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GENERALIZED P-FLATNESS AND P-INJECTIVITY OF MODULES

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Abstract

A right R-module N is called GP-flat if for any $a \in R$, there exists a positive integer n (depending on a) such that the sequence $0 \to N \otimes Ra^n \to N \otimes R$ is exact. A left R-module M is said to be GP-injective if for any $a \in R$, there exists a positive integer n (depending on a) such that every homomorphism from Ra^n to M extends to one from R to M. R is said to be a left GP-coherent (resp. GPF, GPP) ring in case for any $a \in R$, there exists a positive integer n (depending on a) such that Ra^n is finitely presented (resp. flat, projective). We study GP-coherent, GPF, GPP and π-regular rings using GP-flat and GP-injective modules.

Keywords: GP-flat module, GP-injective module, GP-coherent ring, GPF ring, GPP ring, π-regular ring.

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1. Introduction

Recall that a left R-module M is P*-injective* [32] or *divisible* [12] if for any $a \in R$, any homomorphism from Ra to M extends to one from R to M, equivalently, $\mathrm{Ext}^1(R/Ra, M) = 0$. In [33], the concept of P-injectivity is weakened to GP-injectivity. A left R-module M is called GP*-injective* if for any $a \in R$, there exists a positive integer n (depending on a) such that any homomorphism from Ra^n to M extends to one from R to M, equivalently, $\mathrm{Ext}^1(R/Ra^n, M) = 0$. Let us note that GP-injective modules defined here are different from the GP-injective modules defined in [21]. For any left R-module M and any $t \in R$, there exists the isomorphism $\mathrm{Ext}^1(R/Rt, M) \cong r_M l_R(t)/tM$ (see [12, p.148]). So, M is a P-injective left R-module if and only if $r_M l_R(a) = aM$ for any $a \in R$; M is a GP-injective left R-module if and only if for any $a \in R$, there exists a positive integer n (depending on a) such that $r_M l_R(a^n) = a^n M$. R is said to be a *left* P*-injective* (resp. GP*-injective) ring* if R is P-injective (resp. GP-injective) as a left R-module. We also recall that a right R-module N is P*-flat* [22] or *torsionfree* [12] if for any $a \in R$,

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the sequence $0 \to N \otimes Ra \to N \otimes R$ is exact, equivalently, $\mathrm{Tor}_1(N, R/Ra) = 0$. There exists an intimate connection between P-flat and P-injective modules. For example, N is a P-flat right R-module if and only if its character module N^+ is a P-injective left R-module. P-flat, P-injective and GP-injective modules have been extensively studied for many years (see, for example, [6, 9, 10, 12], [18] - [20], [22, 23], [30] - [36]). These modules play important roles in the research on von Neumann regular rings, coherent rings, V-rings and their generalizations. It is natural for us to consider the properties of those modules whose character modules are GP-injective.

On the other hand, we recall that R is a *left PP ring* if every principal left ideal of R is projective. R is a *left* GPP *ring* [13, 24] if for any $a \in R$, there exists a positive integer n (depending on a) such that Ra^n is projective, equivalently, if for any $a \in R$, the left annihilator of a^n is generated by an idempotent for some positive integer n (depending on a). As a generalization of GPP rings and PF rings (i.e., rings which have flat principal left ideals [15]), the concept of GPF rings is first introduced in [1]. R is a *left* GPF *ring* if for any $a \in R$, there exists a positive integer n (depending on a) such that Ra^n is flat. See [1, 13, 14, 17, 24, 25] for more details about GPP and GPF rings.

Motivated by these facts, in this paper, we first introduce a new class of modules, called GP-flat modules, which is a generalization of P-flat modules. We find that the relationship between GP-flat modules and GP-injective modules has an analogue between P-flat modules and P-injective modules. Then several important rings, such as GPF, GPP and π-regular rings, are characterized in terms of GP-flat and GP-injective modules, where R is said to be a *π-regular ring* [13] if for any $a \in R$, there exist $b \in R$ and a positive integer m such that $a^m = a^m b a^m$. Finally, as a generalization of both left P-coherent rings (i.e., rings which have principal left ideals finitely presented [20]) and left GPP rings, we introduce a new class of rings, called left GP-coherent rings, which is also closely related to GP-flat and GP-injective modules.

Next let us describe the contents of the paper in more detail.

In Section 2, after the concept of GP-flat module is introduced, we obtain several characteristic properties of GP-flat modules. For instance, we prove that N is a GP-flat right R-module if and only if N^+ is GP-injective, if and only if for any $a \in R$, there exists a positive integer n such that $Nl_R(a^n) = l_N(a^n)$, if and only if for any $a \in R$, there exists a positive integer n such that any homomorphism from R/a^nR to M factors through a free right R-module. We also show that a simple module over a commutative ring is GP-flat if and only if it is GP-injective. When R is a left P-coherent ring, we obtain some special properties of GP-flat and GP-injective modules.

Section 3 is devoted to the investigation on GPF, GPP and π-regular rings using GP-flat and GP-injective modules. For example, we prove that R is a left GPF ring if and only if every right ideal is GP-flat; R is a π-regular ring if and only if every right R-module is GP-flat. For a commutative ring R, we show that the following are equivalent: (1) R is π-regular ring. (2) R is a GP-injective GPP ring. (3) Every cyclic R-module is GP-injective. (4) Every cyclic R-module is GP-flat.

In Section 4 of this paper, we introduce and study the class of GP-coherent rings. We will call R a *left* GP-*coherent ring* if for any $a \in R$, there exists a positive integer n (depending on a) such that Ra^n is finitely presented. We prove that the following conditions are equivalent for a ring R: (1) R is a left GP-coherent ring. (2) Any direct product of copies of R_R is GP-flat. (3) Any direct limit of injective left R-modules is GP-injective. Finally, we get that GP-coherence coincides with P-coherence if R is a reduced ring.

