+ (-1)^{c+r} H_{k(c(r-1)+d)}$$
$$+ (-1)^{c+n} H_{k(cn+d)}]/1 + V_{kc} + (-1)^c.$$

Proof. i) By the Binet formula for $\{H_n\}$, we have

$$\sum_{i=r}^{n} H_{k(ci+d)} = H_{k(cr+d)} + H_{k(c(r+1)+d)} + \cdots + H_{k(cn+d)}$$

$$= A\alpha^{k(cr+d)} + B\beta^{k(cr+d)} + A\alpha^{k(c(r+1)+d)} + B\beta^{k(c(r+1)+d)}$$
$$+ \cdots + A\alpha^{k(cn+d)} + B\beta^{k(cn+d)}$$

$$= A\alpha^{k(cr+d)} \left[1 + \alpha^{kc} + \alpha^{2kc} + \cdots + \alpha^{kc(n-r)}\right]$$
$$+ B\beta^{k(cr+d)} \left[1 + \beta^{kc} + \beta^{2kc} + \cdots + \beta^{kc(n-r)}\right]$$

$$= A\alpha^{k(cr+d)} \frac{1 - \alpha^{kc(n-r+1)}}{1 - \alpha^{kc}} + B\beta^{k(cr+d)} \frac{1 - \beta^{kc(n-r+1)}}{1 - \beta^{kc}},$$

which, by $(\alpha\beta)^k = -1$ gives us

$$\sum_{i=r}^{n} H_{k(ci+d)} = \left[A\alpha^{k(cr+d)} + B\beta^{k(cr+d)} - \left(A\alpha^{k(c(n+1)+d)} + B\beta^{k(c(n+1)+d)}\right)\right.$$
$$- (-1)^c \left(A\alpha^{k(c(r-1)+d)} + B\beta^{k(c(r-1)+d)}\right)$$
$$\left.+ (-1)^c \left(A\alpha^{k(cn+d)} + B\beta^{k(cn+d)}\right)\right] / 1 - \alpha^{kc} - \beta^{kc} + (-1)^c$$

$$= (H_{k(cr+d)} - H_{k(c(n+1)+d)} - (-1)^c H_{k(c(r-1)+d)}$$
$$+ (-1)^c H_{k(cn+d)})/1 - V_{kc} + (-1)^c.$$

Thus the proof of (i) is complete. The equation (2.12) can be similarly proven. $\square$

2.3. Theorem. *For any integers $r, c\,(c \neq 0)$, d and $n \geq 0$,*

i) (2.13)
$$\sum_{i=r}^{n} iH_{k(ci+d)}$$
$$= \left(nH_{k(c(n+2)+d)} - \left(n+1+2n\,(-1)^c\right)H_{k(c(n+1)+d)}\right.$$
$$+ \left(n + 2\,(n+1)\,(-1)^c\right)H_{k(cn+d)} - (n+1)\,H_{k(c(n-1)+d)}$$
$$- \left(r + 1 + 2r\,(-1)^c\right)H_{k(c(r-1)+d)} + rH_{k(c(r-2)+d)}$$
$$+ \left(r + 2\,(r-1)\,(-1)^c\right)H_{k(cr+d)}$$
$$\left.- (r-1)\,H_{k(c(r+1)+d)}\right) \Big/ \left(1 - V_{kc} + (-1)^c\right)^2,$$

ii) (2.14)
$$\sum_{i=r}^{n} (-1)^{i-1} iH_{k(ci+d)}$$
$$= \left[\left(n\,(-1)^{n+1} - 2\,(n+1)\,(-1)^{c+n}\right)H_{k(cn+d)}\right.$$
$$- (n+1)\,(-1)^n\, H_{k\left(c(n-1)+d\right)}$$
$$+ \left(2n\,(-1)^{c+n+1} - (n+1)\,(-1)^n\right)H_{k\left(c(n+1)+d\right)}$$
$$+ n\,(-1)^{n+1}\, H_{k\left(c(n+2)+d\right)}$$
$$- \left((r-1)\,(-1)^r - 2r\,(-1)^{c+r-1}\right)H_{k\left(c(r-1)+d\right)}$$
$$+ \left(r\,(-1)^{r-1} - 2\,(r-1)\,(-1)^{c+r}\right)H_{kcr+d}$$
$$+ \left(r\,(-1)^{r-1}\right)H_{k(c(r-2)+d)}$$
$$\left.- (r-1)\,(-1)^r\, H_{k\left(c(r+1)+d\right)}\right] \Big/ \left(1 + V_{kc} + (-1)^c\right)^2.$$

