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MODULES WHOSE MAXIMAL SUBMODULES ARE SUPPLEMENTS[‡ §]

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Abstract

We study modules whose maximal submodules are supplements (direct summands). For a locally projective module, we prove that every maximal submodule is a direct summand if and only if it is semisimple and projective. We give a complete characterization of the modules whose maximal submodules are supplements over Dedekind domains.

Keywords: Locally projective module, Supplement submodule.

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1. Introduction

Let R be a unitary ring and M a left R-module. A submodule N of M is called a *supplement* if there exists another submodule L such that N is minimal with respect to the property that $N + L = M$. This is equivalent to $N + L = M$ and $N \cap L \ll N$. A module M is called *supplemented* if every submodule has a supplement. Several authors have been recently attracted by different generalizations of supplemented modules. An interesting example of this situation has been studied in [1], where modules M in which the kernel of any epimorphism from M to a finitely generated module has a supplement are studied. These modules are characterized as modules whose maximal submodules have supplements, (see, [1, Theorem 2.8]). Motivated by these results, we study in this paper, modules in which every maximal submodule is a supplement, and modules in which every maximal submodule is a direct summand. For the sake of brevity, we call them *ms-modules* and *md-modules*, respectively.

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We begin by studying some basic properties of md-modules. In particular, we show that homomorphic images of md-modules are md-modules and that a module M containing an md-module L is also md-module provided that L is not contained in any maximal submodule of M (Proposition 2.2). In general, md-modules need not be closed under extensions. But we show that M is an md-module provided that L and M/L are md-modules where L is a closed submodule of M. These basic results allow us to characterize semilocal rings as those rings in which any module with zero Jacobson radical is an md-module.

In Section 3, we study locally projective md-modules. Locally projective modules were introduced by Huisgen-Zimmermann in [20], and they coincide with the flat strict Mittag-Leffler modules in the sense of Raynaud and Gruson (see [10]). These modules are closely related to pure submodules of direct products of free modules (see [20]). And it has been recently observed by several authors that there exists a strong connection between the existence of nontrivial locally projective modules in the functor category of a ring (in the sense that they are not projective) and the construction of separable modules and the pure semisimplicity of certain subcategories of modules over the ring (see e.g. [8, 9, 11, 12, 21]).

In particular, it is proved in [21] that any ring R which is not left perfect has locally projective left modules which are not projective. Motivated by these relations, we show in Section 3 that any locally projective md-module is semisimple projective. In particular, we deduce that any projective md-module is semisimple.

In Section 4, we characterize the coatomic modules whose maximal submodules are supplement (Theorem 4.3). As a consequence, for a module M over a left perfect ring, we prove that every maximal submodule of M is a supplement if and only if $\operatorname{Rad} K = \operatorname{Rad} M$ for every maximal submodule K of M.

In Section 5, we prove that the class of ms-modules is strictly larger than class of md-modules. We close this paper by studying md-modules over commutative domains. Zöschinger proved that over a Dedekind domain, a submodule of a module is closed if and only it is coclosed. Using this result we obtain that ms-modules and md-modules coincide over Dedekind domains. This allows us to determine completely the structure of md-modules over Dedekind domains.

Throughout this paper, R will be an associative ring with identity and all modules are unital left R-modules. By $N \subseteq M$ we shall mean that N is a submodule of M. Let $L \subseteq M$, L is said to be *small* in M, denoted by $L \ll M$, if $L + K \neq M$ for every proper submodule $K \subseteq M$. Dually, a submodule $L \subseteq M$ is called *essential* in M, denoted by $L \trianglelefteq M$, if $L \cap K \neq 0$ for every nonzero $K \subseteq M$. By $\operatorname{Rad} M$ and $\operatorname{Soc}(M)$, we denote the Jacobson radical and the socle of M, respectively. A submodule L of M is called *closed* in M if $L \trianglelefteq K$ for some $K \subseteq M$, implies $L = K$. Dually, a submodule N of M is called *coclosed* in M if $N/K \ll M/K$ implies $K = N$ for every submodule K of N.

It is easy to see that a maximal submodule of a module is either essential or a direct summand. Therefore a module is an md-module if and only if every maximal submodule is a closed submodule.

2. Modules whose maximal submodules are direct summands

In this section we shall prove some closure properties of md-modules.

