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COMPARISON CRITERIA FOR THE OSCILLATION OF MIXED-TYPE IMPULSIVE DIFFERENCE EQUATIONS WITH CONTINUOUS ARGUMENTS

B. Karpuz*†, Ö. Öcalan* and S. Öztürk*

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Abstract

The main objective of this paper is to present a comparison criteria for the oscillation of solutions to mixed-type impulsive difference equation with continuous arguments, without imposing sign restrictions on the coefficients.

Keywords: Difference equations, Continuous arguments, Impulse effects, Mixed-type, Nonoscillation, Oscillation.

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1. Introduction

This paper is concerned with the oscillatory nature of solutions of the following impulsive difference equation (IDE) with continuous arguments:

$$(1.1)\qquad \begin{cases} \Delta_\rho x(t) + p(t)x(t-\tau) + q(t)x(t+\sigma) = 0 & \text{for } t \in [t_0,\infty)\backslash\{\theta_k\}_{k\in\mathbb{N}} \\ x(\theta_k^+) = \lambda_k x(\theta_k) & \text{for } k \in \mathbb{N}, \end{cases}$$

where $t_0 \in \mathbb{R}$, $\rho \in (0,\infty)$, $p,q \in C\big([t_0,\infty),\mathbb{R}\big)$, $\tau,\sigma \in [0,\infty)$, $\{\lambda_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ and $\{\theta_k\}_{k\in\mathbb{N}} \subset [t_0,\infty)$ is the increasing unbounded sequence of impulse points. Here, $\Delta_\rho x(t) := x(t+\rho) - x(t)$ for $t \in [t_0,\infty)$ and $x(\theta_k^+)$ denotes the right sided limit of x at the impulse point θ_k for some $k \in \mathbb{N}$, and the left sided limits are defined similarly. It should be noted that all solutions of (1.1) are oscillatory in the absence of a subsequence $\{\theta_{k_\ell}\}_{\ell\in\mathbb{N}}$ such that $\{\lambda_{k_\ell}\}_{\ell\in\mathbb{N}} \subset (-\infty,0)$, since the solution always changes sign at the impulse points $\{\theta_{k_\ell}\}_{\ell\in\mathbb{N}}$.

*Department of Mathematics, Faculty of Sciences and Arts, ANS Campus, 03200 Afyonkarahisar, Turkey. E-mail: (B. Karpuz) bkarpuz@gmail.com (Ö. Öcalan) ozkan@aku.edu.tr (S. Öztürk) ssahin@aku.edu.tr

†Corresponding Author.

In the sequel, for simplicity in the notation, we shall assume that the empty product is the unit.

In general, oscillation theory is focused on the following problems:

(1) Sufficient conditions for the existence of a nonoscillatory solution.
(2) Sufficient conditions for all solutions to be oscillatory.

Significantly different methods are employed for the investigation of problems 1 and 2. For problem 1, it is enough to prove the existence of a solution with a constant sign. In this case, various fixed point methods are applied, or a monotone sequence converging to a nonoscillatory solution is defined. The investigation of problem 2 cannot employ methods characterizing only some solutions of the equation. The proof is therefore usually given by contradiction, i.e., the assumption that there exists a nonoscillatory solution is shown to be absurd given the conditions assumed to hold for the parameters of the equation.

Necessary and sufficient conditions for all solutions of (1.1) to be oscillatory are obtained by comparing with mixed type difference equations with continuous arguments and without impulse effects. The method employed here for this result gives us the advantage of no sign restriction on the coefficients. However, as far as we know, there is no result in the literature to test the oscillation of all solutions of mixed type difference equations with continuous arguments and without impulse effects in the case where the coefficients are of different signs, or of alternating signs.

When there is no sign condition imposed on the coefficients, the monotonicity properties of the nonoscillatory solutions disappear, and the oscillation properties of the solutions therefore become difficult to check. This is why, later on, we shall restrict our attention to (1.1) with fixed coefficients of the same sign, so that we will be able to give explicit oscillation results for the equation. Our method here allows us to consider (1.1) with several coefficients, but for simplicity of notation we shall be interested in (1.1), which involves a single coefficient for the delay and a single coefficient for the advanced term.