Proof. Theorem 2.3 is proved by considering

$$\sum_{i=r}^{n} i x^{i-1} = \frac{n x^{n+1} - (n+1) x^n - (r-1) x^r + r x^{r-1}}{(x-1)^2},$$

and using the Binet formula. $\hfill\square$

2.4. Corollary. *For an odd positive integer m,*

$$(2.15) \qquad \sum_{i=j+1}^{\frac{m-1}{2}} U_{k(2m-4i-1)} = \frac{U_k}{V_k (V_k^2 + 4)} \left[V_{k(2m-4j-3)} + V_k \right],$$

$$(2.16) \qquad \sum_{i=r}^{\frac{m-1}{2}} (-1)^i V_{k(m-2i)} = (-1)^r \frac{U_{k(m-2r+1)}}{U_k},$$

$$(2.17) \qquad \sum_{i=0}^{\frac{m-1}{2}} (-1)^i V_{k(3(m-2i-1))} = (-1)^{\frac{m-1}{2}} + \frac{U_{3km} U_{3k}}{(V_k^2 + 1)^2},$$

$$(2.18) \qquad \sum_{i=0}^{\frac{m-1}{2}} (-1)^{i-1} i V_{k(m-2i)} = \frac{(-1)^{\frac{m-1}{2}} V_k - V_{km}}{V_k^2 + 4},$$

Proof. Substituting $n = \frac{m-1}{2}$, $c = -4$, $d = 2m - 1$ and $r = j + 1$ in (2.11) gives us

$$\sum_{i=j+1}^{\frac{m-1}{2}} U_{k(2m-4i-1)} = \frac{-U_{k(2m-4j-5)} + U_{k(2m-4j-1)} + U_{3k} - U_k}{V_k^2 (V_k^2 + 4)},$$

which, by (2.4), is equal to

$$\frac{U_{2k} V_{k(2m-4j-3)} + U_k V_k^2}{V_k^2 (V_k^2 + 4)} = \frac{U_k}{V_k (V_k^2 + 4)} \left[V_{k(2m-4j-3)} + V_k \right],$$

as claimed in (2.15). Using Lemma 2.1, the identities (2.16)-(2.18) can be proved in a similar way as in the proof of (2.15). $\hfill\square$

2.5. Corollary. *For an even positive integer m,*

$$\sum_{i=j+1}^{\frac{m-2}{2}} U_{k(2m-4i-1)} = \frac{U_k}{V_k (V_k^2 + 4)} \left[V_{k(2m-4j-3)} - V_k \right],$$

$$\sum_{i=r}^{\frac{m-2}{2}} (-1)^i V_{k(m-2i)} = (-1)^r \frac{U_{k(m-2r+1)}}{U_k} - (-1)^{\frac{m}{2}},$$

$$\sum_{i=0}^{\frac{m-2}{2}} (-1)^i V_{k(3(m-2i-1))} = \frac{U_{3km}}{U_k (V_k^2 + 1)},$$

$$\sum_{i=0}^{\frac{m-2}{2}} (-1)^{i-1} i V_{k(m-2i)} = (-1)^{\frac{m}{2}} \frac{m}{2} + \frac{V_{km} - 2 (-1)^{\frac{m}{2}}}{V_k^2 + 4}.$$

$\hfill\square$

3. Sums of products of the terms of $\{V_{kn}\}$

Define the sequence $\left\{ S_n^k\left(m\right)\right\}_{n=0}^{\infty}$ in the following way: for $m > 0$

$$(3.1) \qquad S_0^k\left(m\right) = U_k, \quad S_1^k\left(m\right) = \sum_{i_1=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} (-1)^{i_1}\, V_{k(m-2i_1)}$$

and for $n > 1$,

$$(3.2) \qquad S_n^k\left(m\right) = \sum_{i_n=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \sum_{i_{n-1}=i_n+1}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \cdots \sum_{i_1=i_2+1}^{\left\lfloor \frac{m-1}{2} \right\rfloor} (-1)^{i_1+i_2+\cdots+i_n} \prod_{j=1}^{n} V_{k\left(m-2i_j\right)}.$$

Throughout this section we shall frequently follow the organization of the work [11] while giving our results.

3.1. Theorem. *For an odd positive integer m,*

$$S_1^k\left(m\right) = \sum_{i_1=0}^{\frac{m-1}{2}} (-1)^{i_1}\, V_{k(m-2i_1)} = \frac{U_{k(m+1)}}{U_k},$$

$$S_2^k\left(m\right) = \sum_{i_2=0}^{\frac{m-1}{2}} \sum_{i_1=i_2+1}^{\frac{m-1}{2}} (-1)^{i_1+i_2}\, V_{k(m-2i_2)} V_{k(m-2i_1)}$$

$$= \frac{m+1}{2} - \frac{1}{U_k U_{2k}} U_{km} U_{k(m+1)},$$

and

$$S_3^k\left(m\right) = \sum_{i_3=0}^{\frac{m-1}{2}} \sum_{i_2=i_3+1}^{\frac{m-1}{2}} \sum_{i_1=i_2+1}^{\frac{m-1}{2}} (-1)^{i_1+i_2+i_3}\, V_{k(m-2i_3)} V_{k(m-2i_2)} V_{k(m-2i_1)}$$

$$= \left(\frac{m-1}{2}\right) \frac{U_{k(m+1)}}{U_k} - \frac{1}{U_k U_{2k} U_{3k}} U_{k(m+1)} U_{km} U_{k(m-1)}.$$

