Hacettepe Journal of Mathematics and Statistics
Volume 40 (1) (2011), 77 – 89

APPROXIMATION BY GENUINE q-BERNSTEIN-DURRMEYER POLYNOMIALS IN COMPACT DISKS

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Received 08 : 06 : 2010 : Accepted 20 : 07 : 2010

Abstract

In this paper, the order of simultaneous approximation and Voronovskaja type theorems with quantitative estimate for complex genuine q-Bernstein-Durrmeyer polynomials $(0 < q < 1)$ attached to analytic functions on compact disks are obtained. Our results show that extension of the complex genuine q-Bernstein-Durrmeyer polynomials from real intervals to compact disks in the complex plane extends approximation properties (with quantitative estimates).

Keywords: Complex genuine q-Bernstein–Durrmeyer operators, Voronovskaja's theorem, Approximation in compact disks.

2000 AMS Classification: 30 E 10, 41 A 25.

1. Introduction

Recent studies demonstrate that the theory of q-calculus plays an important role in analytic number theory and theoretical physics. For example, various applications of this theory have appeared in the study of hypergeometric series [3], in approximation theory [24], while other important applications have been related with quantum theory [20]. In this paper, with the help of techniques from the q-calculus, we study the approximation properties of a general family of genuine q-Bernstein–Durrmeyer operators, which are well-known positive linear operators in approximation theory.

Genuine Bernstein–Durrmeyer operators were first considered by W. Chen [4] and T. N. T. Goodman and A. Sharma [16] around 1987. In recent years, genuine Bernstein–Durrmeyer operators have been investigated intensively by a number of authors. Among the many articles written on the U_n, we mention here only the ones by H. Gonska *et al.* [15], by P. Parvanov and B. Popov [23], by T. Sauer [25], by S. Waldron [26], and the

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book of R. Páltánea [22]. Recently, the genuine q-Bernstein–Durrmeyer operators was introduced and studied in [18].

Approximation properties of complex Bernstein polynomials in compact disks were studied by many authors, see [17], [9] and references therein. In [9] a Voronovskaja-type result with a quantitative estimate for complex Bernstein polynomials in compact disks was obtained. In [14] similar results for Bernstein-Stancu and Kantorovich-Stancu polynomials were obtained, while in [1] similar results for Bernstein-Schurer polynomials were proved. Very recently, G. A. Anastassiou and S. G. Gal [2] studied the order of simultaneous approximation and Voronovskaja kind results with quantitative estimate for complex Bernstein-Durrmeyer polynomials attached to analytic functions on compact disks. The results of paper [2] put in evidence the over-convergence phenomenon for Bernstein-Durrmeyer polynomials, namely the extensions of approximation properties (with quantitative estimates) from real intervals to compact disks in the complex plane. On the other hand complex Bernstein type operators based on q-integers were studied by S. Ostrovska [21], S. G. Gal [9] and N. I. Mahmudov [19].

The goal of this paper is to obtain approximation results for complex genuine q-Bernstein-Durrmeyer polynomials on compact disks in the case $0 < q < 1$. First we present upper estimates in approximation and we prove a Voronovskaja type convergence theorem in compact disks in $\mathbb{C}$, centered at the origin, with quantitative estimate of this convergence. These results allow us to obtain the exact degrees in simultaneous approximation by complex genuine q-Bernstein-Durrmeyer polynomials. Our results show that the extension of the complex genuine q-Bernstein-Durrmeyer polynomials from real intervals to compact disks in the complex plane extends approximation properties (with quantitative estimates).

2. Auxiliary results

Let $q > 0$. For any $n \in \mathbb{N} \cup \{0\}$, the *$q$-integer* $[n]_q$ is defined by

$$[n]_q := 1 + q + \cdots + q^{n-1}, \; [0]_q := 0;$$

and the *q-factorial* $[n]_q!$ by

$$[n]_q! := [1]_q [2]_q \cdots [n]_q, \; [0]_q! := 1.$$

For integers $0 \le k \le n$, the *q-binomial* is defined by

$$\left[\begin{array}{c} n \\ k \end{array} \right]_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

For $q = 1$ we obviously get $[n]_q = n$, $[n]_q! = n!$, $\left[\begin{array}{c} n \\ k \end{array} \right]_q = \binom{n}{k}$. Moreover,

$$(1-z)_q^n := \prod_{s=0}^{n-1} \left(1 - q^s z\right), \; p_{n,k}(q;z) := \left[\begin{array}{c} n \\ k \end{array} \right]_q z^k (1-z)_q^{n-k}, \; z \in \mathbb{C}.$$

For fixed $q > 0, q \neq 1$, we define the q-derivative $D_q f(z)$ of f by

$$D_q f(z) = \begin{cases} \dfrac{f(qz) - f(z)}{(q-1)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

The q-analogue of integration in the interval $[0, A]$ (see [3]) is defined by

$$\int_0^A f(t) d_q t := A(1-q) \sum_{n=0}^{\infty} f\left(Aq^n\right) q^n, \; 0 < q < 1.$$

Let $\mathbb{D}_R$ be a disc $\mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$ in the complex plane $\mathbb{C}$. Denote by $H(\mathbb{D}_R)$ the space of all analytic functions on $\mathbb{D}_R$. For $f \in H(\mathbb{D}_R)$ it follows that $f(z) = \sum_{m=0}^{\infty} a_m z^m$.