2.1. Proposition. *The class of md-modules is closed under arbitrary direct sums and homomorphic images.*

Proof. Let $M = \sum_{i \in I} M_i$, where M_i is an md-module for each $i \in I$. Let K be a maximal submodule of M. Then $M_i \nsubseteq K$ for some $i \in I$, so that $M = M_i + K$ and $M_i \cap K$ is a maximal submodule of M_i. Since M_i is an md-module, there is a submodule $L \subseteq M_i$ such that $M_i = L \oplus M_i \cap K$ for some $L \subseteq M$. Then it is straightforward to see that the sum $M = K + L$ is direct. Hence M is an md-module.

Let M be an md-module and $f : M \to N$ any homomorphism of left R-modules. Note that md-modules are closed under isomorphisms, since being an md-module is a lattice-theoretical notion. Therefore we can assume that $f(M)$ is of the form M/L for some submodule L of M. Let K/L be a maximal submodule of M/L. As K is also a maximal submodule of M and M is an md-module, there exists a submodule S of M such that $M = K \oplus S$. Now it is clear that the sum $K/L + (S + L)/L = M/L$ is a direct sum. Hence M/L is an md-module. □

Let M and N be R-modules. Then, N is said to be an *M-generated* module if there is an epimorphism $f : M^{(\Lambda)} \to N$ for some index set Λ.

From Proposition 2.1, we obtain the following.

2.2. Corollary. *Any M-generated module of an md-module is an md-module.* □

2.3. Proposition. *Let M be an R-module and $N \subseteq M$. Suppose N is an md-module and M/N has no maximal submodules. Then M is an md-module.*

Proof. Let K be a maximal submodule of M. If $N \subseteq K$, then K/N would be a maximal submodule of M/N, which is impossible, so we must have $M = N + K$. Since $M/K \cong N/(N \cap K)$ is simple, $N \cap K$ is a maximal submodule of N. Since N is an md-module, $N \cap K \oplus L = N$ for some simple submodule $L \subseteq N$. Then $M = K + N = K + K \cap N + L = K + L$. Since L is simple, $K \cap L = 0$. That is, K is a direct summand of M, and so M is an md-module. □

Let M be a module with no maximal submodules, i.e. satisfying $\operatorname{Rad} M = M$, then M is an md-module (take $N = 0$ in the above Proposition).

In general, a submodule of an md-module need not be an md-module. For example, the $\mathbb{Z}$-module ${}_{\mathbb{Z}}\mathbb{Q}$ is an md-module, because it has no maximal submodules. On the other hand, ${}_{\mathbb{Z}}\mathbb{Q}$ does not contain any nonzero proper md-submodules, because proper submodules of ${}_{\mathbb{Z}}\mathbb{Q}$ have a proper radical and moreover are indecomposable since ${}_{\mathbb{Z}}\mathbb{Q}$ is uniform. However, we have the following result for particular submodules.

2.4. Proposition. *Let M be an md-module. Then any coclosed submodule N of M with* $\operatorname{Soc}(M) \subseteq N$ *is an md-module.*

Proof. Let K be a maximal submodule of N. Since N is coclosed, we have $N/K + T/K = M/K$ for some proper submodule $T/K \subseteq M/K$. Then $(N/K) \cap (T/K) = 0$ because N/K is a simple module. Now we get $M/K = N/K \oplus T/K$ and so $N \cap T = K$. Then $N/K \cong M/T$ is also simple, hence T is a maximal submodule of M. Since M is an md-module, $M = T \oplus S$ for some simple submodule S of M. Then $S \subseteq \operatorname{Soc}(M) \subseteq N$. By the modular law, we get $N = N \cap T \oplus S = K \oplus S$. That is, K is a direct summand of N. Hence N is an md-module. □

Let M be an R-module. If U and M/U are md-modules for some $U \subset M$, then M need not be an md-module. To see this, let p be a prime integer, $M = \mathbb{Z}/p^2\mathbb{Z}$ and $U = pM$. Then U and M/U are both simple modules, hence md-modules. Clearly, U is a maximal submodule of M and U is not a direct summand of M. Hence M is not an md-module.