Throughout this paper, R represents an associative ring with identity and all modules are unitary. M_R ($_RM$) denotes a right (left) R-module. For a subset X of R, the left (resp. right) annihilator of X in N_R (resp. $_RM$) is denoted by $l_N(X)$ (resp. $r_M(X)$). If $X = \{a\}$, we usually abbreviate this to $l_N(a)$ (resp. $r_M(a)$). For an R-module M, the dual module $\mathrm{Hom}_R(M, R)$ is denoted by M^*, and the character module M^+ is defined by $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. Let M and N be R-modules. $\mathrm{Hom}(M, N)$ (resp. $\mathrm{Ext}^n(M, N)$) means $\mathrm{Hom}_R(M, N)$ (resp. $\mathrm{Ext}^n_R(M, N)$), and similarly $M \otimes N$ (resp. $\mathrm{Tor}_n(M, N)$) denotes $M \otimes_R N$ (resp. $\mathrm{Tor}^R_n(M, N)$) for an integer $n \geq 1$ unless otherwise specified. For unexplained concepts and notations, we refer the reader to [2, 9, 18, 26, 30].

2. GP-flat modules and GP-injective modules

We start with the following

2.1. Definition. A right R-module N is said to be *generalized P-flat* (GP-*flat* for short) if for any $a \in R$, there exists a positive integer n (depending on a) such that the sequence $0 \to N \otimes Ra^n \to N \otimes R$ is exact, equivalently, for any $a \in R$, there exists a positive integer n (depending on a) such that $\mathrm{Tor}_1(N, R/Ra^n) = 0$.

Obviously, every P-flat module is GP-flat. But the converse is not true as shown in Example 3.4.

Now we give some characteristic properties of GP-flat modules.

2.2. Theorem. *The following are equivalent for a right R-module N:*

(1) *N is* GP-*flat.*
(2) *N^+ is* GP-*injective.*
(3) *For any $a \in R$, there exists a positive integer n such that $Nl_R(a^n) = l_N(a^n)$.*
(4) *For any $a \in R$, there exists a positive integer n such that $\sigma : N \otimes l_R(a^n) \to l_N(a^n)$ is an epimorphism, where σ is defined by $\sigma(a \otimes b) = ab$ for $a \in N$ and $b \in l_R(a^n)$.*
(5) *For any $a \in R$, there exists a positive integer n such that any homomorphism from R/a^nR to N factors through a free right R-module.*
(6) *There exists an exact sequence $0 \to K \to F \to N \to 0$ with F free such that for any $a \in R$, there exists a positive integer n satisfying $Ka^n = K \cap Fa^n$.*

Proof. (1)⇔(2) Let $a \in R$ and n be a positive integer. Then the sequence $0 \to N \otimes Ra^n \to N \otimes R$ is exact if and only if the sequence $(N \otimes R)^+ \to (N \otimes Ra^n)^+ \to 0$ is exact, if and only if the sequence $\mathrm{Hom}(R, N^+) \to \mathrm{Hom}(Ra^n, N^+) \to 0$ is exact. So, N is GP-flat if and only if N^+ is GP-injective.

(1)⇔(3) By the proof of [12, Proposition 1], for $a \in R$ and a positive integer n, we have the isomorphism $\mathrm{Tor}_1(N, R/Ra^n) \cong l_N(a^n)/Nl_R(a^n)$. So $\mathrm{Tor}_1(N, R/Ra^n) = 0$ if and only if $Nl_R(a^n) = l_N(a^n)$.

(3)⇔(4) is obvious by only noting that $\mathrm{im}(\sigma) = Nl_R(a^n)$.

(1)⇒(5) Let $a \in R$. By (1), there is a positive integer n such that $\mathrm{Tor}_1(N, R/Ra^n) = 0$. In addition, there exists an exact sequence $0 \to K \to F \xrightarrow{\pi} N \to 0$ with F free, which induces the exact sequence

$$0 = \mathrm{Tor}_1(N, R/Ra^n) \to K \otimes (R/Ra^n) \to F \otimes (R/Ra^n).$$

Let $f : R/a^nR \to N$ be any homomorphism. Then there exists $x \in F$ such that $\pi(x) = f(\overline{1})$. Hence $\pi(xa^n) = \pi(x)a^n = f(\overline{1})a^n = f(\overline{a^n}) = 0$, and so $xa^n \in K$. Since $xa^n \otimes \overline{1} = x \otimes \overline{a^n} = 0$ in $F \otimes (R/Ra^n)$, we have $xa^n \otimes \overline{1} = 0$ in $K \otimes (R/Ra^n)$. By [9, Lemma 6.1, p.33], there is $k \in K$ such that $xa^n = ka^n$. Define $\delta : R/a^nR \to F$ by $\delta(\overline{r}) = (x - k)r$ for $r \in R$. It is easy to check that δ is well-defined and $\pi\delta = f$. So f factors through F.