2.1. Definition. Let $0 < q \leq 1$. For $f : [0,1] \to \mathbb{C}$, we define the following genuine q-Bernstein-Durrmeyer operator

$$\begin{aligned} U_{n,q}(f;z) &:= f(0)p_{n,0}(q;z) + f(1)p_{n,n}(q;z) \\ &+[n-1]\sum_{k=1}^{n-1} q^{1-k} p_{n,k}(q;z) \int_0^1 p_{n-2,k-1}(q;qt) f(t)\, d_q t, \end{aligned} \tag{2.1}$$

where for $n = 1$ the sum is empty, i.e., equal to 0.

Here $U_{n,q}(f;z)$ are linear operators reproducing linear functions, and thus interpolating every function $f \in C[0,1]$ at 0 and 1. For $q = 1$ we recapture the genuine complex Bernstein-Durrmeyer polynomials.

2.2. Lemma. *$U_{n,q}(t^m;z)$ is a polynomial of degree less than or equal to $\min(m,n)$ and*

$$U_{n,q}(e_m;z) = \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^{m} c_s(m)[n]_q^s B_{n,q}(e_s;z). \qquad \square$$

Also, the following lemma holds.

2.3. Lemma. *For all $m, n \in \mathbb{N}$ the identity*

$$\frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^{m} c_s(m)[n]_q^s = 1,$$

holds.

Proof. This follows from the interpolation properties of $U_{n,q}(e_m;z)$ and $B_{n,q}(e_s;z)$ at the end points. Indeed,

$$\begin{aligned} 1 &= U_{n,q}(e_m;1) \\ &= \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^{m} c_s(m)[n]_q^s B_{n,q}(e_s;1) \\ &= \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^{m} c_s(m)[n]_q^s. \end{aligned} \qquad \square$$

Lemma 2.3 implies that for all $m, n \in \mathbb{N}$ and $|z| \leq r$ we have

$$\begin{aligned} |U_{n,q}(e_m;z)| &\leq \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^{m} c_s(m)[n]_q^s |B_{n,q}(e_s;z)| \\ &\leq \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^{m} c_s(m)[n]_q^s r^s \\ &\leq r^m. \end{aligned} \tag{2.2}$$

Notice that if $P_m(z)$ is a polynomial of degree m, then by the Bernstein inequality we have

$$|D_q P_m(z)| \leq \left\| P_m' \right\|_r \leq \frac{m}{r} |P_m(z)|. \tag{2.3}$$

For our purpose first we need a recurrence formula for $U_{n,q}(e_m;z)$.

2.4. Lemma. *For all $m,n \in \mathbb{N}\cup\{0\}$ and $z\in\mathbb{C}$ we have*

$$U_{n,q}\left(e_{m+1};z\right)=\frac{q^{m}z\left(1-z\right)}{\left[n+m\right]_{q}}D_{q}U_{n,q}\left(e_{m};z\right)+\frac{q^{m}\left[n\right]z+\left[m\right]_{q}}{\left[n+m\right]_{q}}U_{n,q}\left(e_{m};z\right). \tag{2.4}$$

Proof. By a simple calculation we obtain

$$\begin{aligned}z\left(1-z\right)D_{q}\left(p_{n,k}\left(q;z\right)\right)&=\left(\left[k\right]_{q}-\left[n\right]_{q}z\right)p_{n,k}\left(q;z\right),\\ t\left(1-qt\right)D_{q}\left(p_{n,k}\left(q;qt\right)\right)&=p_{n,k}\left(q;qt\right)\left(\left[k\right]-\left[n\right]qt\right).\end{aligned}$$