Proof. i) Substituting $n = \frac{m-1}{2}$, $c = -2$, $d = m$ and $r = 0$ in (2.12), we get

$$S_1^k\left(m\right) = \sum_{i_1=0}^{\frac{m-1}{2}} (-1)^{i_1}\, V_{k(m-2i_1)}$$

$$= \frac{V_{km} + (-1)^{\frac{m-1}{2}} V_k + V_{k(m+2)} + (-1)^{\frac{m-1}{2}} V_k}{(V_k^2 + 4)}.$$

Since $V_{-k} = V_k$ and by (2.1), we get

$$S_1^k\left(m\right) = \frac{V_{km} + V_{k(m+2)}}{(V_k^2 + 4)} = \sum_{i_1=0}^{\frac{m-1}{2}} (-1)^{i_1}\, V_{k(m-2i_1)} = \frac{U_{k(m+1)}}{U_k}.$$

ii) Using (2.6) and (2.16), we have

$$S_2^k(m) = \sum_{i_2=0}^{\frac{m-1}{2}} \sum_{i_1=i_2+1}^{\frac{m-1}{2}} (-1)^{i_1+i_2} V_{k(m-2i_2)} V_{k(m-2i_1)}$$

$$= -\frac{1}{U_k} \sum_{i_2=0}^{\frac{m-1}{2}} V_{k(m-2i_2)} U_{k(m-2i_2-1)}$$

$$= \frac{1}{U_k} \sum_{i_2=0}^{\frac{m-1}{2}} \left(U_k - U_{k(2m-4i_2-1)} \right).$$

Taking $j = -1$ in (2.15), we get

$$\sum_{i=0}^{\frac{m-1}{2}} U_{k(2m-4i-1)} = \frac{U_{k(m+1)} U_{km}}{U_{2k}}.$$

Then

$$S_2^k(m) = \frac{m+1}{2} - \frac{1}{U_k U_{2k}} U_{km} U_{k(m+1)}.$$

iii) Using (2.16) and (2.6), we write

$$S_3^k(m) = \sum_{i_3=0}^{\frac{m-1}{2}} \sum_{i_2=i_3+1}^{\frac{m-1}{2}} \sum_{i_1=i_2+1}^{\frac{m-1}{2}} (-1)^{i_1+i_2+i_3} V_{k(m-2i_3)} V_{k(m-2i_2)} V_{k(m-2i_1)}$$

$$= \frac{1}{U_k} \sum_{i_3=0}^{\frac{m-1}{2}} \sum_{i_2=i_3+1}^{\frac{m-1}{2}} (-1)^{i_3+1} V_{k(m-2i_3)} V_{k(m-2i_2)} U_{k(m-2i_2-1)}$$

$$= \frac{1}{U_k} \sum_{i_3=0}^{\frac{m-1}{2}} \sum_{i_2=i_3+1}^{\frac{m-1}{2}} (-1)^{i_3+1} V_{k(m-2i_3)} \left(U_{k(2m-4i_2-1)} - U_k \right)$$

$$= \frac{1}{U_k} \sum_{i_3=0}^{\frac{m-1}{2}} (-1)^{i_3+1} V_{k(m-2i_3)} \sum_{i_2=i_3+1}^{\frac{m-1}{2}} \left(U_{k(2m-4i_2-1)} - U_k \right).$$

From (2.15), we have

$$S_3^k(m) = \frac{1}{U_k} \sum_{i_3=0}^{\frac{m-1}{2}} (-1)^{i_3+1} V_{k(m-2i_3)}$$

$$\times \left(\frac{U_k}{V_k(V_k^2+4)} \left(V_{k(2m-4i_3-3)} + V_k \right) + \left(i_3 - \frac{m-1}{2} \right) U_k \right)$$

$$= \frac{1}{U_k} \sum_{i_3=0}^{\frac{m-1}{2}} (-1)^{i_3} V_{k(m-2i_3)}$$

$$\times \left(-\frac{U_k}{V_k(V_k^2+4)} \left(V_{k(2m-4i_3-3)} + V_k \right) + \left(\frac{m-1}{2} - i_3 \right) U_k \right)$$

$$= -\frac{1}{V_k \left(V_k^2 + 4\right)} \sum_{i_3=0}^{\frac{m-1}{2}} (-1)^{i_3} V_{k(m-2i_3)} V_{k(2m-4i_3-3)}$$

$$+ \left(\frac{m-1}{2} - \frac{1}{V_k^2 + 4}\right) \sum_{i_3=0}^{\frac{m-1}{2}} (-1)^{i_3} V_{k(m-2i_3)}$$

$$- \sum_{i_3=0}^{\frac{m-1}{2}} (-1)^{i_3} i_3 V_{k(m-2i_3)}.$$

From (2.16)- (2.18) and (2.1), we get

$$S_3^k (m)$$

$$= \frac{1}{V_k \left(V_k^2 + 4\right)} \left(\frac{U_{k(m-2)}}{U_k} + (-1)^{\frac{m-1}{2}} \left(V_k^2 + 1\right) - \frac{U_{3km} U_{3k}}{U_k^2 \left(V_k^2 + 1\right)^2} - (-1)^{\frac{m-1}{2}}\right)$$