2.5. Proposition. *Let M be an R-module and L be a closed submodule of M. If L and M/L are md-modules, then M is an md-module.*

Proof. Let K be a maximal submodule of M. If $K+L=M$, then $M/K \cong L/(L\cap K)$ is simple, so $L\cap K$ is a maximal submodule of L. Since L is an md-module, $L = L\cap K \oplus S$ for some simple submodule $S \subseteq L$. Then $M = K+L = K+L\cap K+S = K+S$ and $K\cap S = 0$, so that K is a direct summand of M. If $L \subseteq K$, then K/L is a maximal submodule of M/L, so K/L is a direct summand of M/L. That is, $M/L = K/L \oplus N/L$ for some submodule N/L of M/L. Since N/L is simple, L is a maximal submodule of N. As L is closed in M, $L\cap S = 0$ for some nonzero $S \subseteq N$. So $L\oplus S = N$ with S a simple submodule of M. We get $M = K+N = K+L+S = K+S$ and $K\cap S = 0$. So K is a direct summand of M. Hence M is an md-module. □

For a module M let $s(M)$ be the sum of all simple submodules of M that are direct summands of M.

2.6. Theorem. *For an R-module M, the following are equivalent.*

(1) *M is an md-module,*
(2) *$M/s(M)$ has no maximal submodules,*
(3) *$M/\operatorname{Soc}(M)$ has no maximal submodules.*

Proof. (1) $\Longrightarrow$ (2) Let M be an md-module and suppose K is a maximal submodule of M such that $s(M) \subseteq K$. Then $M = K\oplus S$ for some simple submodule $S \subseteq M$. Hence $S \subseteq s(M) \subseteq K$, a contradiction. Therefore $M/s(M)$ has no maximal submodules.

(2) $\Longrightarrow$ (3) Clear, because any submodule of M containing $\operatorname{Soc}(M)$ also contains $s(M)$.

(3) $\Longrightarrow$ (1) Clearly $\operatorname{Soc}(M)$ is an md-module. Then (3) and Proposition 2.3 implies that M is an md-module. □

Note that, if M is a finitely generated module, then every submodule is contained in a maximal submodule. In this case, M is an md-module if and only if it is semisimple by Theorem 2.6. In particular, R is a semisimple (artinian) ring if and only if ${}_RR$ is an md-module. Therefore, R is a semilocal ring if and only if $R/J(R)$ is an md-module.

2.7. Proposition. *Let M be a module such that $s(M)$ is finitely generated. Then M is an md-module if and only if $M = s(M)\oplus N$, where $N \subseteq M$ with $N = \operatorname{Rad} N = \operatorname{Rad} M$.*

Proof. First note that the (composition) length $l(s(M))$ of $s(M)$ is finite. The proof is by induction on the length $l(s(M))$ of $s(M)$. First suppose $l(s(M)) = 0$. Then clearly $s(M) = 0$, so that M has no maximal submodules, because M is an md-module. Then $\operatorname{Rad} M = M$, and so we are done. Suppose $l(s(M)) = n > 0$ and each md-submodule of M with length less than n has the desired decomposition. Let K be a maximal submodule of M. Then $M = K\oplus S$ for some $S \subseteq s(M)$. Now, K is an md-module by Proposition 2.1 and $l(s(K)) = n-1$. By the induction hypothesis, $K = s(K)\oplus N$ where $\operatorname{Rad} N = N$. Then $M = S\oplus K = S\oplus s(K)\oplus N = s(M)\oplus N$, and this completes the proof.

For the converse, note that a module with no maximal submodules is an md-module. Now if $M = s(M)\oplus N$ with $N = \operatorname{Rad} N$, then both $s(M)$ and N are md-modules. Hence M is an md-module by Proposition 2.1. □

3. Locally projective modules

Let R be a ring and let us denote $\mathrm{Soc}({}_RR)$ by S. As S is a two-sided ideal, R/S has a canonical ring structure. Moreover, for any R-module M, we have that M/SM is an R/S-module. Let us note that a module M is semisimple projective if and only if $M = SM$, where SM is the R-submodule of M generated by the products of elements of S by elements of M.

The proof of the following lemma is straightforward.

3.1. Lemma. *Let M be a left R-module, X an R/S-module and $f : M \to X$ a homomorphism of R-modules. Then $SM \subseteq \mathrm{Ker}(f)$.* □

Let F be a module. We recall that F is called *locally projective* if for any epimorphism $p : X \to Y$, any homomorphism $g : F \to Y$, and any finitely generated submodule Z of F, there exists a homomorphism $h : F \to X$ such that $p \circ h \mid_Z = g \mid_Z$ (see e.g. [20]).

Every projective module is locally projective. But the converse is far from being true. It was proved in [20, Examples 2.3(1)] that any pure submodule of a projective module is locally projective. This means, for instance, that if F is a flat module and $q : R^{(I)} \to F$ is an epimorphism, then $\mathrm{Ker}(q)$ is always locally projective. But it cannot be projective if we choose a flat module having projective dimension bigger than one. In fact, a main result of [21, Theorem 10] asserts that if R is a ring which is not left perfect, then there always exists a locally projective left R-module which is not projective.