(5)⇒(1) Let $a \in R$. Then there exists a positive integer n such that any homomorphism from R/a^nR to N factors through a free right R-module. There exists an exact sequence $0 \to K \to F \xrightarrow{\pi} N \to 0$ with F free, which yields the exact sequence

$$0 = \mathrm{Tor}_1(F, R/Ra^n) \to \mathrm{Tor}_1(N, R/Ra^n) \to K \otimes (R/Ra^n) \to F \otimes (R/Ra^n).$$

Next we show that the sequence $0 \to K \otimes (R/Ra^n) \to F \otimes (R/Ra^n)$ is exact. Let $k \otimes \overline{1} \in K \otimes (R/Ra^n)$ be such that $k \otimes \overline{1} = 0$ in $F \otimes (R/Ra^n)$. Then there exists $p \in F$ such that $k = pa^n$ by [9, Lemma 6.1, p.33]. Define $f : R/a^nR \to N$ by $f(\overline{r}) = \pi(p)r$ for $r \in R$. Then f is well-defined. By (5), there exist a free right R-module G, homomorphisms $\beta : R/a^nR \to G$ and $\gamma : G \to N$ such that $f = \gamma\beta$. Therefore there exists $\delta : G \to F$ such that $\gamma = \pi\delta$. So $k = pa^n = (p - \delta\beta(\overline{1}))a^n$. Since $\pi(p - \delta\beta(\overline{1})) = f(\overline{1}) - \gamma\beta(\overline{1}) = 0$, $p - \delta\beta(\overline{1}) \in K$. Hence $k \otimes \overline{1} = (p - \delta\beta(\overline{1}))a^n \otimes \overline{1} = (p - \delta\beta(\overline{1})) \otimes \overline{a^n} = 0$ in $K \otimes (R/Ra^n)$, as desired. Thus $\mathrm{Tor}_1(N, R/Ra^n) = 0$, and so N is GP-flat.

(1)⇔(6) There exists an exact sequence $0 \to K \to F \to N \to 0$ with F free. Let $a \in R$ and n be a positive integer. By [26, Exercise 2.7, p.27], there exists $\psi : \mathrm{Tor}_1(N, R/Ra^n) \to (K \cap Fa^n)/Ka^n$ such that the following diagram with exact rows is commutative:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Tor}_1(N, R/Ra^n) & \longrightarrow & K \otimes (R/Ra^n) & \longrightarrow & F \otimes (R/Ra^n) \\
 & & \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \beta} \\
0 & \longrightarrow & (K \cap Fa^n)/Ka^n & \longrightarrow & K/Ka^n & \longrightarrow & F/Fa^n,
\end{array}$$

where α and β are isomorphisms. So $\mathrm{Tor}_1(N, R/Ra^n) \cong (K \cap Fa^n)/Ka^n$ by the Five Lemma. Thus N is GP-flat if and only if $Ka^n = K \cap Fa^n$. □

2.3. Proposition. *The following are true for a ring R:*

(1) *If N is a* GP*-flat right R-module, then $\oplus N$ is* GP*-flat.*
(2) *If M is a P-flat right R-module and N is a* GP*-flat right R-module, then $M \oplus N$ is* GP*-flat.*
(3) *If M is a* GP*-injective left R-module, then $\oplus M$ and ΠM are* GP*-injective.*
(4) *If M is a P-injective left R-module and N is a* GP*-injective left R-module, then $M \oplus N$ is* GP*-injective.*
(5) *The class of all* GP*-injective left R-modules and the class of all* GP*-flat right modules are closed under pure submodules.*

Proof. (1), (2), (3) and (4) are routine.

(5) Let N be a pure submodule of a GP-injective left R-module M. For any $a \in R$, there exists a positive integer n such that $\mathrm{Ext}^1(R/Ra^n, M) = 0$. So we have the exact sequence

$$\mathrm{Hom}(R/Ra^n, M) \to \mathrm{Hom}(R/Ra^n, M/N) \to \mathrm{Ext}^1(R/Ra^n, N) \to 0.$$

But the sequence $\mathrm{Hom}(R/Ra^n, M) \to \mathrm{Hom}(R/Ra^n, M/N) \to 0$ is exact since R/Ra^n is finitely presented. So $\mathrm{Ext}^1(R/Ra^n, N) = 0$, i.e., N is GP-injective.

Let A be a pure submodule of a GP-flat right R-module B. Then the exact sequence $0 \to A \to B \to B/A \to 0$ induces the split exact sequence $0 \to (B/A)^+ \to B^+ \to A^+ \to 0$. Thus A^+ is GP-injective since B^+ is GP-injective by Theorem 2.2. So A is GP-flat by Theorem 2.2 again. □

2.4. Proposition. *The following are true for a ring R:*

(1) *For a right ideal I of R, if R/I is a* GP*-flat right R-module, then for any $a \in I$, there exist a positive integer n and $r \in I$ such that $a^n = ra^n$.*

(2) *If I is a* GP*-injective right ideal of R, then R/I is a* GP*-flat right R-module. The converse holds if R is a right P-injective ring.*
(3) *If any maximal right ideal of R is* GP*-injective, then every simple right R-module is* GP*-flat.*
(4) *A simple module over a commutative ring is* GP*-flat if and only if it is* GP*-injective.*

Proof. (1) For any $a \in I$, there exists a positive integer n such that $(R/I)l_R(a^n) = l_{R/I}(a^n)$ by Theorem 2.2. Since $\overline{1} \in l_{R/I}(a^n)$, $\overline{1} \in (R/I)l_R(a^n)$. So there exists $t \in l_R(a^n)$ such that $\overline{1} = \overline{1}t = \overline{t}$. Thus $1 - t \in I$, and hence $a^n = (1-t)a^n$.

(2) If I is a GP-injective right ideal of R, then for any $a \in R$, there exists a positive integer n such that $l_I r_R(a^n) = Ia^n$. Since $I \cap Ra^n \subseteq l_I r_R(a^n) = Ia^n$, we have $I \cap Ra^n = Ia^n$. So R/I is a GP-flat right R-module by Theorem 2.2.