$$+ \left(\frac{m-1}{2} - \frac{1}{V_k^2 + 4}\right) \left(\frac{V_{k(m+2)} + V_{km}}{V_k^2 + 4}\right) - \frac{1}{V_k^2 + 4} \left((-1)^{\frac{m-1}{2}} V_k - V_{km}\right)$$

$$= -\frac{U_{3km}}{U_{2k} \left(V_k^2 + 4\right) \left(V_k^2 + 1\right)} + \left(\frac{m-1}{2} - \frac{1}{V_k^2 + 4}\right) \frac{U_{k(m+1)}}{U_k}$$

$$+ \frac{1}{V_k^2 + 4} \left(\frac{U_{k(m-2)}}{U_k V_k} + \frac{V_{km} U_{2k}}{U_k V_k}\right)$$

$$= -\frac{U_{3km}}{U_{2k} \left(V_k^2 + 4\right) \left(V_k^2 + 1\right)} + \left(\frac{m-1}{2} - \frac{1}{V_k^2 + 4}\right) \frac{U_{k(m+1)}}{U_k} + \frac{U_{k(m+2)}}{\left(V_k^2 + 4\right) U_{2k}}$$

$$= \left(\frac{m-1}{2}\right) \frac{U_{k(m+1)}}{U_k} - \frac{U_{3km}}{U_{2k} \left(V_k^2 + 4\right) \left(V_k^2 + 1\right)} + \frac{U_{km}}{\left(V_k^2 + 4\right) U_{2k}}$$

$$= \left(\frac{m-1}{2}\right) \frac{U_{k(m+1)}}{U_k} - \frac{1}{U_k U_{2k} U_{3k}} U_{k(m+1)} U_{km} U_{k(m-1)}.$$

Thus we have the conclusion. □

3.2. Theorem. *For even positive integer m,*

$$S_1^k (m) = \sum_{i_1=0}^{\frac{m-2}{2}} (-1)^{i_1} V_{k(m-2i_1)} = \frac{U_{k(m+1)}}{U_k} - (-1)^{\frac{m}{2}},$$

$$S_2^k (m) = \sum_{i_2=0}^{\frac{m-2}{2}} \sum_{i_1=i_2+1}^{\frac{m-2}{2}} (-1)^{i_1+i_2} V_{k(m-2i_2)} V_{k(m-2i_1)}$$

$$= \frac{m-2}{2} + (-1)^{\frac{m}{2}} \frac{U_{k(m+1)}}{U_k} + \frac{U_{km} U_{k(m+1)}}{U_k U_{2k}},$$

$$S_3^k (m) = \sum_{i_3=0}^{\frac{m-2}{2}} \sum_{i_2=i_3+1}^{\frac{m-2}{2}} \sum_{i_1=i_2+1}^{\frac{m-2}{2}} (-1)^{i_1+i_2+i_3} V_{k(m-2i_3)} V_{k(m-2i_2)} V_{k(m-2i_1)}$$

$$= \left(\frac{m-4}{2}\right) \left((-1)^{\frac{m}{2}} - \frac{U_{k(m+1)}}{U_k}\right)$$

$$+ U_{km} U_{k(m+1)} \left((-1)^{\frac{m}{2}} - \frac{U_{k(m-1)}}{U_k U_{2k} U_{3k}}\right).$$

Proof. The proof is similar to the proof of Theorem 3.1. □

In [14], Stanica gave the generating function for powers of the terms of the sequence $\{W_n\}$, $W(m, x) = \sum_{i=0}^{\infty} W_i^m x^i$ as follows:

3.3. Theorem. *For $n \geq 0$ and odd positive integer m,*

$$(3.3) \qquad W(m, x) = \sum_{i=0}^{\frac{m-1}{2}} (-AB)^i \binom{m}{i}$$

$$\times \frac{A^{m-2i} - B^{m-2i} + (-q)^i \left(B^{m-2i}\alpha^{m-2i} - A^{m-2i}\beta^{m-2i}\right) x}{1 - (-q)^i V_{m-2i}x - q^m x^2},$$

and for even positive integer m,

$$(3.4) \qquad W(m, x) = \sum_{i=0}^{\frac{m}{2}-1} (-AB)^i \binom{m}{i}$$

$$\times \frac{B^{m-2i} + A^{m-2i} - (-q)^i \left(B^{m-2i}\alpha^{m-2i} + A^{m-2i}\beta^{m-2i}\right) x}{1 - (-q)^i V_{m-2i}x + q^m x^2}$$

$$+ \binom{m}{\frac{m}{2}} \frac{(-AB)^{\frac{m}{2}}}{1 - (-q)^{\frac{m}{2}}x}. \qquad \square$$

3.4. Theorem. *For $n \geq 0$ and an odd positive integer m,*

$$(3.5) \qquad S_n^k(m) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{\lfloor \frac{n}{2} \rfloor - i} \theta(i, m, n) \begin{bmatrix} m+1 \\ n-2i \end{bmatrix}_{U_k},$$

and for an even positive integer m,

$$S_n^k(m) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{n-2i} (-1)^{i + n\left(\frac{m}{2}+1\right) + \frac{j}{2}(j+m+1)} \theta(i, m, n) \begin{bmatrix} m+1 \\ j \end{bmatrix}_{U_k},$$

where $\theta(i, m, n) = \left(\begin{array}{c} \lfloor \frac{m+1}{2} \rfloor - n + i \\ i \end{array}\right) + \left(\begin{array}{c} \lfloor \frac{m+1}{2} \rfloor - n + i - 1 \\ i - 1 \end{array}\right).$