The notion of locally projective module coincides with that of flat strict Mittag-Leffler module in the sense of Raynaud and Gruson [10], and their existence has been shown to have a strong relation with the decomposition properties of modules into direct summands (see e.g. [11, 12]). Bearing in mind this connection, we will prove in this section that any locally projective md-module is trivial in the sense that it is a direct sum of simple projective modules.

We first need to prove the following lemma.

3.2. Lemma. *Let F be a locally projective module. Then any finitely generated direct summand of F is projective.*

Proof. Let N be a finitely generated direct summand of F and $p : R^{(n)} \to N$ an epimorphism. Let us denote by $u : N \to F$ the inclusion and let $\pi : F \to N$ be an epimorphism such that $\pi \circ u = 1_N$. As F is locally projective and N is finitely generated, there exists a homomorphism $h : F \to R^{(n)}$ such that $p \circ h \mid_N = \pi \mid_N$. But this means that N is a direct summand of $R^{(n)}$ and therefore, projective. □

We can now state the main result of this section.

3.3. Theorem. *Every locally projective md-module is semisimple projective.*

Proof. Let F be a locally projective md-module. We need to show that $SF = F$. Assume on the contrary that $SF \neq F$ and let us choose $0 \neq x \in F \setminus SF$. Let $p : R^{(I)} \to F$ be an epimorphism for some index set I. As F is locally projective, there exists a homomorphism $h : F \to R^{(I)}$ such that $p \circ h(x) = x$.

We claim that $\mathrm{Gor}(h) \subseteq (J + S)^{(I)}$. Otherwise, if we call $\pi : R^{(I)} \to R^{(I)}/(J + S)^{(I)}$ the canonical projection, we have that $\pi \circ h \neq 0$. And, as $\mathrm{Rad}(R^{(I)}/(J + S)^{(I)}) = 0$, this means that there exists an epimorphism $q : R^{(I)}/(J + S)^{(I)} \to C$ onto a simple module C such that $q \circ \pi \circ h \neq 0$. Our hypothesis implies now that C is a direct summand of F, which must be projective by Lemma 3.2. Hence $C \subseteq SF$. But this is a contradiction, since otherwise $q \circ \pi \circ h = 0$.

Let us now choose a finite subset $K \subseteq I$ such that $h(x) \subseteq R^{(K)}$. Say that $h(x) = \sum_{i\in K} r_i e_i$ where $r_i \in R$. Again, for any $i \in K$, we may choose a finite subset $K_i \subseteq I$ such that $h \circ p(e_i) \subseteq R^{(K_i)}$. Let us set $K' = K \cup \left(\bigcup_{i\in K} K_i\right)$. Then, for any $i \in K$, we can find elements $r_{ij} \in R$ such that $h \circ p(e_i) = \sum_{j\in K'} r_{ij} e_j$. Thus we get that

$$h(x) = hph(x) = hp\left(\sum_{i\in K} r_i e_i\right) = \sum_{i\in K} r_i hp(e_i) = \sum_{i\in K} r_i \left(\sum_{j\in K_i} r_{ij} e_j\right).$$

So, if we denote by $\phi : R^{(K')} \to R^{(K')}$ the endomorphism whose matrix with respect to the basis $\{e_j\}_{j\in K'}$ is (r_{ij}), we get that $\phi \circ h(x) = h(x)$. Let us enlarge the row vector $(r_i)_K$ to a vector in $R^{(K')}$ by setting $r_j = 0$ if $j \in K' \setminus K$. We deduce from the above equality that $(r_j)_{j\in K'} = (r_j)_{j\in K'} \cdot (r_{ij})_{i,j\in K'}$. So if we denote by $I_{K'}$ the identity matrix of size K', then $(r_j)_{j\in K'} \cdot (I_{K'} - (r_{ij})_{i,j\in K'}) = 0$.

On the other hand, as we know that $\mathrm{Gor}(h) \subseteq (J+S)^{(I)}$, and S is a two-sided ideal of R, we deduce that all entries of the matrix $(r_{ij}+S)_{i,j\in K'}$ belong to the Jacobson radical of R/S and therefore, it is a quasi-regular matrix by [2, Corollary 17.13]. This means that the matrix $I_{K'} - (r_{ij}+S)$ is invertible in the matrix ring $M_{K'}(R/S)$ and thus, the row matrix $(r_i)_{i\in K'} = (0+S)$ is in $M_{K'}(R/S)$, i.e. $r_i \in S$ for any $i \in K$. But this means that $h(x) \in S^{(I)}$ and, as any simple quotient of F is a direct summand, we deduce that $x = p \circ h(x) \in SF$. A contradiction, since we were assuming that $x \notin SF$. □

In particular, we get the following corollary.