Conversely, if R/I is a GP-flat right R-module, then by Theorem 2.2, for any $a \in R$, there exists a positive integer n such that any homomorphism from R/a^nR to R/I factors through a free right R-module. So $\mathrm{Hom}(R/a^nR, R) \to \mathrm{Hom}(R/a^nR, R/I)$ is epic. On the other hand, since R is right P-injective, the exact sequence $0 \to I \to R \to R/I \to 0$ gives rise to the exact sequence

$$\mathrm{Hom}(R/a^nR, R) \to \mathrm{Hom}(R/a^nR, R/I) \to \mathrm{Ext}^1(R/a^nR, I) \to \mathrm{Ext}^1(R/a^nR, R) = 0.$$

Thus $\mathrm{Ext}^1(R/a^nR, I) = 0$, and so I is GP-injective.

(3) is clear by (2).

(4) Let $\{S_i\}_{i\in I}$ be the irredundant set of representatives of all simple R-modules and E the injective envelope of $\oplus_{i\in I} S_i$. Then E is an injective cogenerator by [2, Corollary 18.19]. Let S be a simple R-module. For any $a \in R$ and any positive integer n, there is the isomorphism

$$\mathrm{Ext}^1(R/a^nR, \mathrm{Hom}(S, E)) \cong \mathrm{Hom}(\mathrm{Tor}_1(R/a^nR, S), E).$$

Note that $\mathrm{Hom}(S, E) \cong S$ by the proof of [29, Lemma 2.6]. Thus $\mathrm{Tor}_1(R/a^nR, S) = 0$ if and only if $\mathrm{Ext}^1(R/a^nR, S) = 0$, and so S is GP-flat if and only if S is GP-injective. $\square$

The following propositions exhibit that GP-flat and GP-injective modules have several special properties when the ring in question is P-coherent. Recall from [20] that R is a *left P-coherent ring* if every principal left ideal of R is finitely presented.

2.5. Proposition. *Let R be a left P-coherent ring. Then,*

(1) *A left R-module M is* GP*-injective if and only if M^+ is* GP*-flat.*
(2) *Every* GP*-flat right R-module is P-flat if and only if every* GP*-injective left R-module is P-injective.*
(3) *R is a left* GPP *ring if and only if R is a left* GPF *ring.*

Proof. (1) Let M be a left R-module, n a positive integer and $a \in R$. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^+ \otimes Ra^n & \longrightarrow & M^+ \otimes R & \longrightarrow & M^+ \otimes (R/Ra^n) & \longrightarrow & 0 \\ & & \downarrow \theta_1 & & \downarrow \theta_2 & & \downarrow \theta_3 & & \\ 0 & \longrightarrow & \mathrm{Hom}(Ra^n, M)^+ & \longrightarrow & \mathrm{Hom}(R, M)^+ & \longrightarrow & \mathrm{Hom}(R/Ra^n, M)^+ & \longrightarrow & 0. \end{array}$$

Note that Ra^n is finitely presented since R is left P-coherent. So θ_1, θ_2 and θ_3 are isomorphisms by [5, Lemma 2]. Thus the first row is exact if and only if the second row is exact, if and only if the sequence $0 \to \mathrm{Hom}(R/Ra^n, M) \to \mathrm{Hom}(R, M) \to \mathrm{Hom}(Ra^n, M) \to 0$ is exact. So M is GP-injective if and only if M^+ is GP-flat.

(2) $\Longrightarrow$ Let M be any GP-injective left R-module. Then M^+ is GP-flat by (1), and so M^+ is P-flat. Thus M is P-injective by [20, Theorem 2.7].

$\Longleftarrow$ Let N be any GP-flat right R-module. Then N^+ is GP-injective by Theorem 2.2, and so N^+ is P-injective. Hence N is P-flat.

(3) is obvious. □

2.6. Lemma. *The following are equivalent for a ring R:*

(1) *R is a left* GP*-injective ring.*
(2) *For any $a \in R$, there exists a positive integer n such that $a^nR = r_R l_R(a^n)$*
(3) *For any $a \in R$, there exists a positive integer n such that R/a^nR is torsionless.*

Proof. (1) $\Longleftrightarrow$ (2) is clear.

(2) $\Longleftrightarrow$ (3) follows from [2, Lemma 25.2]. □

2.7. Proposition. *The following are equivalent for a left P-coherent ring R:*

(1) *R is a left* GP*-injective ring.*
(2) *Every injective right R-module is* GP*-flat.*
(3) *Every flat left R-module is* GP*-injective.*
(4) *For any $a \in R$, there exists a positive integer n such that R/a^nR embeds in a free right R-module.*

Proof. (1) $\Longrightarrow$ (2) Let E be an injective right R-module. Then there exists an exact sequence $0 \to E \to \Pi({}_RR)^+$. Note that $\oplus_R R$ is GP-injective by (1) and Proposition 2.3 (3), so $\Pi({}_RR)^+ \cong (\oplus_R R)^+$ is GP-flat by Proposition 2.5 (1). Thus E is GP-flat.

(2) $\Longrightarrow$ (3) Let M be a flat left R-module. Then M^+ is injective, and so M^+ is GP-flat by (2). Thus M is GP-injective by Proposition 2.5 (1).

(3) $\Longrightarrow$ (1) is clear.

(1) $\Longrightarrow$ (4) For any $a \in R$, there exists a positive integer n such that R/a^nR is torsionless by Lemma 2.6. So there exists an exact sequence $0 \to R/a^nR \to \Pi R_R$. By [20, Theorem 2.7], ΠR_R is P-flat. So R/a^nR embeds in a free right R-module by [35, Theorem 4.3].