Proof. We give the proof for an odd integer m. From (3.3), we write

$$(3.6) \qquad U_{kn}(m, x) = \left(\frac{U_k}{\sqrt{V_k^2 + 4}}\right)^{\frac{m-1}{2}} \sum_{j=0}^{\frac{m-1}{2}} \binom{m}{j} \frac{U_{k(m-2j)}x}{1 - (-1)^j V_{k(m-2j)}x - x^2}.$$

Relation (3.6), which hold for odd m, leads to

$$D_{m+1}^k(x) = \prod_{j=0}^{\frac{m-1}{2}} \left(1 - (-1)^j V_{k(m-2j)}x - x^2\right) = \sum_{i=0}^{m+1} d_{m+1,i}x^i,$$

where $d_{m+1,i} = (-1)^{\frac{i(i+1)}{2}} \begin{bmatrix} m+1 \\ i \end{bmatrix}_{U_k}$. After multiplication of all the factors in $D_{m+1}^k(x)$, we obtain

$$d_{m+1,0} = S_0^k(m), \quad d_{m+1,i} = \sum_{l=0}^{\lfloor \frac{i}{2} \rfloor} \left(\begin{array}{c} \frac{m+1}{2} - (i - 2l) \\ l \end{array}\right) (-1)^{i+l} S_{i-2l}^k(m),$$

where $i = 1, 2, \ldots, m + 1$. Since $\binom{n}{m} = (-1)^m \binom{m-n-1}{m}$, we rewrite the last identity for $n > 0$ as follows:

$$d_{m+1,2n-1} = -\sum_{i=1}^{n} \binom{n + i - \frac{m+5}{2}}{n - i} S_{2i-1}^{k}(m)$$

$$d_{m+1,2(n-1)} = \sum_{i=1}^{n} \binom{n + i - \frac{m+7}{2}}{n - i} S_{2(i-1)}^{k}(m).$$

By the binomial inversion theorem (for more details, see [10]),

$$(3.7) \qquad a_n = \sum_{i=1}^{n} \binom{n + i + r}{n - i} b_i$$

holds if and only if

$$b_n = \sum_{i=1}^{n} (-1)^{i+n} \left(\binom{2n + r}{n - i} - \binom{2n + r}{n - i - 1} \right) a_i,$$

where r is any integer. After this, by taking $a_n = d_{m+1,2n-1}$, $b_i = -S_{2i-1}^{k}$ and $r = -\frac{m+5}{2}$ in (3.7), we obtain

$$(3.8) \quad
\begin{aligned}
S_{2n-1}^{k}(m) &= \sum_{i=1}^{n} (-1)^{-i+n+1} \left[\binom{2n - \frac{m+5}{2}}{n - i} - \binom{2n - \frac{m+5}{2}}{n - i - 1} \right] d_{m+1,2i-1} \\
&= \sum_{i=1}^{n} (-1) \left[\binom{-n - i + \frac{m+3}{2}}{n - i} - \binom{-n - i + \frac{m+1}{2}}{n - i - 1} \right] d_{m+1,2i-1} \\
&= \sum_{i=1}^{n} (-1)^{(2i-1)i+1} \left[\binom{-n - i + \frac{m+3}{2}}{n - i} - \binom{-n - i + \frac{m+1}{2}}{n - i - 1} \right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \begin{bmatrix} m + 2 \\ 2i - 1 \end{bmatrix}_{U_k}.
\end{aligned}$$

Similarly if we take $a_n = d_{m+1,2(n-1)}$, $b_i = -S_{2(i-1)}^{k}$ and $r = -\frac{m+7}{2}$ in (3.7), then we get

$$(3.9) \quad
\begin{aligned}
S_{2(n-1)}^{k}(m) &= \sum_{i=1}^{n} (-1)^{(2i-1)i+1} \left[\binom{-n - i + \frac{m+5}{2}}{n - i} - \binom{-n - i + \frac{m+5}{2}}{n - i - 1} \right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \begin{bmatrix} m + 1 \\ 2(i - 1) \end{bmatrix}_{U_k}.
\end{aligned}$$

Combining (3.8) and (3.9), we obtain (3.5).