3.4. Corollary. *Any projective md-module is semisimple.* □

4. Maximal submodules that are supplements

In this section we shall study modules whose maximal submodules are supplements, and call them *ms-modules* for short. Clearly any direct summand is a supplement, and hence md-modules are ms-modules. We shall prove that the converse need not be true in general.

It can be verified easily that the properties in Proposition 2.1 and Proposition 2.3 hold also for ms-modules.

Recall that a module is called *coatomic* provided that every submodule is contained in a maximal submodule. First, we shall characterize coatomic ms-modules. Then we will obtain a characterization of ms-modules over left perfect rings. We begin with following:

4.1. Lemma. *Let M be a coatomic module and N be a coclosed submodule of M. Then N is coatomic.*

Proof. Suppose $\mathrm{Rad}(N/K) = N/K$ for some $K \subseteq N$. Then $N/K \subseteq \mathrm{Rad}(M/K) \ll M/K$. Then $N/K \ll M/K$, and hence $N = K$ because N is coclosed. Therefore N is coatomic. □

4.2. Lemma. *Let M be a module with* $\mathrm{Rad}\, M = 0$. *Then M is an ms-module if and only if it is an md-module.*

Proof. If $\mathrm{Rad}\, M = 0$ then supplements and direct summands in M are the same. □

4.3. Theorem. *Let R be any ring and M be a coatomic R-module. Then M is an ms-module if and only if the following conditions hold:*

(i) *Every maximal submodule N of M is coatomic and* $\mathrm{Rad}\, N = \mathrm{Rad}\, M$,

(ii) $M/\operatorname{Rad} M$ *is semisimple.*

Proof. Suppose M is an ms-module and K is a maximal submodule of M. Then K is a supplement in M, so K is coatomic by Lemma 4.1, and $\operatorname{Rad} K = K \cap \operatorname{Rad} M = \operatorname{Rad} M$ by [19, 41.1], which proves (i). Now (ii) follows from Lemma 4.2 and the fact that coatomic md-modules are semisimple (see, Theorem 2.6).

Conversely, let K be a maximal submodule of M. Then $K/\operatorname{Rad} M$ is a direct summand of $M/\operatorname{Rad} M$ by (ii), so $K + L = M$ and $K \cap L = \operatorname{Rad} M$ for some submodule $L \subseteq M$. Since K is coatomic and $\operatorname{Rad} K = \operatorname{Rad} M$, we have $K \cap L = \operatorname{Rad} K \ll K$, that is K is a supplement of L in M. Hence M is an ms-module. □

A ring R is called a *left max ring* if $\operatorname{Rad} M \ll M$ for every left R-module M. Equivalently, R is a left max ring if and only every (nonzero) left R-module is coatomic. Also, R is a *left perfect ring* if R is a left max ring and $R/\operatorname{Rad} R$ is semisimple as a left R-module (see [2]). For every module M over a left perfect ring, we have $M/\operatorname{Rad} M$ is semisimple.

Now, from Theorem 4.3 we obtain the following corollary.

4.4. Corollary. *Let R be a left perfect ring and M be an R-module. Then M is an ms-module if and only if* $\operatorname{Rad} K = \operatorname{Rad} M$ *for every maximal submodule K of M.* □

An R-module M is called *π-projective* if for every two submodules U, V of M with $U + V = M$, there exists $f \in \operatorname{End}(M)$ with $\operatorname{Gor}(f) \subseteq U$ and $\operatorname{Gor}(1 - f) \subseteq V$.

A projective module P together with an epimorphism $f : P \to M$ such that $\operatorname{Ker}(f) \ll P$ is called a *projective cover* of M. A ring R is semiperfect if and only if every simple left R-module has a projective cover, if and only if the left (right) R-module R is supplemented (see [19, 42.6]).

