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THE RING AND ALGEBRA OF INTUITIONISTIC SETS

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Abstract

The aim of this study is to define "intuitionistic rings", "intuitionistic sigma rings", "intuitionistic algebras" and "intuitionistic sigma algebras", and on the other hand to adapt some well known ring and algebra theorems in classical sets to intuitionistic sets.

Keywords: Intuitionistic algebra, Intuitionistic ring.

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1. Introduction

After the introduction of the concept of fuzzy set by Zadeh [3], the idea of intuitionistic fuzzy sets was given by Krassimir T. Atanassov [1], and some notes on intuitionistic sets and intuitionistic points were given by Doğan Çoker [2]. In this paper, the concepts of intuitionistic ring (IR), intuitionistic sigma (σ) ring (ISR), intuitionistic algebra (IA) and intuitionistic σ-algebra (ISA) are first defined. On the other hand the validity of some suitable theorems of ring and algebra theory in classical sets for intuitionistic sets has been investigated and some theorems are first given.

2. Preliminaries

2.1. Definition. [2] Let X be a nonempty fixed set. An *intuitionistic set* A is an object having the form

$$A = \langle x,\ A_1,\ A_2 \rangle,$$

where A_1 and A_2 are subsets of X satisfying $A_1 \cap A_2 = \varnothing$. The set A_1 is called the set of members of A, while A_2 is called the set of nonmembers of A.

2.2. Definition. [2] Let X be a nonempty fixed set and A, B intuitionistic sets of the form $A = \langle x,\ A_1,\ A_2 \rangle$ and $B = \langle x,\ B_1,\ B_2 \rangle$. Then

(a) $A \cup B = \langle x,\ A_1 \cup B_1,\ A_2 \cap B_2 \rangle$,
(b) $A \cap B = \langle x,\ A_1 \cap B_1,\ A_2 \cup B_2 \rangle$,

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(c) $A - B = A \cap \overline{B}$,
(d) $\overline{A} = \langle x,\ A_2,\ A_1 \rangle$,
(e) $A \subseteq B \iff A_1 \subseteq B_1,\ A_2 \supseteq B_2$.

2.3. Definition. [2] $\underset{\sim}{\varnothing} = \langle x,\ \varnothing,\ X \rangle$ and $\underset{\sim}{X} = \langle x,\ X,\ \varnothing \rangle$.

2.4. Theorem. *Let $\{A_i : i \in J\}$ be an arbitrary family of intuitionistic sets in X, where $A_i = \langle x,\ A_i^{(1)},\ A_i^{(2)} \rangle$, and let $B = \langle x,\ B_1,\ B_2 \rangle$ be a fixed intuitionistic set in X. Then:*
(i) $B \cap \big(\bigcup_i A_i\big) = \bigcup_i \big(B \cap A_i\big)$,
(ii) $B \cup \big(\bigcap_i A_i\big) = \bigcap_i \big(B \cup A_i\big)$,
(iii) $B - \big(\bigcup_i A_i\big) = \bigcap_i \big(B - A_i\big)$,
(iv) $B - \big(\bigcap_i A_i\big) = \bigcup_i \big(B - A_i\big)$.

Proof. (i) The union of the sets A_i is $\bigcup_i A_i = \big\langle x,\ \bigcup_i A_i^{(1)},\ \bigcap_i A_i^{(2)} \big\rangle$. Hence,

$$\begin{aligned} B \cap \Big(\bigcup_i A_i\Big) &= \Big\langle x,\ B_1 \cap \Big(\bigcup_i A_i^{(1)}\Big),\ B_2 \cup \Big(\bigcap_i A_i^{(2)}\Big) \Big\rangle \\ &= \Big\langle x,\ \bigcup_i \Big(B_1 \cap A_i^{(1)}\Big),\ \bigcap_i \Big(B_2 \cup A_i^{(2)}\Big) \Big\rangle \\ &= \bigcup_i \big\langle x,\ B_1 \cap A_i^{(1)},\ B_2 \cup A_i^{(2)} \big\rangle \\ &= \bigcup_i (B \cap A_i) \end{aligned}$$

(ii) Similar to (i).