For the case when m is even, the proof is completed by considering (3.4) as for the calculation of $S_n^{k}(m)$ for odd m. $\qquad\square$

3.5. Lemma. *For a positive integer s and a positive even integer m, it holds that*

$$(3.10) \qquad \begin{bmatrix} m + 1 \\ s \end{bmatrix}_{U_k} + (-1)^{\frac{m}{2}+s} \begin{bmatrix} m + 1 \\ s - 1 \end{bmatrix}_{U_k} = \frac{U_{k\left(\frac{m}{2}+1-s\right)}}{U_{k\left(\frac{m}{2}+1\right)}} \begin{bmatrix} m + 2 \\ s \end{bmatrix}_{U_k}.$$

For a positive integer s and a positive odd integer m, it holds that

$$\begin{bmatrix} m \\ s \end{bmatrix}_{U_k} + (-1)^{\frac{m-1}{2}+s} \begin{bmatrix} m \\ s - 1 \end{bmatrix}_{U_k} = \frac{U_{k\left(\frac{m+1}{2}-s\right)}}{U_{k\left(\frac{m+1}{2}\right)}} \begin{bmatrix} m + 1 \\ s \end{bmatrix}_{U_k}.$$

Proof. For an even positive integer m and a positive integer s, substituting $m = \frac{m}{2} - s + 1$, $n = \frac{m}{2} + 1$ in (2.9), we get

$$(3.11) \quad U_{k\left(\frac{m}{2}-s+1\right)}V_{k\left(\frac{m}{2}+1\right)} = U_{k(m-s+2)} + (-1)^{\frac{m}{2}+s} U_{ks}.$$

From (3.11), we get

$$\begin{bmatrix} m+1 \\ s \end{bmatrix}_{U_k} + (-1)^{\frac{m}{2}+s} \begin{bmatrix} m+1 \\ s-1 \end{bmatrix}_{U_k}$$

$$= \frac{U_{k(m+1)}U_{km}...U_{k(m-s+2)}}{(U_k U_{2k} \cdots U_{ks})} + (-1)^{\frac{m}{2}+s} \frac{U_{k(m+1)}U_{km} \cdots U_{k(m-s+3)}}{\left(U_k U_{2k} \cdots U_{k(s-1)}\right)}$$

$$= \frac{U_{k(m+2)}U_{k(m+1)} \cdots U_{k(m-s+3)}}{(U_k U_{2k} \cdots U_{ks})} \left[\frac{U_{k(m-s+2)} + (-1)^{\frac{m}{2}+s} U_{ks}}{U_{k(m+2)}} \right]$$

$$= \frac{U_{k(m+2)}U_{k(m+1)} \cdots U_{k(m-s+3)}}{(U_k U_{2k} \cdots U_{ks})} \left[\frac{U_{k\left(\frac{m}{2}-s+1\right)}V_{k\left(\frac{m}{2}+1\right)}}{U_{k(m+2)}} \right]$$

$$= \frac{U_{k(m+2)}U_{k(m+1)} \cdots U_{k(m-s+3)}}{(U_k U_{2k} \cdots U_{ks})} \frac{U_{k\left(\frac{m}{2}-s+1\right)}}{U_{k\left(\frac{m}{2}+1\right)}}$$

$$= \frac{U_{k\left(\frac{m}{2}+1-s\right)}}{U_{k\left(\frac{m}{2}+1\right)}} \begin{bmatrix} m+2 \\ s \end{bmatrix}_{U_k}.$$

Thus the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Define

$$(3.12) \quad \sigma_m^k(t) := \sum_{j=0}^{m-t} (-1)^{\frac{j}{2}(j+m+1)} \begin{bmatrix} m+1 \\ j \end{bmatrix}_{U_k},$$

where m is an even positive integer and t is any integer.

3.6. Lemma. *For an even positive integer m and any integer t,*

    i) $\sigma_m^k(t) = 0$, *for* $t \leq -1$ *or* $t \geq m+1$,

    ii) $\sigma_m^k(m-t) = \sigma^k(t)$,

    iii) $\sigma_m^k(0) = 1$, $\sigma_m^k(1) = 1 + \frac{1}{U_k}(-1)^{\frac{m-2}{2}} U_{k(m+1)}$,

    $\sigma_m^k(2) = 1 - \frac{1}{U_k U_{2k}} V_{k\left(\frac{m+2}{2}\right)} U_{k(m+1)} U_{k\left(\frac{m-2}{2}\right)}$,

    $\sigma_m^k(3) = 1 - \frac{1}{U_k U_{2k} U_{3k}} (-1)^{\frac{m}{2}} U_{k(m+1)} \left(U_{2k}U_{3k} - V_{k\left(\frac{m+2}{2}\right)} U_{km} U_{k\left(\frac{m-4}{2}\right)} \right).$

Proof. In order to get the proof of i) and ii), one can follow the method of proof of in [12, Lemma 16 i) and ii)], since they are similar statements.

iii) The identities for $\sigma_m^k(0)$ and $\sigma_m^k(1)$ are directly implied by $\sigma_m^k(-1) = 0$. Using ii) and (3.12), we have

$$\sigma_m^k(2) = \sum_{j=0}^{2} (-1)^{\frac{j}{2}(j+m+1)} \begin{bmatrix} m+1 \\ j \end{bmatrix}_{U_k}$$

$$= 1 + (-1)^{\frac{(m+2)}{2}} \frac{U_{k(m+1)}}{U_k} - (-1)^m \frac{U_{k(m+1)}U_{km}}{U_k U_{2k}}$$

$$= 1 - \frac{U_{k(m+1)}}{U_k U_{2k}} \left(U_{km} + (-1)^{\frac{m}{2}} \right)$$

$$= 1 - \frac{U_{k(m+1)}}{U_k U_{2k}} \left(U_{k\left(\frac{m}{2}-1\right)} V_{k\left(\frac{m}{2}+1\right)} \right),$$