The explicit oscillation results we will give depend on making comparison with known differential equations. Hence, we find it useful the recall that

$$\limsup_{t\to\infty}\int_{t-\tau}^{t} p(\eta)\mathrm{d}\eta > 1 \text{ or } \liminf_{t\to\infty}\int_{t-\tau}^{t} p(\eta)\mathrm{d}\eta > \frac{1}{\mathrm{e}}$$

implies oscillation of all solutions to the delay differential equation

$$x'(t) + p(t)x(t-\tau) = 0 \quad \text{for } t \in [t_0, \infty),$$

where $p \in C([t_0,\infty),[0,\infty))$ and $\tau > 0$. The same conditions also prevent

$$x'(t) + p(t)x(t-\tau) \leq 0 \quad \text{for } t \in [t_0, \infty)$$

and

$$x'(t) - p(t)x(t+\tau) \geq 0 \quad \text{for } t \in [t_0, \infty)$$

from admitting eventually positive solutions, where the parameters are the same as given above (see [1, § 2 and § 3]).

By a *solution* of (1.1), we mean a function $x : [t_0-\tau,\infty) \to \mathbb{R}$ such that x is continuous on (θ_k,θ_{k+1}) for all $k \in \mathbb{N}$ and satisfies (1.1), and that $x(\theta_k^{\pm})$ exists as a finite constant with $x(\theta_k^-) = x(\theta_k)$ for all $k \in \mathbb{N}$. From now on, to make the definition of the solution consistent, we shall assume that $t \notin \{\theta_k\}_{k\in\mathbb{N}}$ implies $t+\rho \notin \{\theta_k\}_{k\in\mathbb{N}}$, $t-\tau \notin \{\theta_k\}_{k\in\mathbb{N}}$ for all $t+\sigma \notin \{\theta_k\}_{k\in\mathbb{N}}$.

Together with the impulsive difference equation (1.1), it is customary to specify an *initial condition* of the form

$$x = \varphi \quad \text{on } [t_0 - \tau, t_0 + \max\{\sigma, \rho\}], \tag{1.2}$$

where the *initial function* φ is a prescribed real-valued function on the interval $[t_0 - \tau, t_0 + \max\{\sigma, \rho\}]$ such that φ is continuous on the interval $[t_0 - \tau, t_0)$, and is piecewise continuous on $[t_0, t_0 + \max\{\sigma, \rho\}]\backslash\{\theta_k\}_{k\in\mathbb{N}}$ satisfying the *consistency condition*

$$\begin{cases} \Delta_\rho \varphi(t_0) + p(t_0)\varphi(t_0 - \tau) + q(t_0)\varphi(t_0 + \sigma) = 0 & \text{for } t_0 \neq \theta_0 \\ \varphi(\theta_0^+) = \lambda_0 \varphi(\theta_0) & \text{for } t_0 = \theta_0. \end{cases} \tag{1.3}$$

By the method of steps, one can easily conclude that (1.1) admits a unique solution x which satisfies the initial condition (1.2) and the consistency condition (1.3). For convenience, we denote this solution by $x = x(\,\cdot\,, t_0, \varphi)$. As is customary, a solution x of (1.1) is called *nonoscillatory* if it is eventually of fixed sign, otherwise, it is called *oscillatory*.

Readers are referred to [3, 4, 5] for the oscillation of difference equations with continuous arguments and without impulses, and [2, 6, 7] for impulsive difference equations with continuous arguments. In this paper, we shall make use of the method introduced in [8] for differential equations.

2. Comparison criteria for IDEs

The primary assumptions for this section are as follows:

- a_1 There exists a constant $\alpha > 0$ such that $\prod_{t-\rho\leq\theta_\ell<t} \lambda_\ell \equiv \alpha$ for all $t \in [t_0 + \rho, \infty)$.
- a_2 There exists a constant $\beta > 0$ such that $\prod_{t-\tau\leq\theta_\ell<t} \lambda_\ell \equiv \beta$ for all $t \in [t_0 + \tau, \infty)$.
- a_3 There exists a constant $\gamma > 0$ such that $\prod_{t\leq\theta_\ell<t+\sigma} \lambda_\ell \equiv \gamma$ for all $t \in [t_0, \infty)$.