(4) $\Longrightarrow$ (1) follows from Lemma 2.6. □

3. GPF rings, GPP rings and π-regular rings

We first characterize GPF rings using GP-flat modules as follows.

3.1. Theorem. *The following are equivalent for a ring R:*

(1) *R is a left* GPF *ring.*
(2) *Every submodule of any P-flat right R-module is* GP*-flat.*
(3) *Every right ideal of R is* GP*-flat.*
(4) *For any right R-module N and any $a \in R$, there exists a positive integer n such that $\mathrm{Tor}_2(N, R/Ra^n) = 0$.*

Proof. (1) $\Longrightarrow$ (2) Let N be a submodule of a P-flat right R-module M and $a \in R$. Then there exists a positive integer n such that Ra^n is flat. Consider the following commutative diagram:

$$\begin{array}{ccc} N \otimes Ra^n & \xrightarrow{\gamma} & N \otimes R \\ \downarrow{\scriptstyle \alpha} & & \downarrow \\ M \otimes Ra^n & \xrightarrow{\beta} & M \otimes R. \end{array}$$

Since Ra^n is flat and M is P-flat, α and β are monomorphisms, and so γ is a monomorphism. Thus N is GP-flat.

(2) $\Longrightarrow$ (4) For any right R-module N, there exists an exact sequence $0 \to K \to F \to N \to 0$ with F free. Then K is GP-flat by (2). For any $a \in R$, there is a positive integer n such that $\mathrm{Tor}_1(K, R/Ra^n) = 0$. Consider the induced exact sequence

$$0 = \mathrm{Tor}_2(F, R/Ra^n) \to \mathrm{Tor}_2(N, R/Ra^n) \to \mathrm{Tor}_1(K, R/Ra^n) = 0.$$

So $\mathrm{Tor}_2(N, R/Ra^n) = 0$.

(4) $\Longrightarrow$ (3) For any right ideal I of R and any $a \in R$, there exists a positive integer n such that $\mathrm{Tor}_2(R/I, R/Ra^n) = 0$ by (4). In addition, the exact sequence $0 \to I \to R \to R/I \to 0$ induces the exact sequence

$$0 = \mathrm{Tor}_2(R/I, R/Ra^n) \to \mathrm{Tor}_1(I, R/Ra^n) \to \mathrm{Tor}_1(R, R/Ra^n) = 0.$$

So $\mathrm{Tor}_1(I, R/Ra^n) = 0$. Thus I is GP-flat.

(3) $\Longrightarrow$ (1) Let I be any right ideal of R and $a \in R$. Then I is GP-flat, and so there exists a positive integer n such that the sequence $0 \to I \otimes Ra^n \xrightarrow{g} I \otimes R$ is exact. Consider the following commutative diagram:

$$\begin{array}{ccc} I \otimes Ra^n & \xrightarrow{f} & R \otimes Ra^n \\ \downarrow{\scriptstyle g} & & \downarrow \\ I \otimes R & \xrightarrow{h} & R \otimes R. \end{array}$$

Since g and h are monic, f is monic. Hence Ra^n is flat, so R is a left GPF ring. □

The equivalence of (1) through (3) in the following theorem has been established by the author (see [19, Theorem 2.1]). But we here include a different proof.

3.2. Theorem. *The following are equivalent for a ring R:*

(1) *R is a left* GPP *ring.*
(2) *Every quotient of any P-injective left R-module is* GP*-injective.*
(3) *Every quotient of any injective left R-module is* GP*-injective.*
(4) *For any left R-module M and any $a \in R$, there exists a positive integer n such that* $\mathrm{Ext}^2(R/Ra^n, M) = 0$.

Proof. (1) $\Longrightarrow$ (2) Let X be any P-injective left R-module and N any submodule of X. We will show that X/N is GP-injective. Let $a \in R$. Then there exists a positive integer n such that Ra^n is projective by (1). Let $\iota : Ra^n \to R$ be the inclusion and $\pi : X \to X/N$ the canonical map. For any $f : Ra^n \to X/N$, there exists $g : Ra^n \to X$ such that $\pi g = f$. So there is $h : R \to X$ such that $h\iota = g$ since X is P-injective. It follows that $(\pi h)\iota = f$, and (2) holds.

(2) $\Longrightarrow$ (3) is trivial.

(3) $\Longrightarrow$ (4) Let M be any left R-module. Then there exists an exact sequence $0 \to M \to E \to L \to 0$ with E injective. By (3), L is GP-injective. So for any $a \in R$, there exists a positive integer n such that $\mathrm{Ext}^1(R/Ra^n, L) = 0$. From the induced exact sequence

$$0 = \mathrm{Ext}^1(R/Ra^n, L) \to \mathrm{Ext}^2(R/Ra^n, M) \to \mathrm{Ext}^2(R/Ra^n, E) = 0,$$

we have $\mathrm{Ext}^2(R/Ra^n, M) = 0$.

(4) $\Longrightarrow$ (1) Let M be any left R-module and $a \in R$. Then there exists a positive integer n such that $\mathrm{Ext}^2(R/Ra^n, M) = 0$ by (4). The exact sequence $0 \to Ra^n \to R \to R/Ra^n \to 0$ yields the exact sequence

$$0 = \mathrm{Ext}^1(R, M) \to \mathrm{Ext}^1(Ra^n, M) \to \mathrm{Ext}^2(R/Ra^n, M) = 0.$$

Thus $\mathrm{Ext}^1(Ra^n, M) = 0$, and so Ra^n is projective. Hence R is a left GPP ring. □

Now we turn to the characterizations of π-regular rings. Recall from [13] that R is a *π-regular ring* if for any $a \in R$, there exist $b \in R$ and a positive integer m such that $a^m = a^m b a^m$. A left R-module M is called *cotorsion* [8] if $\mathrm{Ext}^1(F, C) = 0$ for any flat left R-module F.