4.5. Proposition. *Let R be a semiperfect ring and M a π-projective R-module. Then M is an ms-module if and only if M is an md-module. In particular, ${}_RR$ is an ms-module if and only if it is semisimple.*

Proof. Necessity is clear. Now suppose M is an ms-module and let N be a maximal submodule of M. Then by hypothesis $M = N + L$ and $N \cap L \ll N$ for some $L \subseteq M$. Since R is semiperfect, the simple R-module M/N has a projective cover. So that N has a supplement L' in L by [16, Lemma 4.40]. Then N and L' are mutual supplements. Hence N is a direct summand of M by [3, 20.9]. □

5. An example

As we have mentioned, in general an ms-module need not be an md-module. In the following two lemmas we shall prove the existence of such a module.

5.1. Lemma. *Let R be a ring and M be an R-module. Suppose M has a simple submodule U such that $U \trianglelefteq M$ and M/U is semisimple but not simple. Then M is an ms-module but not an md-module.*

Proof. It is clear from the hypothesis that $\operatorname{Soc}(M) = U$ and $U \subseteq L$ for every nonzero proper submodule L of M. In particular, U is contained in every maximal submodule of M, and hence $U \subseteq \operatorname{Rad} M$. Since $(\operatorname{Rad} M)/U = \operatorname{Rad}(M/U) = 0$, we have $\operatorname{Rad} M = U$. By the same argument we have $\operatorname{Rad} N = U$ for every submodule N of M which contains U properly. Let K be a maximal submodule of M. Then $M/U = K/U \oplus T/U$ for some $T/U \subseteq M/U$ because M/U is semisimple. We get $K + T = M$ and $K \cap T = U = \operatorname{Rad} K$. Clearly U is finitely generated, so $K \cap T = U \ll K$. Therefore K is a supplement of T in M. Hence M is an ms-module. Since every nonzero submodule of M contains U, K is not a direct summand of M, i.e. M is not an md-module. □

5.2. Lemma. *Let R be a complete commutative noetherian local ring with maximal ideal P. Suppose P is not principal. Then there exists an ms-module over R which is not an md-module.*

Proof. Let U be the simple R-module R/P and $E = E(U)$ be the injective hull of U. Let $V = \{e \in E \mid P^2 e = 0\}$. Then V is a submodule of E and $P(V/U) = 0$, so that V/U is a vector space over R/P. Also P/P^2 is a vector space over R/P. The dimension of these vector spaces is the respective composition length. By [18, Corollary p. 154] the composition length of V/U is the same as the composition length of P/P^2. Since P is not principal, the composition length of P/P^2 is at least two (see [17, Proposition 9.3]), so that V/U is not simple. Therefore by Lemma 5.1, V is an ms-module but not an md-module. □

5.3. Example. Let $R = \mathbb{C}[x, y]$, $P = Rx + Ry$ and $S = R/P^2$. Then S is an artinian local ring. Let $M = E_S(R/P)$ be the injective hull of the simple S-module R/P. Then $P^2 M = 0$, so M is an ms-module but not an md-module by Lemma 5.2.

5.4. Corollary. *Let M be an R-module such that $\operatorname{Rad} M$ is a simple essential submodule of M and $M/\operatorname{Rad} M \cong S_1 \oplus S_2$ for simple modules S_1 and S_2. Then M is an ms-module but not an md-module.* □

5.5. Note. A concrete example satisfying the hypothesis of Corollary 5.4 can be found in [15, p. 339].

6. Modules over Commutative Rings

Throughout this section all rings are commutative. In general direct product of simple modules need not be an md-module. For instance, let F be a field and $R = F^I$ where I is an infinite index set. Then R is a direct product of simple R-modules each of which is isomorphic to F. By [13, p. 264] R is not semisimple. Hence R is not an md-module by Theorem 3.3.

In case R is commutative and noetherian, we shall prove that an arbitrary direct product of simple R-modules is an md-module. First we need the following lemma.

6.1. Lemma. *Let R be a ring and A be a finitely generated ideal of R. Let $X = \prod_{i \in I} X_i$ be the direct product of the R-modules X_i. Suppose that $X_i = AX_i$ for all $i \in I$. Then $X = AX$.*

Proof. Let $A = Ra_1 + Ra_2 + \cdots + Ra_k$ for some $k \geqslant 1$, $a_i \in A$, $(1 \leqslant i \leqslant k)$. For every $i \in I$, we have $X_i = AX_i = a_1 X_i + \cdots + a_k X_i$. Let $x = (x_i) \in X$, where $x_i \in X_i$ for all $i \in I$. By assumption, for every $i \in I$ there exists $x_{ij} \in X_i$, $(1 \leqslant j \leqslant k)$ such that $x_i = a_1 x_{i1} + \cdots + a_k x_{ik}$. Then $(x_{ij}) \in X$, $(1 \leqslant j \leqslant k)$ and $x = a_1(x_{i1}) + \cdots + a_k(x_{ik}) \in AX$. Hence $X = AX$. □