Above it is assumed that the empty product is unity. Consider the following difference equation without impulse effects:

$$\Delta_\rho x(t) + \frac{\alpha^{\tau/\rho}}{\beta} p(t)x(t - \tau) + \frac{\gamma}{\alpha^{\sigma/\rho}} q(t)x(t + \sigma) = 0 \text{ for } t \in [t_0, \infty). \tag{2.1}$$

As is customary, a *solution* of the equation (2.1) is a function $x \in C([t_0 - \tau, \infty), \mathbb{R})$ satisfying (2.1) on $[t_0, \infty)$.

2.1. Theorem. *Assume that a_1, a_2 and a_3 hold. If $y = y(\,\cdot\,, t_0, \varphi)$ is a solution of* (2.1)*, then $x = x(\,\cdot\,, t_0, \psi)$, where*

$$\psi(t) = \begin{cases} \varphi(t), & t \in [t_0 - \tau, t_0) \\ \dfrac{1}{\alpha^{t/\rho}} \left[\displaystyle\prod_{t_0\leq\theta_\ell<t} \lambda_\ell \right] \varphi(t), & t \in \big[t_0, t_0 + \max\{\sigma, \rho\}\big], \end{cases}$$

defined by

$$x(t) := \frac{1}{\alpha^{t/\rho}} \left[\prod_{t_0\leq\theta_\ell<t} \lambda_\ell \right] y(t) \textit{ for } t \in [t_0, \infty) \tag{2.2}$$

is a solution of (1.1).

Proof. Let $y = y(\,\cdot\,, t_0, \varphi)$ be the solution of (2.1). We shall prove that x defined by (2.2) satisfies (1.1). It is obvious that x is continuous on each interval (θ_n, θ_{n+1}) for all $n \in \mathbb{N}$.

From (2.2), we get

$$\begin{aligned}\Delta_\rho y(t) =&\frac{\alpha^{(t+\rho)/\rho}}{\prod_{t_0\le\theta_\ell<t+\rho}\lambda_\ell}x(t+\rho)-\frac{\alpha^{t/\rho}}{\prod_{t_0\le\theta_\ell<t}\lambda_\ell}x(t)\\ =&\frac{\alpha^{t/\rho}}{\prod_{t_0\le\theta_\ell<t}\lambda_\ell}x(t+\rho)-\frac{\alpha^{t/\rho}}{\prod_{t_0\le\theta_\ell<t}\lambda_\ell}x(t)\\ =&\frac{\alpha^{t/\rho}}{\prod_{t_0\le\theta_\ell<t}\lambda_\ell}\Delta_\rho x(t)\end{aligned}\tag{2.3}$$

for all $t\in[t_0,\infty)$. Moreover, for all $t\in[t_0,\infty)$, we have

$$\begin{aligned}y(t-\tau) =&\frac{\alpha^{(t-\tau)/\rho}}{\prod_{t_0\le\theta_\ell<t-\tau}\lambda_\ell}x(t-\tau)\\ =&\frac{\beta\alpha^{t/\rho}}{\alpha^{\tau/\rho}\prod_{t_0\le\theta_\ell<t}\lambda_\ell}x(t-\tau),\end{aligned}\tag{2.4}$$

and

$$\begin{aligned}y(t+\sigma) =&\frac{\alpha^{(t+\sigma)/\rho}}{\prod_{t_0\le\theta_\ell<t+\sigma}\lambda_\ell}x(t+\sigma)\\ =&\frac{\alpha^{\sigma/\rho}\alpha^{t/\rho}}{\gamma\prod_{t_0\le\theta_\ell<t}\lambda_\ell}x(t+\sigma).\end{aligned}\tag{2.5}$$