It follows that

$$\begin{aligned}&z\left(1-z\right)D_{q}U_{n,q}\left(e_{m};z\right)\\
&=\left[n\right]_{q}z\left(1-z\right)z^{n-1}+\left[n-1\right]_{q}\sum_{k=1}^{n-1}q^{1-k}\left(\left[k\right]_{q}-\left[n\right]_{q}z\right)p_{n,k}\left(q;z\right)\\
&\qquad\times\int_{0}^{1}p_{n-2,k-1}\left(q;qt\right)t^{m}d_{q}t\\
&=\left[n\right]_{q}z^{n}+\left[n-1\right]_{q}\sum_{k=1}^{n-1}q^{1-k}p_{n,k}\left(q;z\right)\\
&\qquad\times\int_{0}^{1}\left(q\left[k-1\right]_{q}+1\right)p_{n-2,k-1}\left(q;qt\right)t^{m}d_{q}t-z\left[n\right]_{q}U_{n,q}\left(e_{m};z\right)\\
&=\left[n\right]_{q}z^{n}-z^{n}+\left[n-1\right]_{q}\sum_{k=1}^{n-1}q^{1-k}p_{n,k}\left(q;z\right)q\\
&\qquad\times\int_{0}^{1}\left(\left[k-1\right]_{q}-\left[n-2\right]_{q}qt+\left[n-2\right]_{q}qt\right)p_{n-2,k-1}\left(q;qt\right)t^{m}\,d_{q}t\\
&\qquad-z\left[n\right]_{q}U_{n,q}\left(e_{m};z\right)+U_{n,q}\left(e_{m};z\right)\\
&=\left[n\right]_{q}z^{n}-z^{n}+\left[n-1\right]_{q}\sum_{k=1}^{n-1}q^{1-k}p_{n,k}\left(q;z\right)q\\
&\qquad\times\int_{0}^{1}\left(\left[k-1\right]_{q}-\left[n-2\right]_{q}qt\right)p_{n-2,k-1}\left(q;qt\right)t^{m}d_{q}t\\
&\qquad+\left[n-1\right]_{q}\left[n-2\right]_{q}q^{2}\sum_{k=1}^{n-1}q^{1-k}p_{n,k}\left(q;z\right)\\
&\qquad\times\int_{0}^{1}p_{n-2,k-1}\left(q;qt\right)t^{m+1}d_{q}t-z\left[n\right]_{q}U_{n,q}\left(e_{m};z\right)+U_{n,q}\left(e_{m};z\right)\\
&=\left[n\right]_{q}z^{n}-z^{n}-\left[n-2\right]_{q}q^{2}z^{n}+\left[n-1\right]_{q}\sum_{k=1}^{n-1}q^{1-k}p_{n,k}\left(q;z\right)\\
&\qquad\times\int_{0}^{1}\left(D_{q}p_{n-2,k-1}\left(q;qt\right)\right)qt\left(1-qt\right)t^{m}d_{q}t\\
&\qquad+\left[n-2\right]_{q}q^{2}U_{n,q}\left(e_{m+1};z\right)-z\left[n\right]_{q}U_{n,q}\left(e_{m};z\right)+U_{n,q}\left(e_{m};z\right).\end{aligned}\tag{2.5}$$

We integrate by parts, setting $\delta(t) = t(1-t)\left(\frac{t}{q}\right)^m$. The integral in the above formula becomes

$$\begin{aligned}
&\int_0^1 D_q p_{n-2,k-1}(q;qt)\, qt(1-qt)\, t^m d_q t \\
&= \delta(t)\, p_{n-2,k-1}(q;qt)\,|_0^1 - \int_0^1 p_{n-2,k-1}(q;qt)\, D_q\delta(t)\, d_q t \\
&= -q^{-m}\int_0^1 p_{n-2,k-1}(q;qt)\, D_q\left(t^{m+1} - t^{m+2}\right) d_q t \\
&= -q^{-m}[m+1]_q \int_0^1 p_{n-2,k-1}(q;qt)\, t^m d_q t \\
&\quad + q^{-m}[m+2]_q \int_0^1 p_{n-2,k-1}(q;qt)\, t^{m+1} d_q t
\end{aligned} \tag{2.6}$$

Substituting (2.6) in (2.5) we get

$$\begin{aligned}
&z(1-z)\, D_q U_{n,q}(e_m;z) \\
&= [n]_q z^n - z^n - [n-2]_q q^2 z^n + q^{-m}[m+1]_q z^n - q^{-m}[m+2]_q z^n \\
&\quad - q^{-m}[m+1]_q U_{n,q}(e_m;z) + q^{-m}[m+2]_q U_{n,q}(e_{m+1};z) \\
&\quad + [n-2]_q q^2 U_{n,q}(e_{m+1};z) - z[n]_q U_{n,q}(e_m;z) + U_{n,q}(e_m;z) \\
&= -q^{-m}[m+1]_q U_{n,q}(e_m;z) + q^{-m}[m+2]_q U_{n,q}(e_{m+1};z) \\
&\quad + [n-2]_q q^2 U_{n,q}(e_{m+1};z) - z[n]_q U_{n,q}(e_m;z) + U_{n,q}(e_m;z)
\end{aligned}$$

which implies the recurrence in the statement. □

Let

$$\Upsilon_{n,m}(q;z) := U_{n,q}(e_m;z) - z^m,$$

$$\Theta_{n,m}(q;z) := U_{n,q}(e_m;z) - z^m - \frac{m(m-1)}{[n+1]_q} z^{m-1}(1-z).$$

Using the recurrence formula (2.4) we prove two more recurrences.