(iii)
$$\begin{aligned} B - \Big(\bigcup_i A_i\Big) &= B \cap \Big(\overline{\bigcup_i A_i}\Big) \\ &= \langle x,\ B_1,\ B_2 \rangle \cap \Big\langle x,\ \bigcap_i A_i^{(2)},\ \bigcup_i A_i^{(1)} \Big\rangle \\ &= \Big\langle x,\ B_1 \cap \Big(\bigcap_i A_i^{(2)}\Big),\ B_2 \cup \Big(\bigcup_i A_i^{(1)}\Big) \Big\rangle \\ &= \Big\langle x,\ \bigcap_i \big(B_1 \cap A_i^{(2)}\big),\ \bigcup_i \big(B_2 \cup A_i^{(1)}\big) \Big\rangle \\ &= \bigcap_i \big(\big\langle x,\ B_1 \cap A_i^{(2)},\ B_2 \cup A_i^{(1)} \big\rangle\big) \\ &= \bigcap_i \big(\langle x,\ B_1, B_2 \rangle \cap \big\langle x,\ A_i^{(2)}, A_i^{(1)} \big\rangle\big) \\ &= \bigcap_i (B - A_i) \end{aligned}$$

(iv) Similar to (iii). □

3. Intuitionistic rings and intuitionistic algebras

3.1. Definition. Let X be a nonempty fixed set and Ω any family of intuitionistic sets in X. If the family Ω satisfies the following conditions, then the family Ω is called a *ring of intuitionistic sets in* X, or for short an "intuitionistic ring" (IR(X)).

(i) For all $A \in \Omega$, $B \in \Omega$ we have $A \cup B \in \Omega$.
(ii) For all $A \in \Omega$, $B \in \Omega$ we have $A - B \in \Omega$.

If $\bigcup_{n=1}^{\infty} A_n \in \mathrm{IR}(X)$ for all $A_n \in \mathrm{IR}(X)$ then $\mathrm{IR}(X)$ is called a *σ-ring of intuitionistic sets* or shortly an *intuitionistic σ-ring* (ISR(X)).

By the definition, for all $A_i \in \mathrm{IR}(X)$ and for every $n \in N^+$, $\left(\bigcup_{i=1}^{n} A_i\right) \in \mathrm{IR}(X)$ is obvious.

3.2. Definition. Let X be a nonempty fixed set and Θ any family of intuitionistic sets in X. If the family Θ satisfies the following conditions then the family Θ is called an *algebra of intuitionistic sets* in X, or shortly an *intuitionistic algebra* (IA(X)).

(i) If $A, B \in \Theta$ then $(A \cup B) \in \Theta$,
(ii) $\overline{A} \in \Theta$ for each $A \in \Theta$.

If $\left(\bigcup_{n=1}^{\infty} A_n\right) \in \mathrm{IA}(X)$ for all $A_n \in \mathrm{IA}(X)$, then IA is called a *σ-algebra of intuitionistic sets*, or shortly an *intuitionistic σ-algebra* (ISA(X)).

3.3. Theorem. *Every* $\mathrm{IA}(X)$ *is also an* $\mathrm{IR}(X)$.

Proof. Let $A, B \in \mathrm{IA}(X)$. Thus by Definition 3.2, we have $\overline{A} \in \mathrm{IA}(X)$, $(\overline{A} \cup B) \in \mathrm{IA}(X)$ and $\overline{(\overline{A} \cup B)} \in \mathrm{IA}(X)$.

For $\overline{(\overline{A} \cup B)} = (A \cap \overline{B}) = (A - B)$ we have $(A - B) \in \mathrm{IA}(X)$. So, by the definition of $\mathrm{IR}(X)$, $\mathrm{IA}(X)$ is also an $\mathrm{IR}(X)$. □

3.4. Theorem. *Every* $\mathrm{ISA}(X)$ *is also an* $\mathrm{ISR}(X)$.

Proof. Obvious by Theorem 3.3 and Definitions 3.1, 3.2. □

3.5. Theorem. *If* $X \in \mathrm{IR}(X)$ *then* $\mathrm{IR}(X)$ *is an* $\mathrm{IA}(X)$*, and also if* $X \in \mathrm{ISR}(X)$ *then* $\mathrm{ISR}(X)$ *is an* $\mathrm{ISA}(X)$.