Substituting (2.3), (2.4) and (2.5) into (2.1), and canceling the positive term

$$\alpha^{t/\rho}/\prod_{t_0\le\theta_\ell<t}\lambda_\ell,$$

we see that x defined by (2.2) solves the first equation in (1.1). On the other hand, for all $k\in\mathbb{N}$, we have

$$\begin{aligned}x(\theta_k^+) &= \lim_{t\to\theta_k^+}\left(\frac{1}{\alpha^{t/\rho}}\left[\prod_{t_0\le\theta_\ell<t}\lambda_\ell\right]y(t)\right)\\ &= \frac{1}{\alpha^{\theta_k/\rho}}\left[\prod_{t_0\le\theta_\ell\le\theta_k}\lambda_\ell\right]y(\theta_k)=(1-\lambda_k)x(\theta_k),\end{aligned}$$

which shows that x satisfies the second equation in (1.1) too. Similarly, one can show that ψ is the initial function for this solution. The proof is therefore completed. □

The following result can be regarded as the converse of Theorem 2.1, and we state it without proof.

2.2. Remark. Assume that a_1, a_2 and a_3 hold. If $x=x(\,\cdot\,,t_0,\psi)$ is a solution of (1.1), then $y=y(\,\cdot\,,t_0,\varphi)$, where

$$\varphi(t)=\begin{cases}\psi(t), & t\in[t_0-\tau,t_0)\\ \dfrac{\alpha^{t/\rho}}{\prod_{t_0\le\theta_\ell<t}\lambda_\ell}\psi(t), & t\in\big[t_0,t_0+\max\{\sigma,\rho\}\big],\end{cases}$$

defined by

$$y(t)=\frac{\alpha^{t/\rho}}{\prod_{t_0\le\theta_\ell<t}\lambda_\ell}x(t)\text{ for }t\in[t_0,\infty)\tag{2.6}$$

is a solution of (2.1).

2.3. Theorem. *Assume that a_1, a_2 and a_3 hold. Every solution of* (1.1) *is oscillatory if and only if every solution of* (2.1) *is oscillatory.*

Proof. The proof follows from Theorem 2.1, Remark 2.2, and the fact that $\{\lambda_k\}_{k\in\mathbb{N}} \subset (0,\infty)$. □

3. An application of IDEs

In this section, we give the following application which considers the autonomous case.

3.1. Example. Let $\rho \in (0,\infty)$, $p,q \in \mathbb{R}$ and $\tau,\sigma \in [0,\infty)$, and consider

$$\begin{cases} \Delta_\rho x(t) + px(t-\rho\tau) + qx(t+\rho\sigma) = 0 & \text{for } t \in [0,\infty)\backslash\{\rho k\}_{k\in\mathbb{N}}, \\ x(k^+) = \lambda x(k) & \text{for } k \in \rho\mathbb{N}, \end{cases} \tag{3.1}$$

where $\lambda \in (0,\infty)$. Then we have $\alpha = \lambda$, $\beta = \lambda^\tau$ and $\gamma = \lambda^{-\sigma}$, so that the associated difference equation without impulses is

$$\Delta_\rho x(t) + p\lambda^{\tau(1-\rho)/\rho}x(t-\rho\tau) + q\lambda^{\sigma(\rho-1)/\rho}x(t+\rho\sigma) = 0 \text{ for } t \in [0,\infty). \tag{3.2}$$

By the result in [3], we can infer that every solution of (3.2) (and hence of (3.1)) is oscillatory if and only if the following characteristic equation

$$1 - \mathrm{e}^{-\rho s} + p\lambda^{\tau(1-\rho)/\rho}\mathrm{e}^{-\rho\tau s} + q\lambda^{\sigma(\rho-1)/\rho}\mathrm{e}^{\rho\sigma s} = 0 \tag{3.3}$$

has no real roots. For instance, when $\rho = 1/2$, $\lambda = 1/3$, $p = \mathrm{e}-1$, $\tau = 1$, $q = 1/\mathrm{e}$ and $\sigma = 1/2$, then (3.1) admits a nonoscillatory solution. Figure 1 shows that the characteristic equation

$$1 + \frac{1}{\mathrm{e}}\sqrt{3}\mathrm{e}^{s/4} - \frac{1}{3}(4-2\mathrm{e})\mathrm{e}^{-s/2} = 0$$

has a negative root.