2.5. Lemma. *For all $m,n \in N \cup \{0\}$ and $z \in \mathbb{C}$ we have*

$$\begin{aligned}
\Upsilon_{n,m}(q;z) &= \frac{q^{m-1} z(1-z)}{[n+m-1]_q} D_q U_{n,q}(e_{m-1};z) \\
&\quad + \frac{q^{m-1}[n]z + [m-1]_q}{[n+m-1]_q}\Upsilon_{n,m-1}(q;z) + \frac{[m-1]_q}{[n+m-1]_q}(1-z) z^{m-1},
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
\Theta_{n,m}(q;z) &= \frac{q^{m-1} z(1-z)}{[n+m-1]_q} D_q \Upsilon_{n,m}(q;z) \\
&\quad + \frac{q^{m-1}[n]z + [m-1]_q}{[n+m-1]_q}\Theta_{n,m-1}(q;z) + R_{n,m}(q;z),
\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}
R_{n,m}(q;z) &= \frac{(q^n - 1)\left(q[m-1]_q^2 + 2\left([m-1]_q + \cdots + 1\right)\right) - m(m-1)[m-1]_q}{[n+m-1]_q [n+1]_q} \\
&\quad \times z^{m-1}(1-z) + \frac{[m-1]_q}{[n+m-1]_q}\frac{(m-1)(m-2)}{[n+1]_q} z^{m-2}(1-z).
\end{aligned} \tag{2.9}$$

Proof. By simple calculation we obtain

$$
\begin{aligned}
&U_{n,q}(e_m;z) - z^m - \frac{m(m-1)}{[n+1]_q} z^{m-1}(1-z) \\
&\quad = \frac{q^{m-1} z(1-z)}{[n+m-1]_q} D_q\left(U_{n,q}(e_m;z) - z^{m-1}\right) + \frac{q^{m-1}[n]z + [m-1]_q}{[n+m-1]_q} \\
&\qquad \times \left(U_{n,q}(e_m;z) - z^{m-1} - \frac{(m-1)(m-2)}{[n+1]_q} z^{m-2}(1-z)\right) \\
&\qquad + R_{n,m}(q;z),
\end{aligned}
$$

where

$$
\begin{aligned}
R_{n,m}(q;z) &= \frac{[m-1]_q}{[n+m-1]_q}(1-z) z^{m-1} - \frac{m(m-1)}{[n+1]_q} z^{m-1}(1-z) \\
&\quad + \frac{q^{m-1}[m-1]_q}{[n+m-1]_q}(1-z) z^{m-1} + \frac{q^{m-1}[n]z + [m-1]_q}{[n+m-1]_q} \\
&\quad \times \frac{(m-1)(m-2)}{[n+1]_q} z^{m-2}(1-z) \\
&:= T_{n,m}(q;z)(1-z) z^{m-1} \\
&\quad + \frac{[m-1]_q}{[n+m-1]_q} \frac{(m-1)(m-2)}{[n+1]_q} z^{m-2}(1-z)
\end{aligned}
$$

Again, by a simple but tedious calculation, we obtain

$$
\begin{aligned}
&T_{n,m}(q;z) \\
&= \frac{[m-1]_q}{[n+m-1]_q} - \frac{m(m-1)}{[n+1]_q} + \frac{q^{m-1}[m-1]_q}{[n+m-1]_q} + \frac{q^{m-1}[n]_q}{[n+m-1]_q} \frac{(m-1)(m-2)}{[n+1]_q} \\
&= \frac{[m-1]_q [n+1]_q - m(m-1)[n+m-1]_q}{[n+m-1]_q [n+1]_q} \\
&\quad + \frac{q^{m-1}[m-1]_q [n+1]_q + q^{m-1}[n]_q (m-1)(m-2)}{[n+m-1]_q [n+1]_q} \\
&= \frac{[m-1]_q [n+1]_q - m(m-1)\left([m-1]_q + q^{m-1}[n]_q\right)}{[n+m-1]_q [n+1]_q} \\
&\quad + \frac{q^{m-1}[m-1]_q [n+1]_q + q^{m-1}[n]_q (m-1) m - 2q^{m-1}[n]_q (m-1)}{[n+m-1]_q [n+1]_q} \\
&= \frac{[m-1]_q [n+1]_q - m(m-1)[m-1]_q}{[n+m-1]_q [n+1]_q} \\
&\quad + \frac{q^{m-1}[m-1]_q [n+1]_q - 2q^{m-1}[n]_q (m-1)}{[n+m-1]_q [n+1]_q} \\
&= \frac{(q^n - 1)\left(q[m-1]_q^2 + 2\left([m-1]_q + \cdots + 1\right)\right) - m(m-1)[m-1]_q}{[n+m-1]_q [n+1]_q}. \qquad \square
\end{aligned}
$$

3. Approximation by complex genuine q-Bernstein-Durrmeyer polynomials

The first main result of the paper is expressed by the following upper estimates.