Proof. Let $X, A \in \mathrm{IR}(X)$. Then, by the equality $\overline{A} = X - A$ we have $\overline{A} \in \mathrm{IR}(X)$. This completes the proof. □

3.6. Theorem. *Let* $(A_i)_{i=1}^{\infty}$ *be a countable family of intuitionistic sets in an* $\mathrm{ISA}(X)$. *Then* $\left(\bigcap_{i=1}^{\infty} A_i\right) \in \mathrm{ISA}(X)$.

Proof. By the definitions;

For each $A_i \in \mathrm{ISA}(X)$, $\overline{A_i} \in \mathrm{ISA}(X)$ and $\left(\bigcup_{i=1}^{\infty} \overline{A_i}\right) \in \mathrm{ISA}(X)$ and $\overline{\left(\bigcup_{i=1}^{\infty} \overline{A_i}\right)} \in \mathrm{ISA}(X)$.

By the equality $\overline{\left(\bigcup_{i=1}^{\infty} \overline{A_i}\right)} = \bigcap_{i=1}^{\infty} A_i$ we have $\left(\bigcap_{i=1}^{\infty} A_i\right) \in \mathrm{ISA}(X)$. □

3.7. Theorem. *Let* X *be a nonempty fixed set and* Φ *a family of intuitionistic sets in* X*. In this case, there is a minimal* $\mathrm{ISA}(X)$ *containing the family* Φ*. This family is called the* Φ*-produced minimal* $\mathrm{ISA}(X)$.

Proof. Let $\Delta = \{\mathrm{ISA}(X) : \mathrm{ISA}(X)$ contains the family $\Phi\}$ and define

$$\mathrm{ISA}(X)^* = \bigcap_{\mathrm{IA}(X) \in \Delta} \mathrm{IA}(X).$$

Now let us prove that the family $\mathrm{ISA}(X)^*$ is an $\mathrm{ISA}(X)$.

(i)
$$\begin{aligned} A, B \in \mathrm{ISA}(X)^* &\Longrightarrow A, B \in \mathrm{ISA}(X)\ \forall\, \mathrm{ISA}(X) \in \Delta \\ &\Longrightarrow (A \cup B) \in \mathrm{ISA}(X)\ \forall\, \mathrm{ISA}(X) \in \Delta \\ &\Longrightarrow (A \cup B) \in \mathrm{ISA}(X)^*. \end{aligned}$$

(ii)
$$\begin{aligned} A \in \mathrm{ISA}(X)^* &\Longrightarrow A \in \mathrm{ISA}(X)\ \forall\, \mathrm{ISA}(X) \in \Delta \\ &\Longrightarrow \overline{A} \in \mathrm{ISA}(X)\ \forall\, \mathrm{ISA}(X) \in \Delta \\ &\Longrightarrow \overline{A} \in \mathrm{ISA}(X)^*. \end{aligned}$$

(iii) We must prove that $\left(\bigcup_{n=1}^{\infty} A_n\right) \in \mathrm{ISA}(X)^*$ for each $A_n \in \mathrm{ISA}(X)^*$, $n \in N^+$.

For each $n \in N^+$ and $A_n \in \mathrm{ISA}(X)^*$, $A_n \in \mathrm{ISA}(X)$ for each $\mathrm{ISA}(X) \in \Delta$.

For each $n \in N^+$ and $\mathrm{ISA}(X) \in \Delta$ we have $\left(\bigcup_{n=1}^{\infty} A_n\right) \in \mathrm{ISA}(X)$.

Hence, $\left(\bigcup_{n=1}^{\infty} A_n\right) \in \mathrm{ISA}(X)^*$. □

3.8. Theorem. *Let* $\mathrm{IA}_1(X)$ *and* $\mathrm{IA}_2(X)$ *be intuitionistic algebras in* X*. Then,* $\mathrm{IA}_1(X) \cap \mathrm{IA}_2(X)$ *is an intuitionistic* σ*-algebra in* X*.*

Proof. (i) For every $A, B \in (\mathrm{IA}_1(X) \cap \mathrm{IA}_2(X))$, we must prove that $(A \cup B) \in (\mathrm{IA}_1(X) \cap \mathrm{IA}_2(X))$.