Figure 1. The graph of the characteristic equation (3.3) **with parameters $\rho = 1/2$, $\lambda = 1/3$, $p = \mathrm{e}-1$, $\tau = 1$, $q = 1/\mathrm{e}$ and $\sigma = 1/2$**

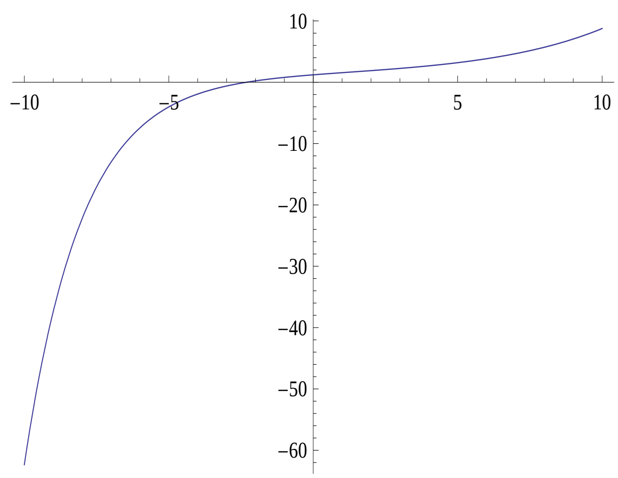

Table 1 illustrates the oscillation of the solutions of (3.1) by changing the parameters λ, p and q one by one when ρ, τ and σ are fixed to be $1/2$, 1 and $1/2$, respectively.

Table 1. A table illustrating the oscillation/nonoscillation behaviour of solutions for particular values of the parameters ρ, λ, p, τ, q, and σ

ρ	λ	p	τ	q	σ	Oscillation/Nonoscillation
$\frac{1}{2}$	$\frac{1}{3}$	$\mathrm{e}-1$	$\frac{1}{\mathrm{e}}$	1	$\frac{1}{2}$	Nonoscillation
$\frac{1}{2}$	$\frac{2}{3}$	$\mathrm{e}-1$	$\frac{1}{\mathrm{e}}$	1	$\frac{1}{2}$	Oscillation
$\frac{1}{2}$	$\frac{2}{3}$	$-(\mathrm{e}-1)$	$\frac{1}{\mathrm{e}}$	1	$\frac{1}{2}$	Nonoscillation
$\frac{1}{2}$	$\frac{2}{3}$	$-(\mathrm{e}-1)$	$-\frac{1}{\mathrm{e}}$	1	$\frac{1}{2}$	Oscillation

4. Comparison criteria for IDEs with Differential Equations

In this section to give explicit oscillation results for (1.1). We will confine our attention to the case where either $p,q\in C([t_0,\infty),[0,\infty))$ or $p,q\in C([t_0,\infty),(-\infty,0])$ holds. This is due to the ensuing technical difficulties mentioned previously.

4.1. IDEs with positive coefficients. Our first result for this section reads as follows.

4.1. Theorem. *In addition to a_1, a_2 and a_3, assume that $p,q\in C([t_0,\infty),[0,\infty))$. If the differential inequality*

$$x'(t)+\frac{\alpha^{\tau/\rho}}{\rho\beta}\Big(\min_{s\in[t,t+\rho]}p(s)\Big)x(t-\tau)+\frac{\gamma}{\rho\alpha^{\sigma/\rho}}\Big(\min_{s\in[t,t+\rho]}q(s)\Big)x(t+\sigma)\leq 0,\ t\in[t_0,\infty) \tag{4.1}$$

has no eventually positive solutions, then every solution of (1.1) *is oscillatory.*