3.1. Theorem. *Let $1 \leq r < R$. Then for all $|z| \leq r$ we have*

$$|U_{n,q}(f;z) - f(z)| \leq \frac{r(1+r)}{[n+1]_q} \sum_{m=2}^{\infty} |a_m| \, m(m-1) \, r^{m-2}.$$

Proof. From the recurrence formula (2.7) and the inequality (2.2) for $m \geq 2$ we get

$$\begin{aligned}|\Upsilon_{n,m}(q;z)| &\leq \frac{r(1+r)}{[n+1]_q} \frac{m-1}{r} \|U_{n,q}(e_{m-1})\|_r \\ &\qquad + r|\Upsilon_{n,m-1}(q;z)| + \frac{[m-1]_q}{[n+1]_q} r^{m-1}(1+r) \\ &\leq \frac{(1+r)(m-1)}{[n+1]_q} r^{m-1} + r|\Upsilon_{n,m-1}(q;z)| + \frac{[m-1]_q}{[n+1]_q} r^{m-1}(1+r) \\ &\leq 2(m-1)\frac{r(1+r)}{[n+1]_q} r^{m-2} + r|\Upsilon_{n,m-1}(q;z)|.\end{aligned}$$

By writing the last inequality for $m = 2, \ldots,$ we easily obtain, step by step the following

$$\begin{aligned}|\Upsilon_{n,m}(q;z)| &\leq r\Big(r|\Upsilon_{n,m-2}(q;z)| + \frac{2(m-2)}{[n+1]_q} r(1+r) r^{m-3}\Big) \\ &\qquad + 2(m-1)\frac{r(1+r)}{[n+1]_q} r^{m-2} \\ &= r^2|\Upsilon_{n,m-2}(q;z)| + 2\frac{r(1+r)}{[n+1]_q} r^{m-2}(m-1+m-2) \\ &\leq \cdots \leq \frac{r(1+r)}{[n+1]_q} m(m-1) r^{m-2}.\end{aligned}$$

It follows that

$$\begin{aligned}|U_{n,q}(f;z) - f(z)| &\leq \sum_{m=2}^{\infty} |a_m| \, |\Upsilon_{n,m}(q;z)| \\ &\leq \frac{r(1+r)}{[n+1]_q} \sum_{m=2}^{\infty} |a_m| \, m(m-1) \, r^{m-2}. \quad \square\end{aligned}$$

The second main result of the paper is the Voronovskaja theorem with a quantitative estimate for the complex version of genuine q-Bernstein-Durrmeyer polynomials.

3.2. Theorem. *Let $R > 1$, $f \in H(\mathbb{D}_R)$.*

(i) *For any $r \in [1, R)$ we have*

$$\left| U_{n,q}(f;z) - f(z) - \frac{z(1-z)}{[n+1]_q} f''(z) \right| \leq \frac{6r^2(1+r)^2}{[n+1]_q^2} \sum_{m=3}^{\infty} |a_m| \, m(m-1)^3 \, r^{m-4}$$

for all $|z| \leq r$ and $n \in \mathbb{N}$.

(ii) *The following Voronovskaja-type result in the closed unit disk holds*

$$\left| U_{n,q}(f;z) - f(z) - \frac{z(1-z)}{[n+1]_q} f''(z) \right| \leq \frac{8|z(1-z)|}{[n+1]_q^2} \sum_{m=3}^{\infty} |a_m| \, m(m-1)^3$$

for all $|z| \leq 1$ and $n \in \mathbb{N}$.

Proof. (i) The formula (2.8) implies, for $m \geq 2$,

$$\begin{aligned}\left|\Theta_{n,m}(q;z)\right| &\leq \frac{r(1+r)}{[n+m-1]_q}\left|\Upsilon'_{n,m-1}(q;z)\right| \\ &\quad + \frac{q^{m-1}[n]_q r + [m-1]_q}{[n+m-1]_q}\left|\Theta_{n,m-1}(q;z)\right| + \left|R_{n,m}(q;z)\right| \\ &\leq \frac{r(1+r)}{[n+1]_q}\left|\Upsilon'_{n,m-1}(q;z)\right| + r\left|\Theta_{n,m-1}(q;z)\right| + \left|R_{n,m}(q;z)\right|.\end{aligned}$$

Now we will estimate $\left|\Upsilon'_{n,m-1}(q;z)\right|$ for $m \geq 2$. Taking into account that $\Upsilon_{n,m-1}(z)$ is a polynomial of degree $\leq (m-1)$, we obtain

$$\begin{aligned}\left|\Upsilon'_{n,m-1}(q;z)\right| &\leq \frac{m-1}{r}\left\|\Upsilon_{n,m-1}(q;\cdot)\right\|_r \\ &\leq \frac{m-1}{r}\frac{r(1+r)}{[n+1]_q}(m-1)(m-2)r^{m-3} \\ &= \frac{2(m-1)^2(m-2)}{[n+1]_q}(1+r)r^{m-3}.\end{aligned}$$