3.3. Theorem. *The following are equivalent for a ring R:*

(1) *R is a π-regular ring.*
(2) *Every left R-module is* GP*-injective.*
(3) *Every cotorsion left R-module is* GP*-injective.*
(4) *Every right R-module is* GP*-flat.*
(5) *Every cotorsion right R-module is* GP*-flat.*
(6) *R is a left* GP*-injective ring and every quotient of any* GP*-injective left R-module is* GP*-injective.*

Proof. (1) $\Longleftrightarrow$ (2) follows from [36, Theorem 3].

(2) $\Longrightarrow$ (3), (2) $\Longrightarrow$ (6) and (4) $\Longrightarrow$ (5) are trivial.

(3) $\Longrightarrow$ (4) Let N be any right R-module. Since N^+ is pure-injective and hence cotorsion, N^+ is GP-injective by(3). So N is GP-flat by Theorem 2.2.

(5) $\Longrightarrow$ (2) Let M be any left R-module. Then M^+ is GP-flat by (5). Thus M^{++} is GP-injective by Theorem 2.2. Note that M is a pure submodule of M^{++}, so M is GP-injective by Proposition 2.3 (5).

(6) $\Longrightarrow$ (2) For any left R-module M, there exists an exact sequence $\oplus_R R \to M \to 0$. Since ${}_R R$ is GP-injective, $\oplus_R R$ is GP-injective by Proposition 2.3 (3). So M is GP-injective by (6). □

3.4. Example. Let $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{Z}_2 \right\}$. It is easily checked that every element of R is either nilpotent or idempotent or invertible. So R is a π-regular ring. But R is clearly not a von Neumann regular ring. Thus there exists a GP-flat right R-module which is not P-flat by Theorem 3.3 and [22, Theorem 2], and there exists a GP-injective left R-module which is not P-injective by Theorem 3.3 and [32, Lemma 2].

The following remark may be viewed as a further illustration of the usefulness of GP-flat modules.

3.5. Remark. Let $R = \begin{pmatrix} S & {}_SN_T \\ 0 & T \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a \in S, b \in N, c \in T \right\}$ be the formal triangular matrix ring constructed from a pair of rings S, T and a bimodule ${}_SN_T$. We claim that N is a GP-flat right T-module if R is a left GPP ring. Similarly, N is a GP-flat left S-module if R is a right GPP ring.

In fact, assume that R is a left GPP ring, then for any $t \in T$, there exists a positive integer n such that $R\begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix}^n = \begin{pmatrix} 0 & Nt^n \\ 0 & Tt^n \end{pmatrix}$ is a projective left R-module. By [11, Proposition 4.5], $N \otimes_T Tt^n \to N \otimes_T T$ is a monomorphism. So N is a GP-flat right T-module.

In the same fashion, we can prove that N is a GP-flat left S-module if R is a right GPP ring.

Next we discuss how GPP rings are close to π-regular rings.

3.6. Theorem. *The following are true for a ring R:*

(1) *If R is a left* GPP *ring, then for any right annihilator A in R and any $a \in A$, there exists a positive integer n such that $a^n \in Aa^n$.*
(2) *If R is a left* GPP *left* GP*-injective ring, then for any $a \in R$, there exist $b \in R$ and a positive integer n such that $a^n = aba^n$.*
(3) *If R is a left* GPP *left P-injective ring, then R is a π-regular ring.*
(4) *If R is a left* GPP *ring and every maximal right ideal of R is P-injective, then R is a π-regular ring.*

Proof. (1) Let A be a right annihilator in R and $a \in A$. Since R is left GPP, there exists a positive integer n such that $l_R(a^n) = Re$ for some $e^2 = e \in R$. Thus $a^n = (1-e)a^n$. Note that $(1-e)R = r_R(Re) = r_Rl_R(a^n) \subseteq r_Rl_R(a) \subseteq r_Rl_R(A) = A$. Hence $a^n \in Aa^n$.

(2) Let $a \in R$. Then there exists a positive integer m such that $r_Rl_R(a^m) = a^mR$ by Lemma 2.6. Let $c = a^m$ and $A = r_Rl_R(a^m)$. By (1), there exist a positive integer t and $u \in R$ such that $c^t = cuc^t$. Let $n = mt$. Then $a^n = a(a^{m-1}u)a^n$.

(3) Let M be any left R-module. Then there exists an exact sequence $\oplus_R R \to M \to 0$. Since $\oplus_R R$ is P-injective, M is GP-injective by Theorem 3.2. Thus R is a π-regular ring by Theorem 3.3.

(4) Let $I = \upsilon_1R + \upsilon_2R + \cdots + \upsilon_nR$ be a finitely generated proper right ideal of R. Then there exists a maximal right ideal K containing I. For any $\upsilon \in K$, the inclusion $\upsilon R \to K$ extends to $f : R \to K$ since K is P-injective. So $f(\upsilon) = \upsilon$. By [26, Theorem 3.57], there exists $\theta : R \to K$ such that $\theta(\upsilon_i) = \upsilon_i$ $(i = 1, 2, \cdots, n)$. Thus $(1 - \theta(1))I = 0$ and so $l_R(I) \neq 0$.