Proof. Assume the contrary that x is an eventually positive solution of (1.1), then we see that y defined by (2.2) satisfies (2.1). Assume that $x(t),x(t-\rho),x(t-\tau)>0$ for all $t\in[t_1,\infty)$ for some $t_1\in[t_0,\infty)$. Set

$$z(t):=\int_t^{t+\rho}y(\eta)\mathrm{d}\eta\quad\text{for } t\in[t_1,\infty). \tag{4.2}$$

Clearly, we have $z(t)>0$ and $z'(t)=\Delta_\rho y(t)\leq 0$ for all $t\in[t_1,\infty)$. Integrating (2.1) from t to $t+\rho$, we have

$$\begin{aligned}0=&\Delta_\rho z(t)+\frac{\alpha^{\tau/\rho}}{\beta}\int_t^{t+\rho}p(\eta)y(\eta-\tau)\mathrm{d}\eta+\frac{\gamma}{\alpha^{\sigma/\rho}}\int_t^{t+\rho}q(\eta)y(\eta+\sigma)\mathrm{d}\eta\\ \geq&\Delta_\rho z(t)+\frac{\alpha^{\tau/\rho}}{\beta}\Big(\min_{s\in[t,t+\rho]}p(s)\Big)z(t-\tau)+\frac{\gamma}{\alpha^{\sigma/\rho}}\Big(\min_{s\in[t,t+\rho]}q(s)\Big)z(t+\sigma)\end{aligned} \tag{4.3}$$

for all $t\in[t_1,\infty)$. Now, set

$$w(t):=\int_t^{t+\rho}z(\eta)\mathrm{d}\eta\quad\text{for } t\in[t_1,\infty), \tag{4.4}$$

then we obtain $w(t)>0$, $w'(t)=\Delta_\rho z(t)$ and $w(t)\leq\rho z(t)$ for all $t\in[t_1,\infty)$. Therefore, from (4.3), we get

$$w'(t)+\frac{\alpha^{\tau/\rho}}{\rho\beta}\Big(\min_{s\in[t,t+\rho]}p(s)\Big)w(t-\tau)+\frac{\gamma}{\rho\alpha^{\sigma/\rho}}\Big(\min_{s\in[t,t+\rho]}q(s)\Big)w(t+\sigma)\leq 0$$

for all $t \in [t_1, \infty)$, i.e., the eventually positive w solves (4.1). This contradiction completes the proof. □

As an immediate consequence of Theorem 4.1, we can give the following corollary.

4.2. Corollary. *In addition to* a_1, a_2 *and* a_3, *assume that* $p, q \in C([t_0, \infty), [0, \infty))$ *and that* $\tau > \rho$. *Then either*

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} \Big(\min_{s\in[\eta,\eta+\rho]} p(s)\Big) \mathrm{d}\eta > \frac{\rho\beta}{\alpha^{\tau/\rho}}, \text{ or}$$
$$\liminf_{t\to\infty} \int_{t-\tau}^{t} \Big(\min_{s\in[\eta,\eta+\rho]} p(s)\Big) \mathrm{d}\eta > \frac{\rho\beta}{\alpha^{\tau/\rho}\mathrm{e}} \tag{4.5}$$

implies that every solution of (1.1) *is oscillatory.*

Proof. Assume on the contrary that x is an eventually positive solution of (1.1), then we see that y defined by (2.2) satisfies (2.1). Assume that for some $t_1 \in [t_0, \infty)$ e have $x(t), x(t-\rho), x(t-\tau) > 0$ for all $t \in [t_1, \infty)$. Then we have

$$x'(t) + \frac{\alpha^{\tau/\rho}}{\rho\beta}\Big(\min_{s\in[t,t+\rho]} p(s)\Big) x(t-\tau) \le 0 \text{ for } t \in [t_1, \infty),$$

which cannot admit eventually positive solutions if any of the conditions in (4.5) hold (see [1, Theorem 2.3.1 and Theorem 3.4.3]). This completes the proof. □