This implies that

$$\frac{r(1+r)}{[n+1]_q}\left|\Upsilon'_{n,m-1}(q;z)\right| \leq \frac{2m(m-1)(m-2)}{[n+1]_q^2}(1+r)^2 r^{m-2}.$$

On the other hand

$$\begin{aligned}\left|R_{n,m}(q;z)\right| &\leq \frac{r^{m-2}(1+r)}{[n+m-1]_q[n+1]_q}\Big([m-1]_q(m-1)(m-2) \\ &\quad + 3(1-q^n)[m-1]_q(m-1)r + m(m-1)[m-1]_q r\Big) \\ &\leq \frac{4r^{m-2}(1+r)^2}{[n+1]_q^2}m(m-1)[m-1]_q\end{aligned}$$

Finally we obtain

$$\left|\Theta_{n,m}(q;z)\right| \leq r\left|\Theta_{n,m-1}(q;z)\right| + \frac{6(1+r)^2 r^{m-2}}{[n+1]_q^2}m(m-1)^2$$

for all $m \geq 1$, $n \in N$ and $|z| \leq r$. But $\Theta_{n,0}(q;z) = \Theta_{n,1}(q;z) = 0$, for any $z \in \mathbb{C}$, and therefore by writing the last inequality for $m = 1, 2, \ldots$, we easily obtain, step by step the following:

$$\left|\Theta_{n,m}(q;z)\right| \leq \frac{6(1+r)^2 r^{m-2}}{[n+1]_q^2}\sum_{j=2}^{m} j(j-1)^2 \leq \frac{6(1+r)^2 r^{m-2}}{[n+1]_q^2}m(m-1)^3.$$

As a conclusion, we obtain

$$\begin{aligned}\left|U_{n,q}(f;z) - f(z) - \frac{z(1-z)}{[n+1]_q}f''(z)\right| &\leq \sum_{m=3}^{\infty}|a_m|\left|\Theta_{n,m}(q;z)\right| \\ &\leq \frac{6r^2(1+r)^2}{[n+1]_q^2}\sum_{m=3}^{\infty}|a_m|m(m-1)^3 r^{m-4}.\end{aligned}$$

Note that since $f^{iv}(z) = \sum_{m=4}^{\infty} a_m m(m-1)(m-2)(m-3)z^{m-4}$ and the series is absolutely convergent in $|z| \leq r$, this implies that $\sum_{m=4}^{\infty} a_m m(m-1)^3 r^{m-4} < \infty$, and proves part (i) of the theorem.

(ii) The formula (2.8) implies

$$|\Theta_{n,m}(q;z)| \leq \frac{|z(1-z)|}{[n+m-1]_q}\left|\Upsilon'_{n,m-1}(q;z)\right| + |\Theta_{n,m-1}(q;z)| + |R_{n,m}(q;z)|$$
$$\leq \frac{|z(1-z)|}{[n+1]_q}\left|\Upsilon'_{n,m-1}(q;z)\right| + |\Theta_{n,m-1}(q;z)| + |R_{n,m}(q;z)|.$$

Now we will estimate $\left|\Upsilon'_{n,m-1}(q;z)\right|$ for $m \geq 2$. Taking into account that $\Upsilon_{n,m-1}(q;z)$ is a polynomial of degree $\leq (m-1)$, we obtain

$$\left|\Upsilon'_{n,m-1}(q;z)\right| \leq (m-1)\left\|\Upsilon_{n,m-1}(q;\cdot)\right\|_1$$
$$\leq (m-1)\frac{2}{[n+1]_q}(m-1)(m-2)$$
$$\leq \frac{2m(m-1)(m-2)}{[n+1]_q}.$$

On the other hand

$$|R_{n,m}(q;z)| \leq \frac{4|z(1-z)|}{[n+m-1]_q[n+1]_q}m(m-1)[m-1]_q$$
$$\leq \frac{4|z(1-z)|}{[n+1]_q^2}m(m-1)^2.$$

Finally we obtain

$$|\Theta_{n,m}(q;z)| \leq |\Theta_{n,m-1}(q;z)| + \frac{8|z(1-z)|}{[n+1]_q^2}m(m-1)^2$$

for all $m \geq 2$, $n \in N$ and $|z| \leq 1$. But $\Theta_{n,0}(q;z) = \Theta_{n,1}(q;z) = 0$, for any $z \in \mathbb{C}$, and therefore by writing the last inequality for $m = 2, 3, 4, \ldots$, we easily obtain, step by step the following

$$|\Theta_{n,m}(q;z)| \leq \frac{8|z(1-z)|}{[n+1]_q^2}\sum_{j=2}^{m} j(j-1)^2 \leq \frac{8|z(1-z)|}{[n+1]_q^2}m(m-1)^3.$$

As a conclusion, we obtain

$$\left|U_{n,q}(f;z) - f(z) - \frac{z(1-z)}{[n+1]_q}f''(z)\right| \leq \sum_{m=3}^{\infty}|a_m|\,|\Theta_{n,m}(q;z)|$$
$$\leq \frac{8|z(1-z)|}{[n+1]_q^2}\sum_{m=3}^{\infty}|a_m|\,m(m-1)^3.$$

Note that since $f^{iv}(z) = \sum_{m=4}^{\infty} a_m m(m-1)(m-2)(m-3)z^{m-4}$, and the series is absolutely convergent in $|z| \leq r$, it easily follows that $\sum_{m=4}^{\infty}|a_m|\,m(m-1)(m-2)^2 < \infty$. □

Finally we will obtain the exact order of approximation by complex genuine Bernstein-Durrmeyer polynomials and their derivatives. In this sense we present the following results.