and

$$\sigma_m^k(3) = \sigma_m^k(2) - (-1)^{\frac{(m-2)}{2}} \frac{U_{k(m-1)}U_{km}U_{k(m+1)}}{U_k U_{2k} U_{3k}}$$

$$= 1 - (-1)^{\frac{m}{2}} \frac{U_{k(m+1)}}{U_k} - \frac{U_{k(m+1)}U_{km}}{U_k U_{2k}} + (-1)^{\frac{m}{2}} \frac{U_{k(m-1)}U_{km}U_{k(m+1)}}{U_k U_{2k} U_{3k}}$$

$$= 1 - (-1)^{\frac{m}{2}} \frac{U_{k(m+1)}}{U_k} \left(U_{2k}U_{3k} - U_{k(m-1)}U_{km} + (-1)^{\frac{m}{2}} U_{3k}U_{km} \right)$$

$$= 1 - (-1)^{\frac{m}{2}} \frac{U_{k(m+1)}}{U_k} \left(U_{2k}U_{3k} - U_{km}\left(U_{k(m-1)} - (-1)^{\frac{m}{2}} U_{3k} \right) \right)$$

$$= 1 - (-1)^{\frac{m}{2}} \frac{U_{k(m+1)}}{U_k} \left(U_{2k}U_{3k} - U_{km}U_{k\left(\frac{m-4}{2}\right)}V_{k\left(\frac{m+2}{2}\right)} \right),$$

as desired.   □

3.7. Lemma. *For an even positive integer m and any integer t,*

$$\sigma_m^k(t) - \sigma_m^k(t-2) = (-1)^{\frac{t}{2}(t+m+1)} \begin{bmatrix} m+2 \\ t \end{bmatrix}_{U_k} \frac{U_{k\left(\frac{m}{2}-t+1\right)}}{U_{k\left(\frac{m}{2}+1\right)}}.$$

Proof. For $t < 2$, the claim follows from the definition of Fibonomial coefficients and Lemma 3.6. For $m \geq 2$, we have

$$\sigma_m^k(t) - \sigma_m^k(t-2)$$

$$= \sigma_m^k(m-t) - \sigma_m^k(m-t+2)$$

$$= \sum_{j=0}^{t} (-1)^{\frac{j}{2}(j+m+1)} \begin{bmatrix} m+1 \\ j \end{bmatrix}_{U_k} - \sum_{j=0}^{t-2} (-1)^{\frac{j}{2}(j+m+1)} \begin{bmatrix} m+1 \\ j \end{bmatrix}_{U_k}$$

$$= (-1)^{\frac{t}{2}(t+m+1)} \left(\begin{bmatrix} m+1 \\ t \end{bmatrix}_{U_k} + (-1)^{\frac{m}{2}+t} \begin{bmatrix} m+1 \\ t-1 \end{bmatrix}_{U_k} \right).$$

By Lemma 3.5, the proof is complete.   □

3.8. Lemma. *For an even positive integer m and any integer t,*

$$\sigma_m^k(t) - \sigma_m^k(t-4) = (-1)^{\frac{t}{2}(t+m+1)} \begin{bmatrix} m+4 \\ t \end{bmatrix}_{U_k}$$

$$\times \frac{U_{k\left(\frac{m}{2}-t+2\right)}}{U_{k\left(\frac{m}{2}+1\right)}U_{k(m+3)}U_{k(m+4)}} \omega(t,m),$$

where $\omega(t,m) = U_{k\left(\frac{m}{2}+1-t\right)}V_{k\left(\frac{m}{2}+2-t\right)}U_{k(m+3)} - V_k U_{km}U_{k(m-1)}.$

Proof. By Lemma 3.7, we have for any integer t that

$$\sigma_m^k(t) - \sigma_m^k(t-4)$$

$$= \sigma_m^k(t) - \sigma_m^k(t-2) + \sigma_m^k(t-2) - \sigma_m^k(t-4)$$

$$= (-1)^{\frac{t}{2}(t+m+1)} \begin{bmatrix} m+2 \\ t \end{bmatrix}_{U_k} \frac{U_{k\left(\frac{m}{2}-t+1\right)}}{U_{k\left(\frac{m}{2}+1\right)}}$$

$$+ (-1)^{\frac{t-2}{2}(t+m-1)} \begin{bmatrix} m+2 \\ t-2 \end{bmatrix}_{U_k} \frac{U_{k\left(\frac{m}{2}-t+3\right)}}{U_{k\left(\frac{m}{2}+1\right)}}$$

$$= (-1)^{\frac{t}{2}(t+m+1)} \frac{1}{U_{k\left(\frac{m}{2}+1\right)}} \left(U_{k\left(\frac{m}{2}-t+1\right)} \begin{bmatrix} m+2 \\ t \end{bmatrix}_{U_k} - U_{k\left(\frac{m}{2}-t+3\right)} \begin{bmatrix} m+2 \\ t-2 \end{bmatrix}_{U_k} \right)$$

$$= (-1)^{\frac{t}{2}(t+m+1)} \begin{bmatrix} m+4 \\ t \end{bmatrix}_{U_k} \frac{1}{U_{k(m+3)} U_{k(m+4)}}$$

$$\times \left(U_{k\left(\frac{m}{2}-t+1\right)} U_{k(m+3-t)} - U_{k\left(\frac{m}{2}-t+3\right)} U_{kt} U_{k(t-1)} \right).$$