$$\begin{aligned} A, B \in (\mathrm{IA}_1(X) \cap \mathrm{IA}_2(X)) &\Longrightarrow A, B \in \mathrm{IA}_1(X) \text{ and } A, B \in \mathrm{IA}_2(X) \\ &\Longrightarrow (A \cup B) \in \mathrm{IA}_1(X) \text{ and } (A \cup B) \in \mathrm{IA}_2(X) \\ &\Longrightarrow (A \cup B) \in (\mathrm{IA}_1(X) \cap \mathrm{IA}_2(X)). \end{aligned}$$

(ii) For every $A \in (\mathrm{IA}_1(X) \cap \mathrm{IA}_2(X))$, we must prove that $\overline{A} \in (\mathrm{IA}_1(X) \cap \mathrm{IA}_2(X))$.

$$\begin{aligned} A \in (\mathrm{IA}_1(X) \cap \mathrm{IA}_2(X)) &\Longrightarrow A \in \mathrm{IA}_1(X) \text{ and } A \in \mathrm{IA}_2(X) \\ &\Longrightarrow \overline{A} \in \mathrm{IA}_1(X) \text{ and } \overline{A} \in \mathrm{IA}_2(X) \\ &\Longrightarrow \overline{A} \in (\mathrm{IA}_1(X) \cap \mathrm{IA}_2(X)). \end{aligned}$$

(iii) For $A_n \in (\mathrm{IA}_1(X) \cap \mathrm{IA}_2(X))$, $n \in N^+$, we have $\left(\bigcup_{n=1}^{\infty} A_n\right) \in (\mathrm{IA}_1(X) \cap \mathrm{IA}_2(X))$ since $\mathrm{IA}_1(X)$ and $\mathrm{IA}_2(X)$ are intuitionistic σ-algebras.

From (i), (ii) and (iii), $(\mathrm{IA}_1(X) \cap \mathrm{IA}_2(X))$ is an intuitionistic σ-algebra. □

3.9. Corollary. $\Omega = \left\{\underset{\sim}{\varnothing}\right\}$ *is an intuitionistic ring.*

Proof. For $\underset{\sim}{\varnothing} = \langle x, \varnothing, X \rangle$,

(i) $\underset{\sim}{\varnothing} \cup \underset{\sim}{\varnothing} = \langle x, \varnothing \cup \varnothing, X \cap X \rangle = \langle x, \varnothing, X \rangle = \underset{\sim}{\varnothing} \in \Omega$.

(ii) $\underset{\sim}{\varnothing} - \underset{\sim}{\varnothing} = \underset{\sim}{\varnothing} \cap \overline{\underset{\sim}{\varnothing}} = \langle x, \varnothing, X \rangle \cap \langle x, X, \varnothing \rangle = \langle x, \varnothing \cap X, X \cup \varnothing \rangle = \langle x, \varnothing, X \rangle = \underset{\sim}{\varnothing} \in \Omega$.

By (i) and (ii), $\Omega = \left\{\underset{\sim}{\varnothing}\right\}$ is an intuitionistic ring. □

3.10. Corollary. *For* $X \neq \varnothing$*, let* $P\left(\underset{\sim}{X}\right)$ *be the family of all intuitionistic sets in* X*. Then,* $P\left(\underset{\sim}{X}\right)$ *is an* $\mathrm{ISA}(X)$*.*

Proof. Let $A = \langle x, A_1, A_2 \rangle$ and $B = \langle x, B_1, B_2 \rangle$ be any two intuitionistic sets in X.

(i) $A \cup B = \langle x, A_1 \cup B_1, A_2 \cap B_2 \rangle \in P\left(\underset{\sim}{X}\right)$,

(ii) $\overline{A} = \langle x, A_2, A_1 \rangle \in P\left(\underset{\sim}{X}\right)$,

(iii) For $A_n = \langle x, A_n^{(1)}, A_n^{(2)} \rangle \in P\left(\underset{\sim}{X}\right)$, $n \in N^+$, $\bigcup\limits_{n=1}^{\infty} A_n = \bigcup\limits_{n=1}^{\infty} \left(\langle x, A_n^{(1)}, A_n^{(2)} \rangle\right) = \langle x, \bigcup\limits_{n=1}^{\infty} A_n^{(1)}, \bigcap\limits_{n=1}^{\infty} A_n^{(2)} \rangle \in P\left(\underset{\sim}{X}\right)$. □

3.11. Example. Let A be an intuitionistic set in X of the form $A = \langle x, A_1, \varnothing \rangle$. Then $M = \{A, \overline{A}\}$ is an ISA(X).