3.3. Theorem. *Let $0 < q_n \leq 1$ be with $\lim_{n\to\infty} q_n = 1$, $R > 1$, $f \in H(\mathbb{D}_R)$. If f is not a polynomial of degree ≤ 1, then for any $r \in [1, R)$ we have*

$$\|U_{n,q_n}(f) - f\|_r \geq \frac{1}{[n+1]_{q_n}}C_r(f), \; n \in \mathbb{N},$$

holds, where the constant $C_r(f) > 0$ depends on f, r and on the sequence $\{q_n\}_{n\in\mathbb{N}}$ but is independent of n.

Proof. For all $z \in \mathbb{D}_R$ and $n \in N$ we get

$$U_{n,q_n}(f;z) - f(z) = \frac{1}{[n+1]_{q_n}} \Big\{ z(1-z) f''(z) + [n+1]_{q_n} \times \Big(U_{n,q_n}(f;z) - f(z) - \frac{z(1-z)}{[n+1]_{q_n}} f''(z) \Big) \Big\}.$$

It follows that

$$\|U_{n,q_n}(f) - f\|_r \geq \frac{1}{[n+1]_{q_n}} \Big\{ \|e_1(1-e_1) f''\|_r - [n+1]_{q_n} \Big\| U_{n,q_n}(f) - f - \frac{e_1(1-e_1)}{[n+1]_{q_n}} f'' \Big\|_r \Big\}.$$

Since by hypothesis f is not a polynomial of degree ≤ 1 in $\mathbb{D}_R$ we have $\|e_1(1-e_1) f''\|_r > 0$. Indeed, assuming the contrary it follows that $z(1-z) f''(z) = 0$ for all $z \in \overline{\mathbb{D}}_r$ that is $f''(z) = 0$ for all $z \in \overline{\mathbb{D}}_r$. Thus, f is linear, which contradicts the hypothesis.

Now, by Theorem 3.2 we have

$$[n+1]_{q_n} \left| U_{n,q_n}(f;z) - f(z) - \frac{z(1-z)}{[n+1]_{q_n}} f''(z) \right| \leq \frac{6r^2(1+r)^2}{[n+1]_{q_n}} \sum_{m=3}^{\infty} |a_m| m(m-1)^3 r^{m-4} \to 0 \text{ as } n \to \infty.$$

Consequently, there exists n_1 (depending only on f and r) such that for all $n \geq n_1$ we have

$$\|e_1(1-e_1) f''\|_r - [n+1]_{q_n} \Big\| U_{n,q_n}(f) - f - \frac{e_1(1-e_1)}{[n+1]_{q_n}} f'' \Big\|_r \geq \frac{1}{2} \|e_1(1-e_1) f''\|_r,$$

which implies

$$\|U_{n,q_n}(f) - f\|_r \geq \frac{1}{2[n+1]_{q_n}} \|e_1(1-e_1) f''\|_r, \quad \text{for all } n \geq n_1.$$

For $1 \leq n \leq n_1 - 1$ we have

$$\|U_{n,q_n}(f) - f\|_r \geq \frac{1}{[n+1]_{q_n}} \Big([n+1]_{q_n} \|U_{n,q_n}(f) - f\|_r \Big) = \frac{1}{[n+1]_{q_n}} M_{r,n,q_n}(f) > 0,$$

which finally implies that

$$\|U_{n,q_n}(f) - f\|_r \geq \frac{1}{[n+1]_{q_n}} C_{r,q_n}(f),$$

for all n, with $C_{r,q_n}(f) = \min\big\{ M_{r,1,q_n}(f), \ldots, M_{r,n_1-1,q_n}(f), \frac{1}{2}\|e_1(1-e_1) f''\|_r \big\}$, which ends the proof. □

3.4. Corollary. *Let* $0 < q_n \leq 1$ *be with* $\lim_{n\to\infty} q_n = 1$, $R > 1$, $f \in H(\mathbb{D}_R)$. *If* f *is not a polynomial of degree* ≤ 1, *then for any* $r \in [1, R)$ *we have*

$$\|U_{n,qn}(f) - f\|_r \asymp \frac{1}{[n+1]_{q_n}}, \ n \in N,$$

holds, where the constants in the equivalence $\asymp$ *depends on* f, r *and on the sequence* $\{q_n\}_{n\in\mathbb{N}}$, *but is independent of* n. □

Notice that we have the following saturation.