On the other hand, for any $a \in R$, there exists a positive integer m such that Ra^m is projective. So Ra^m is a direct summand of R by [3, Theorem 5.4]. Thus R is a π-regular ring. □

3.7. Theorem. *The following are equivalent for a commutative ring R:*

(1) *R is a π-regular ring.*
(2) *R is a* GP*-injective* GPP *ring.*
(3) *Every cyclic R-module is* GP*-injective.*
(4) *Every cyclic R-module is* GP*-flat.*

Proof. (2) $\Longrightarrow$ (1) For any $a \in R$, there exist a positive integer n and $b \in R$ such that $a^n = aba^n$ by Theorem 3.6 (2). So $a^n = (ab)(ab)a^n = \cdots = a^nb^na^n$. Thus R is a π-regular ring.

(3) $\Longrightarrow$ (1) Let $a \in R$. Then Ra is GP-injective. So there exists a positive integer n such that the inclusion $Ra^n \to Ra$ extends to a homomorphism $R \to Ra$. Thus there is $b \in R$ such that $a^n = a^n ba$. So $a^n = a^n(ba)^2 = \cdots = a^n(ba)^n = a^n b^n a^n$. Hence R is a π-regular ring.

(4) $\Longrightarrow$ (1) For any $a \in R$, R/aR is GP-flat by (4). So there exist a positive integer n and $c \in R$ such that $a^n = aca^n$ by Proposition 2.4 (1). Thus $a^n = a^n c^n a^n$, and so R is a π-regular ring.

The rest are clear by Theorem 3.3. □

4. GP-coherent rings

4.1. Definition. A ring R is called a *left generalized P-coherent* (GP-*coherent* for short) if for any $a \in R$, there exists a positive integer n (depending on a) such that Ra^n is finitely presented.

Obviously, the concept of left GP-coherent rings is a generalization of both left P-coherent rings and left GPP rings. The following examples show that this generalization is proper.

4.2. Example. Let K be a field with a subfield L such that $\dim_L K = \infty$, and there exists a field isomorphism $\varphi : K \to L$ (for instance, $K = \mathbb{Q}(x_1, x_2, x_3, \cdots), L = \mathbb{Q}(x_2, x_3, \cdots)$). Let $R = K \times K$ with multiplication

$$(x, y)(x', y') = (xx', \varphi(x)y' + yx'),\ x, y, x', y' \in K.$$

Let $a = (0, 1) \in R$. Then $l_R(a)$ is not finitely generated (see [18, Example 4.46 (e)]). Thus R is not left P-coherent. On the other hand, it is easy to see that R has exactly three right ideals: (0), R, and $(0, K) = (0, 1)R$. So R is a local ring with the Jacobson radical $J(R) = (0, K)$. Thus every element of R is either nilpotent or invertible. So R is a left GP-coherent ring.

4.3. Example. Let R be the 2×2 full matrix ring over $\mathbb{Z}[x]$. Then R is not a left GPP ring by [14, Example 4]. But $\mathbb{Z}[x]$ is a coherent ring by [18, Example 4.61 (b)]. So R is a left coherent ring, in particular, a left GP-coherent ring.

Recall that a left R-module M is *FP-injective* [27] or *absolutely pure* [7] if $\mathrm{Ext}^1(N, M) = 0$ for any finitely presented left R-module N.

The following theorem characterizes left GP-coherent rings in terms of, among others, GP-flat and GP-injective modules.

4.4. Theorem. *The following are equivalent for a ring R:*

(1) *R is a left* GP-*coherent ring.*
(2) *For any $a \in R$, there exists a positive integer n such that $l_R(a^n)$ is a finitely generated left ideal.*
(3) *For any $a \in R$, there exists a positive integer n such that $(R/a^nR)^*$ is a finitely generated left R-module.*
(4) *Any direct product of P-flat right R-modules is* GP-*flat.*
(5) *Any direct product of copies of R_R is* GP-*flat.*
(6) *Any direct limit of (P-)injective left R-modules is* GP-*injective.*
(7) *For any projective left R-module M, M^* is* GP-*flat.*
(8) *For any FP-injective left R-module M and any $a \in R$, there exists a positive integer n such that* $\mathrm{Ext}^2(R/Ra^n, M) = 0$.

Proof. (1) $\Longleftrightarrow$ (2) is obvious by the exact sequence $0 \to l_R(a^n) \to R \to Ra^n \to 0$.

(2) $\Longleftrightarrow$ (3) follows from the standard isomorphism: $(R/a^nR)^* \cong l_R(a^n)$.

(1) $\Longrightarrow$ (4) Let $\{M_i\}$ be a family of P-flat right R-modules and $a \in R$. Then there exists a positive integer n such that Ra^n is finitely presented by (1). Consider the following commutative diagram:

$$\begin{array}{ccc} (\Pi M_i) \otimes Ra^n & \xrightarrow{\varphi} & (\Pi M_i) \otimes R \\ {\scriptstyle\alpha}\downarrow & & \downarrow \\ \Pi(M_i \otimes Ra^n) & \xrightarrow{\beta} & \Pi(M_i \otimes R). \end{array}$$

Since α is an isomorphism by [28, Lemma 13.2] and β is a monomorphism, φ is a monomorphism. So ΠM_i is GP-flat.

(4) $\Longrightarrow$ (5) is trivial.

(5) $\Longrightarrow$ (1) Let $a \in R$. By (5), there exists a positive integer n such that $(\Pi R_R) \otimes Ra^n \stackrel{\gamma}{\to} (\Pi R_R) \otimes R$ is a monomorphism. Consider the following commutative diagram:

$$\begin{array}{ccc} (\Pi R_R) \otimes Ra^n & \xrightarrow{\gamma} & (\Pi R_R) \otimes R \\ {\scriptstyle\delta}\downarrow & & \downarrow \\ \Pi(R \otimes Ra^n) & \longrightarrow & \Pi(R \otimes R). \end{array}$$

Then δ is a monomorphism. But δ is also an epimorphism by [28, Lemma 13.1]. So δ is an isomorphism. Thus Ra^n is finitely presented by [28, Lemma 13.2]. Hence R is a left GP-coherent ring.