6.2. Theorem. *Let R be a noetherian ring and let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a collection of simple R-modules. Then $M = \prod_{\lambda \in \Lambda} U_\lambda$ is an md-module.*

Proof. Let $\{P_i\}_{i \in I}$ be the collection of distinct maximal ideals P_i of R such that for every $i \in I$ there exists $\lambda \in \Lambda$ with $P_i U_\lambda = 0$. For each $i \in I$ let $\Lambda_i = \{\lambda \in \Lambda \mid P_i U_\lambda = 0\}$. Let K be a maximal submodule of M and P the maximal ideal of R such that $PM \subseteq K$. Since $PM \neq M$, we have $U_\lambda \neq PU_\lambda$ for some $\lambda \in \Lambda$ by Lemma 6.1. Since U_λ is simple and PU_λ is a proper submodule of U_λ, we have $PU_\lambda = 0$, so that $P = P_j$ for some $j \in I$. Again by Lemma 6.1, if $L = \prod_{\lambda \in \Lambda'} U_\lambda$, where $\Lambda' = \bigcup\{\Lambda_i \mid i \in I \setminus \{j\}\}$, then $PL = L$.

Hence, $L \subseteq K$. Now let $L' = \prod_{\lambda \in \Lambda_j} U_\lambda$. Then $P_j L' = 0$, so that L' is semisimple, also $M = L \oplus L'$. Then $K = L \oplus (K \cap L')$ and $K \cap L'$ is a direct summand of L'. Therefore K is a direct summand of M. Hence, M is an md-module. □

We shall now characterize the md-modules over Dedekind domains. We begin with the following lemma which is due to Zöschinger. Using this lemma we shall prove that ms-modules and md-modules coincide over Dedekind domains.

6.3. Lemma. [22, Lemma 3.3] *Let R be Dedekind domain, M an R-module and $V \subseteq M$. Then V is coclosed if and only if V is closed.*

Let M be any module and $N \subseteq M$. A submodule K of M is called a *complement of N* if K is maximal in the collection of submodules L of M such that $L \cap N = 0$. A submodule T of M is called a *complement* if there is a submodule N of M such that T is a complement of N. A submodule of M is a complement if and only if it is closed (see [7, p.6]).

6.4. Proposition. *Let R be a Dedekind domain and M an R-module. Then M is an ms-module if and only if M is an md-module.*

Proof. We only need to prove the necessity. Let N be a maximal submodule of M. Since M is an ms-module, N is a supplement in M. So N is a complement in M by Lemma 6.3, i.e. $N \cap L = 0$ for some $L \subseteq M$ and N is maximal with respect to this property. Now $L \neq 0$ because $M \cap 0 = 0$. Therefore $N + L = M$, i.e. N is a direct summand of M. □

6.5. Lemma. [1, Lemma 4.4] *Let R be a Dedekind domain. For an R-module M the following are equivalent.*

(1) *M is injective.*
(2) *M is divisible.*
(3) *$M = PM$ for every maximal ideal P of R.*
(4) *M does not contain any maximal submodule.* □

Let R be a Dedekind domain and M an R-module. For a maximal ideal P of R, the submodule $T_P(M) = \{m \in M \mid P^n m = 0 \text{ for some positive integer } n\}$ is called the *P-primary component* of M. If $M = T_P(M)$ for some maximal ideal P of R, then M is called a *P-primary module*. For a torsion module M we always have $M = \bigoplus_{P \in \Omega} T_P(M)$, where Ω is the set of all maximal ideals of R (see [4, 10.6.9]).

The divisible part of a module M is denoted by $D(M)$. By Lemma 6.5, we have $M = D(M) \oplus M'$ for some $M' \subseteq M$. If M is a divisible module, then M has no maximal submodules, and so $\operatorname{Rad} M = M$. Therefore, $D(M) \subseteq \operatorname{Rad} M$ for every R-module M.