3.5. Corollary. *If $f \in H(\mathbb{D}_R)$, $R > 1$, then $|U_{n,q_n}(f;z) - f(z)| = o\left(1/[n]_{q_n}\right)$ for an infinite number of points having an accumulation point on the disc $\mathbb{D}_R$ if and only if f is linear.*

Proof. By Theorem 3.2, we get $\lim_{n\to\infty} [n]_{q_n} (U_{n,q_n}(f;z) - f(z)) = z(1-z) f''(z) = 0$ for an infinite number of points having an accumulation point on $\mathbb{D}_R$. Since $z(1-z) f''(z) \in H(\mathbb{D}_R)$, by the Unicity Theorem for analytic functions we get

$$z(1-z) f''(z) = z(1-z) \sum_{m=2}^{\infty} a_m m(m-1) z^{m-2} = 0,$$

and therefore, $a_m = 0$, $m = 2, 3, \ldots$ Thus, f is linear and the corollary is proved. □

For the derivatives of complex genuine Bernstein-Durrmeyer polynomials we can state the following result.

3.6. Theorem. *Let $0 < q_n \le 1$ satisfy $\lim_{n\to\infty} q_n = 1$, $f \in H(\mathbb{D}_R)$, $R > 1$ and $1 \le r < r_1 < R$. If f is not a polynomial of degree $\le \max\{1, p-1\}$, then we have*

$$\left\| U^{(p)}_{n,q_n}(f) - f^{(p)} \right\|_r \asymp \frac{1}{[n+1]_{q_n}},$$

where the constant in the equivalence $\asymp$ depends only on f, r, r_1, p and on the sequence $\{q_n\}_{n\in\mathbb{N}}$.

Proof. Let Γ be a circle of radius $r_1 > r \ge 1$ and center 0. We have

$$\begin{aligned}
&U^{(p)}_{n,q_n}(f;z) - f^{(p)}(z) \\
&= \frac{1}{[n+1]_{q_n}} \Bigg\{ \left[z(1-z) f''(z)\right]^{(p)} + [n+1]_{q_n} \\
&\qquad\qquad \times \left(U_{n,q_n}(f;z) - f(z) - \frac{z(1-z)}{[n+1]_{q_n}} f''(z) \right)^{(p)} \Bigg\} \\
&= \frac{1}{[n+1]_{q_n}} \Bigg\{ \left[z(1-z) f''(z)\right]^{(p)} \\
&\qquad\qquad + n \frac{p!}{2\pi i} \int_\Gamma \frac{U_{n,q_n}(f;v) - f(v) - \frac{v(1-v)}{[n+1]_{q_n}} f''(v)}{(v-z)^{p+1}} dv \Bigg\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left\| U^{(p)}_{n,q_n}(f;z) - f^{(p)}(z) \right\|_r \\
&\quad \ge \frac{1}{[n+1]_{q_n}} \Bigg\{ \left\| \left[e_1(1-e_1) f''(z)\right]^{(p)} \right\|_r \\
&\qquad\qquad + [n+1]_{q_n} \left\| \frac{p!}{2\pi i} \int_\Gamma \frac{U_{n,q_n}(f;v) - f(v) - \frac{v(1-v)}{[n+1]_{q_n}} f''(v)}{(v-z)^{p+1}} dv \right\|_r \Bigg\}
\end{aligned}$$

where by using Theorem 3.2 we get

$$[n+1]_{q_n}\left\|\frac{p!}{2\pi i}\int_{\Gamma}\frac{U_{n,q_n}(f;v)-f(v)-\frac{v(1-v)}{[n+1]_{q_n}}f''(v)}{(v-z)^{p+1}}dv\right\|_r$$

$$\leq [n+1]_{q_n}\frac{p!}{2\pi i}\frac{2\pi r_1}{(r_1-r)^{p+1}}\left\|U_{n,q_n}(f)-f-\frac{e_1(1-e_1)}{[n+1]_{q_n}}f''(v)\right\|_{r_1}$$

$$\leq \frac{M_{r_1}(f)\,p!r_1}{[n+1]_{q_n}(r_1-r)^{p+1}}.$$

But by the hypothesis on f we have

$$\left\|\left[e_1(1-e_1)f''(z)\right]^{(p)}\right\|_r>0.$$

Indeed, assuming the contrary it follows that $z(1-z)f''(z)$ is a polynomial of degree $\leq p-1$. Now, if $p=1$ and $p=2$ then the analyticity of f obviously implies that f necessarily is a polynomial of degree $\leq 1=\max(1,p-1)$, which contradicts the hypothesis. If $p>2$ then the analyticity of f obviously implies that f necessarily is a polynomial of degree $\leq p-1=\max(1,p-1)$, which again contradicts the hypothesis.

Continuing by reasoning exactly as in the proof of Theorem 3.3, we immediately get the desired conclusion. □

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