(1) $\Longleftrightarrow$ (6) Let $a \in R$, n be a positive integer and $\{M_j : j \in J\}$ a family of (P-)injective left R-modules, where J is a directed set. Consider the following commutative diagram:

$$\begin{array}{ccc} \varinjlim \mathrm{Hom}(R, M_j) & \xrightarrow{\alpha} & \varinjlim \mathrm{Hom}(Ra^n, M_j) \\ \downarrow & & \downarrow{\scriptstyle\gamma} \\ \mathrm{Hom}(R, \varinjlim M_j) & \xrightarrow{\psi} & \mathrm{Hom}(Ra^n, \varinjlim M_j). \end{array}$$

Since α is an epimorphism, we have ψ is an epimorphism if and only if γ is an epimorphism, if and only if Ra^n is finitely presented by [16, Proposition 2.5]. Thus R is a left GP-coherent ring if and only if $\varinjlim M_j$ is GP-injective.

(5) $\Longrightarrow$ (7) For any projective left R-module M, there is a projective left R-module N such that $M \oplus N \cong \oplus_R R$. So we have

$$M^* \oplus N^* \cong (\oplus_R R)^* \cong \Pi R_R.$$

Thus M^* is GP-flat since ΠR_R is GP-flat by (5).

(7) $\Longrightarrow$ (5) is obvious by choosing M to be $\oplus_R R$.

(1) $\Longrightarrow$ (8) Let M be any FP-injective left R-module and $a \in R$. Then there exists a positive integer n such that Ra^n is finitely presented by (1), and so $\mathrm{Ext}^1(Ra^n, M) = 0$. The exact sequence $0 \to Ra^n \to R \to R/Ra^n \to 0$ yields the exact sequence

$$0 = \mathrm{Ext}^1(Ra^n, M) \to \mathrm{Ext}^2(R/Ra^n, M) \to \mathrm{Ext}^2(R, M) = 0.$$

Thus $\mathrm{Ext}^2(R/Ra^n, M) = 0$.

(8) $\Longrightarrow$ (1) Let M be any FP-injective left R-module and $a \in R$. Then there exists a positive integer n such that $\mathrm{Ext}^2(R/Ra^n, M) = 0$. The exact sequence $0 \to Ra^n \to R \to R/Ra^n \to 0$ induces the exact sequence

$$0 = \mathrm{Ext}^1(R, M) \to \mathrm{Ext}^1(Ra^n, M) \to \mathrm{Ext}^2(R/Ra^n, M) = 0.$$

Thus $\mathrm{Ext}^1(Ra^n, M) = 0$, and so Ra^n is finitely presented by [7]. Hence R is a left GP-coherent ring. □

4.5. Corollary. *If M^+ is* GP*-flat for any* GP*-injective left R-module M, then R is a left* GP*-coherent ring.*

Proof. Note that $\oplus(R_R)^+$ is GP-injective. So $\Pi(R_R)^{++} \cong (\oplus(R_R)^+)^+$ is GP-flat by hypothesis. Since ΠR_R is a pure submodule of $\Pi(R_R)^{++}$ by [4, Lemma 1 (2)], ΠR_R is GP-flat by Proposition 2.3 (5). So R is left GP-coherent by Theorem 4.4. □

Recall that R is a *reduced ring* if R has no non-zero nilpotent element. Let R be a reduced ring and $a, b \in R$. It is easy to see that $ab = 0$ if $ba = 0$.

4.6. Proposition. *Let R be a reduced ring. Then*

(1) *$l_R(a) = l_R(a^n)$ for any $a \in R$ and any positive integer n.*
(2) *Every* GP*-flat right R-module is P-flat.*
(3) *Every* GP*-injective left R-module is P-injective.*
(4) *R is left* GP*-coherent if and only if R is left P-coherent.*
(5) *R is left* GPP *if and only if R is left* GP*-coherent left* GPF*.*

Proof. (1) Let $x \in l(a^n)$. Then $xa^n = 0$.

If $n = 1$, we are done.

If $n > 1$, then $(xa)a^{n-1} = 0$, and so $a^{n-1}(xa) = 0$. Thus $(xa)(a^{n-2}xa) = 0$, and hence $a^{n-2}(xa)^2 = 0$. Inductively, $(xa)^n = 0$. Therefore $xa = 0$, and so $x \in l_R(a)$. Hence $l_R(a) = l_R(a^n)$.

(2) Let N be any GP-flat right R-module and $a \in R$. Then there exists a positive integer n such that $l_N(a) \subseteq l_N(a^n) = Nl_R(a^n) = Nl_R(a)$ by (1) and Theorem 2.2. So $l_N(a) = Nl_R(a)$. Thus N is P-flat by [22, Lemma 2].

(3) Let M be any GP-injective left R-module and $a \in R$. Then there exists a positive integer n such that $r_M l_R(a) = r_M l_R(a^n) = a^n M \subseteq aM$ by (1). So, $r_M l_R(a) = aM$. Hence M is P-injective.

(4) and (5) are clear by (1) and Theorem 4.4. □

We end this paper by raising two questions, which are motivated by Proposition 2.3 and Theorem 3.7.

4.7. Question. (1) Is the direct sum of any two GP-flat right R-modules (resp. GP-injective left R-modules) again GP-flat (resp. GP-injective)?

(2) In general, is R a π-regular ring if R satisfies any one of the following conditions: (i) Every cyclic right R-module is GP-flat. (ii) Every cyclic left R-module is GP-injective?

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