6.6. Lemma. *Let R be a Dedekind domain and M a reduced and P-primary module for some maximal ideal $P \subseteq R$. Then M is an md-module if and only if M is semisimple.*

Proof. Suppose M is an md-module. Then $M/\operatorname{Soc}(M)$ has no maximal submodules by Proposition 2.6, so $P(M/\operatorname{Soc}(M)) = M/\operatorname{Soc}(M)$ by Lemma 6.5, that is $PM + \operatorname{Soc}(M) = M$, and this gives $P(PM + \operatorname{Soc}(M)) = P^2 M = PM$. Therefore, PM is divisible by Lemma 6.5, but M is reduced so that $PM = 0$. Hence M is an R/P-module, i.e. M is semisimple.

The converse is clear. □

6.7. Theorem. *Let R be a Dedekind domain and M a torsion R-module. The following are equivalent.*

(1) *M is an md-module.*

(2) *$M = M_1 \oplus M_2$ where M_1 is divisible and M_2 is semisimple.*
(3) *Every submodule $U \subseteq M$ with $\operatorname{Rad} M \subseteq U$ is a direct summand of M.*

Proof. (1) $\Longrightarrow$ (2) Let D be the divisible part of M. Then $M = D \oplus N$ for some $N \subseteq M$. Since N is torsion, we have $N = \bigoplus_{P\in\Omega} T_P(N)$, and since M is an md-module $T_P(N)$ is also an md-module for every $P \in \Omega$ by Proposition 2.1. Then $T_P(N)$ is semisimple by Lemma 6.6. Therefore N is semisimple.

(2) $\Longrightarrow$ (3) We have $\operatorname{Rad} M = \operatorname{Rad}(M_1 \oplus M_2) = \operatorname{Rad} M_1 \oplus \operatorname{Rad} M_2 = \operatorname{Rad} M_1 = M_1$. Let $\operatorname{Rad} M \subseteq U \subseteq M$. Then we get $U = M_1 \oplus U \cap M_2$. Since M_2 is semisimple, $M_2 = K \oplus M_2 \cap U$ for some $K \subseteq M_2$. So, $M = M_1 \oplus M_2 = M_1 \oplus K \oplus M_2 \cap U = K \oplus U$.

(3) $\Longrightarrow$ (1) $\operatorname{Rad} M \subseteq P$ for every maximal submodule P of M. So, by hypothesis, every maximal submodule of M is a direct summand. Hence M is an md-module. □

6.8. Lemma. [14, Example 6.34] *Let R be a domain and M be an R-module. Then the torsion submodule $T(M)$ is a closed submodule of M.* □

6.9. Corollary. *Let R be domain and M be an R-module. If $T(M)$ and $M/T(M)$ are md-modules, then M is an md-module.*

If R is a Dedekind domain, then the converse also holds.

Proof. By Lemma 6.8, $T(M)$ is a closed submodule of M. Then M is an md-module by Proposition 2.5.

If R is a Dedekind domain, then $T(M)$ is a coclosed submodule of M by Lemma 6.3 and Lemma 6.8. Since every simple submodule of M is torsion, $\operatorname{Soc}(M) \subseteq T(M)$, so that $T(M)$ is an md-module by Proposition 2.4. Hence, $M/T(M)$ is an md-module by Proposition 2.1. □

6.10. Lemma. *Let R be a Dedekind domain and M a torsion-free R-module. Then M is an md-module if and only if M is divisible.*

Proof. Suppose M is an md-module and let P be a maximal submodule of M. Then $P \oplus S = M$ for some simple submodule S of M. Thus $S \subseteq T(M) = 0$, so $P = M$, a contradiction. Hence M has no maximal submodules, and M is divisible by Lemma 6.5.

Conversely, if M is divisible, then M has no maximal submodules by Lemma 6.5. Hence M is an md-module. □

6.11. Theorem. *Let R be a Dedekind domain and M be an R-module. Then M is an md-module if and only if*

(i) *$T(M) = M_1 \oplus M_2$, where M_1 is semisimple and M_2 is divisible,*
(ii) *$M/T(M)$ is divisible.*

Proof. Suppose M is an md-module. Then $T(M)$ is an md-module by Corollary 6.9, so $T(M)$ has the desired decomposition by Theorem 6.7. Hence $M/T(M)$ is divisible by Lemma 6.10.

To prove the converse, let N be a maximal submodule of M. Then by (ii) we have $N + T(M) = M$. Since M_2 is divisible, $M_2 \subseteq \operatorname{Rad} M \subseteq N$, so $M = N + T(M) = N + M_1$. Then $N + S = M$ for some simple submodule $S \subseteq M_1$. We have $N \cap S = 0$ because S is a simple submodule. Therefore N is a direct summand of M. Hence M is an md-module. □

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