From (2.10), we get

$$\sigma_m^k(t) - \sigma_m^k(t-4)$$

$$= (-1)^{\frac{t}{2}(t+m+1)} \begin{bmatrix} m+4 \\ t \end{bmatrix}_{U_k} \frac{1}{U_{k(m+3)} U_{k(m+4)}}$$

$$\times \left(U_{k\left(\frac{m}{2}-t+1\right)} \left(U_{k(m+4-2t)} U_{k(m+3)} + U_{kt} U_{k(t-1)} \right) - U_{k\left(\frac{m+6}{2}-t\right)} U_{kt} U_{k(t-1)} \right)$$

$$= (-1)^{\frac{t}{2}(t+m+1)} \begin{bmatrix} m+4 \\ t \end{bmatrix}_{U_k} \frac{1}{U_{k(m+3)} U_{k(m+4)}}$$

$$\times \left(U_{k\left(\frac{m}{2}-t+1\right)} U_{k(m+4-2t)} U_{k(m+3)} - V_k U_{k\left(\frac{m}{2}+2-t\right)} U_{kt} U_{k(t-1)} \right)$$

$$= (-1)^{\frac{t}{2}(t+m+1)} \begin{bmatrix} m+4 \\ t \end{bmatrix}_{U_k} \frac{U_{k\left(\frac{m}{2}-t+2\right)}}{U_{k(m+3)} U_{k(m+4)}}$$

$$\times \left(V_{k\left(\frac{m}{2}+2-t\right)} U_{k(m+4-2t)} U_{k(m+3)} - V_k U_{kt} U_{k(t-1)} \right),$$

as claimed.   □

3.9. Theorem. *For any integer m,*

$$\sum_{j=0}^{t} (-1)^{\frac{j}{2}(j+m+1)} \begin{bmatrix} m+1 \\ j \end{bmatrix}_{U_k}$$

$$= \frac{(-1)^{\frac{t}{2}(t+m+1)}}{U_{k\left(\frac{m}{2}+1\right)} U_{k(m+3)} U_{k(m+4)}} \sum_{i=0}^{\left\lfloor \frac{t}{4} \right\rfloor} \begin{bmatrix} m+4 \\ t-4i \end{bmatrix}_{U_k} U_{k\left(\frac{m}{2}+2-t+4i\right)}$$

$$\times \left(U_{k\left(\frac{m}{2}+1-t+4i\right)} V_{k\left(\frac{m}{2}+2-t+4i\right)} U_{k(m+3)} - V_k U_{k(t-4i)} U_{k(t-4i-1)} \right).$$

Proof. By Lemma 3.8, we can write

$$\sum_{i=0}^{\left\lfloor \frac{t}{4} \right\rfloor} \left[\sigma_m^k(t-4i) - \sigma_m^k(t-4(i+1)) \right] = \sigma_m^k(t) - \sigma_m^k\left(t - 4\left(\left\lfloor \frac{t}{4} \right\rfloor + 1 \right) \right).$$

From Lemma 3.6, we have

$$\sigma_m^k\,(t) = \sum_{i=0}^{\left\lfloor \frac{t}{4} \right\rfloor} \left[\sigma_m^k\,(t-4i) - \sigma_m^k\,(t - 4\,(i+1)) \right].$$

Again by Lemma 3.8, we get

$$\sigma_m^k\,(t) = \sum_{i=0}^{\left\lfloor \frac{t}{4} \right\rfloor} (-1)^{\frac{t-4i}{2}(t-4i+m+1)} \begin{bmatrix} m+4 \\ t-4i \end{bmatrix}_{U_k} \frac{U_{k\left(\frac{m}{2}+2-t+4i\right)}}{U_{k\left(\frac{m}{2}+1\right)}U_{k(m+3)}U_{k(m+4)}}$$

$$\times \left(U_{k\left(\frac{m}{2}+1-t+4i\right)}V_{k\left(\frac{m}{2}+2-t+4i\right)}U_{k(m+3)} - V_k U_{k(t-4i)}U_{k(t-4i-1)} \right)$$

$$= \frac{(-1)^{\frac{t}{2}(t+m+1)}}{U_{k\left(\frac{m}{2}+1\right)}U_{k(m+3)}U_{k(m+4)}} \sum_{i=0}^{\left\lfloor \frac{t}{4} \right\rfloor} \begin{bmatrix} m+4 \\ t-4i \end{bmatrix}_{U_k} U_{k\left(\frac{m}{2}+2-t+4i\right)}$$

$$\times \left(U_{k\left(\frac{m}{2}+1-t+4i\right)}V_{k\left(\frac{m}{2}+2-t+4i\right)}U_{k(m+3)} - V_k U_{k(t-4i)}U_{k(t-4i-1)} \right),$$

as claimed. $\square$

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