Proof. (i) $A \cup \overline{A} = \langle x, A_1, \varnothing \rangle \cup \langle x, \varnothing, A_1 \rangle = \langle x, A_1 \cup \varnothing, \varnothing \cap A_1 \rangle = \langle x, A_1, \varnothing \rangle = A \in M$,

(ii) $\overline{A} \in M$ and $\overline{(\overline{A})} = A \in M$. □

3.12. Example. Let $X \neq \varnothing$ and $A = \langle x, A_1, \varnothing \rangle$ be an intuitionistic set in X. In this case, $N = \left\{A, \overline{A}, \underset{\sim}{\varnothing}, \underset{\sim}{X}\right\}$ is an ISA(X).

Proof. (i) The union of any two intuitionistic sets in N is also an intuitionistic set in N:
$A \cup \overline{A} = A$, $A \cup \underset{\sim}{\varnothing} = A$, $A \cup \underset{\sim}{X} = \underset{\sim}{X}$, $\overline{A} \cup \underset{\sim}{\varnothing} = \overline{A}$, $\overline{A} \cup \underset{\sim}{X} = \underset{\sim}{X}$.

(ii) The complement of a intuitionistic set in N is also an intuitionistic set in N:
$\overline{\underset{\sim}{\varnothing}} = \underset{\sim}{X}$, $\overline{\underset{\sim}{X}} = \underset{\sim}{\varnothing}$, $\overline{(\overline{A})} = A$.

(iii) $\underset{\sim}{\varnothing} \cup \underset{\sim}{X} \cup A \cup \overline{A} = \underset{\sim}{X}$.

From (i), (ii), (iii) we have the family N is an ISA(X). □

3.13. Example. Let A be an intuitionistic set having the form $A = \langle x, A_1, A_2 \rangle$. The family $P = \{A, \overline{A}, A \cup \overline{A}, A \cap \overline{A}\}$ is an ISA(X).

Proof. (i) The union of any two intuitionistic sets in P is also an intuitionistic set in P:
$(A \cup \overline{A}) \cup (A \cap \overline{A}) = (A \cup \overline{A})$, $A \cup (A \cap \overline{A}) = A$, $\overline{A} \cup (A \cap \overline{A}) = \overline{A}$.

(ii) The complement of an intuitionistic set in P is also an intuitionistic set in P:
$\overline{(A \cup \overline{A})} = A \cap \overline{A}$, $\overline{(A \cap \overline{A})} = A \cup \overline{A}$.

(iii) A countable union of intuitionistic sets in P is also an intuitionistic set in P:
$(A \cup \overline{A}) \cup (A \cap \overline{A}) \cup A \cup \overline{A} = A \cup \overline{A}$.

From (i), (ii), (iii) we have that the family P is an ISA(X). □

4. Conclusions

In this paper we have adapted some ring and algebra theorems in classical sets to intuitionistic sets. Other theorems couldn't been adapted because not all of the well known properties of the operations union, intersection, difference, complement for classical sets are not valid for intuitionistic sets. One of the main reasons is that the union of an intuitionistic set and its complement does not always equal the intuitionistic universal set $\underset{\sim}{X}$, which contains all of other intuitionistic sets. Another reason is that the intersection of an intuitionistic set and its complement always does not always equal the empty set $\underset{\sim}{\varnothing}$, which is subset all of other empty sets.

References

[1] Atanassov, K., *Intuitionistic Fuzzy Sets* (VII ITKR's Session, Sofia, 1983).
[2] Çoker, D. A *Note on intuitionistic sets and intuitionistic points*, Turkish Journal of Mathematics **20** (3), 343–350, 1996.
[3] Zadeh, L. A., *Fuzzy sets*, Information and Control. **8**, 338–353, 1965.