4.3. Remark. If the coefficients p and q in Theorem 4.2 are nonincreasing, then we can replace (4.1) with

$$x'(t)+\frac{\alpha^{\tau/\rho}}{\rho\beta}\bigg(\int_t^{t+\rho} p(\eta)\,\mathrm{d}\eta\bigg)x(t-\tau)+\frac{\gamma}{\rho\alpha^{\sigma/\rho}}\bigg(\int_t^{t+\rho} q(\eta)\,\mathrm{d}\eta\bigg)x(t+\sigma) \le 0,\ t \in [t_0, \infty). \tag{4.6}$$

In this case, either

$$\limsup_{t\to\infty} \int_{t-\tau}^{t}\int_{\eta}^{\eta+\rho} p(\zeta)\,\mathrm{d}\zeta\mathrm{d}\eta > \frac{\rho\beta}{\alpha^{\tau/\rho}} \text{ or } \liminf_{t\to\infty} \int_{t-\tau}^{t}\int_{\eta}^{\eta+\rho} p(\zeta)\,\mathrm{d}\zeta\mathrm{d}\eta > \frac{\rho\beta}{\alpha^{\tau/\rho}\mathrm{e}}$$

implies that every solution of (1.1) is oscillatory.

4.2. IDEs with negative coefficients. The results of this section can be easily shown, we therefore state the results without proof (see [1]).

4.4. Theorem. *In addition to* a_1, a_2 *and* a_3, *assume that* $p, q \in C([t_0, \infty), (-\infty, 0])$. *If the differential inequality*

$$x'(t)+\frac{\alpha^{\tau/\rho}}{\rho\beta}\Big(\max_{s\in[t,t+\rho]} p(s)\Big)x(t-\tau)+\frac{\gamma}{\rho\alpha^{\sigma/\rho}}\Big(\max_{s\in[t,t+\rho]} q(s)\Big)x(t+\sigma) \ge 0,\ t \in [t_0, \infty) \tag{4.7}$$

has no eventually positive solutions, then every solution of (1.1) *is oscillatory.* □

4.5. Corollary. *In addition to* a_1, a_2 *and* a_3, *assume that* $p, q \in C([t_0, \infty), (-\infty, 0])$. *Then either*

$$\limsup_{t\to\infty} \int_{t-\sigma}^{t} \Big(-\max_{s\in[\eta,\eta+\rho]} q(s)\Big) \mathrm{d}\eta > \frac{\rho\alpha^{\sigma/\rho}}{\gamma} \text{ or}$$
$$\liminf_{t\to\infty} \int_{t-\sigma}^{t} \Big(-\max_{s\in[\eta,\eta+\rho]} q(s)\Big) \mathrm{d}\eta > \frac{\rho\alpha^{\sigma/\rho}}{\gamma\mathrm{e}}$$

implies that every solution of (1.1) *is oscillatory.* □

4.6. Remark. If the coefficients p and q in Theorem 4.4 are nondecreasing, then we can replace (4.7) with

$$x'(t)+\frac{\alpha^{\tau/\rho}}{\rho\beta}\bigg(\int_{t-\rho}^{t}p(\eta)\,\mathrm{d}\eta\bigg)x(t-\tau)+\frac{\gamma}{\rho\alpha^{\sigma/\rho}}\bigg(\int_{t-\rho}^{t}q(\eta)\,\mathrm{d}\eta\bigg)x(t+\sigma)\geq 0,\ t\in[t_0,\infty).$$

In this case, either

$$\limsup_{t\to\infty}\int_{t-\sigma}^{t}\int_{\eta}^{\eta+\rho}\big(-q(\zeta)\big)\mathrm{d}\zeta\mathrm{d}\eta>\frac{\rho\beta}{\alpha^{\tau/\rho}},\ \text{or}$$

$$\liminf_{t\to\infty}\int_{t-\sigma}^{t}\int_{\eta}^{\eta+\rho}\big(-q(\zeta)\big)\mathrm{d}\zeta\mathrm{d}\eta>\frac{\rho\beta}{\alpha^{\tau/\rho}\mathrm{e}}$$

implies that every solution of (1.1) is oscillatory.

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