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BOUR'S THEOREM ON THE GAUSS MAP IN 3-EUCLIDEAN SPACE

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Abstract

In this paper some relations are established between the Laplace-Beltrami operator and the curvatures of helicoidal surfaces in 3-Euclidean space. In addition, Bour's theorem on the Gauss map, and some special examples are given.

Keywords: Rotational surface, Helicoidal surface, Gauss map, Laplace-Beltrami operator, Minimal harmonic surface.

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1. Introduction

Surface theory in three dimensional Euclidean space have been studied for a long time, and many examples of such surfaces have been discovered. Many very useful books have been written on the subject, such as [7, 8].

In classical surface geometry in 3-Euclidean space, it is well known that the right helicoid (resp. catenoid) is the only ruled (resp. rotational) surface which is minimal. Moreover, a pair of these two surfaces has interesting properties. That is, they are both members of a one parameter family of isometric minimal surfaces, and have the same Gauss map. This pair is a typical example for minimal surfaces. On the other hand, the pair consisting of the right helicoid and the catenoid has the following generalization.

1.1. Theorem. ***Bour's Theorem***. *A generalized helicoid is isometric to a surface of revolution so that helices on the helicoid correspond to parallel circles on the surface of revolution* [2].

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In this generalization, the original properties of minimality and preservation of the Gauss map are not generally maintained.

In [6], T. Ikawa showed that a helicoidal surface and a surface of revolution are isometric by Bour's theorem in 3-Euclidean space. He determined pairs of surfaces with an additional condition that they have the same Gauss map in Bour's theorem. In [4], E. Güler showed that a helicoidal surface and a surface of revolution are isometric by Bour's theorem in Minkowski 3-space. Then, he determined the surfaces with light-like profile curve by Bour's theorem in [5].

Concerning helicoidal surfaces in 3-Euclidean space, M. P. do Carmo and M. Dajczer [3] proved that, by using a result of E. Bour [2], there exists a two-parameter family of helicoidal surfaces isometric to a given helicoidal surface. By making use of this parametrization, they found a representation formula for helicoidal surfaces with constant mean curvature. Furthermore they proved that the associated family of Delaunay surfaces is made up by helicoidal surfaces of constant mean curvature.

In this paper, we give Bour's theorem on the *Gauss map* of helicoidal surfaces in 3-Euclidean space. We recall some basic notions of Euclidean geometry and the reader can find a definition of the generalized helicoid in Section 2. In Section 3, the Laplace-Beltrami operator of a minimal helicoidal surface and a minimal surface of revolution are obtained. The mean curvature, the Gaussian curvature and their relations are calculated. Bour's theorem on the Gauss map of the helicoidal surfaces are studied in Section 4. In Section 5, some examples of these surfaces are given.

2. Preliminaries

In this section, we will obtain surfaces of revolutions in 3-Eucidean space. For the remainder of this paper we shall identify a vector (p, q, r) with its transpose $(p, q, r)^t$.

The inner product on the Euclidean space $\mathbb{E}^3$ is

$$\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3,$$

where $\vec{x} = (x_1, x_2, x_3)$, $\vec{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$. The norm of the vector $\vec{x} \in \mathbb{R}^3$ is defined by $\|\vec{x}\| = \sqrt{|\langle \vec{x}, \vec{x} \rangle|}$. The Euclidean vector product $\vec{x} \times \vec{y}$ of $\vec{x}$ and $\vec{y}$ is defined as follows:

$$\vec{x} \times \vec{y} = (x_2 y_3 - y_2 x_3, x_3 y_1 - y_3 x_1, x_1 y_2 - y_1 x_2).$$

Now we define non degenerate surfaces of rotation and generalized helicoids in $\mathbb{E}^3$. For an open interval $I \subset \mathbb{R}$, let $\gamma : I \longrightarrow \Pi$ be a curve in a plane Π in $\mathbb{R}^3$, and let ℓ be a straight line in Π. A *surface of rotation* in $\mathbb{E}^3$ is defined as a non degenerate surface obtained by rotating a curve γ around a line ℓ (these are called the *profile curve* and the *axis*, respectively). Suppose that when a profile curve γ rotates around the axis ℓ, it simultaneously moves parallel to ℓ so that the speed of displacement is proportional to the speed of rotation. Then the resulting surface is called the *generalized helicoid* with axis ℓ and *pitch* a.

We may suppose that ℓ is the line spanned by the vector $(0, 0, 1)$. The orthogonal matrix which fixes the above vector is

$$A(v) = \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ v \in \mathbb{R}.$$

The matrix A can be found by solving the following equations simultaneously; $A\ell = \ell$, $A^t A = A A^t = I_3$, where $I_3 = \mathrm{diag}(1, 1, 1)$ and $\det A = 1$.

When the axis of rotation is ℓ, there is an Euclidean transformation by which the axis ℓ is transformed to the z-axis of $\mathbb{R}^3$. Parametrization of the profile curve γ is given by

$$\gamma(u) = (\psi(u), 0, \varphi(u)), \tag{2.1}$$

where $\psi(u), \varphi(u) : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ are differentiable functions for all $u \in I$. A helicoidal surface in 3-Euclidean space which is spanned by the vector $(0,0,1)$ and with pitch $a \in \mathbb{R} \setminus \{0\}$ as follows

$$H(u,v) = A(v) \cdot \gamma(u) + av(0,0,1).$$

When $a = 0$, then the surface is just a surface of revolution as follows

$$R(u,v) = (\psi(u)\cos v, \psi(u)\sin v, \varphi(u)). \tag{2.2}$$

For a surface $\mathbf{X}(u,v)$, the coefficients of the first and second fundamental forms and the Gauss map are defined in [6].

Next, we will use the parametrization of the profile curve γ as follows

$$\gamma(u) = (u, 0, \varphi(u)).$$

Therefore, a helicoidal surface with axis of rotation z and profile curve γ is given by

$$\mathbf{H}(u,v) = \begin{pmatrix} u\cos v \\ u \sin v \\ \varphi(u) + av \end{pmatrix} \tag{2.3}$$

in 3-Euclidean space, where $u \in I$, $0 \le v < 2\pi$, $a \in \mathbb{R} \setminus \{0\}$.

2.1. Proposition. *The Gauss map of the helicoidal surface in* (2.3) *which is spanned by the vector* $(0,0,1)$*, and with profile curve* $\gamma(u) = (u, 0, \varphi(u))$*, is*

$$\mathbf{e_H} = \frac{1}{(\det \mathbf{I})^{1/2}} \cdot \begin{pmatrix} -u\varphi' \cos v + a \sin v \\ -a\cos v - u\varphi' \sin v \\ u \end{pmatrix} \tag{2.4}$$

in 3-Euclidean space, where $\det \mathbf{I} = EG - F^2 = u^2 + a^2 + u^2\varphi'^2 > 0$*,* $\varphi = \varphi(u)$*,* $\varphi' = \frac{d\varphi}{du}$*,* $a \in \mathbb{R} \setminus \{0\}$*,* $u \in I \subset \mathbb{R}$ *and* $0 \le v < 2\pi$*.* □

The Laplace-Beltrami operator of a smooth function $\phi = \phi(u,v)|_{\mathbf{D}}$ ($\mathbf{D} \subset \mathbb{R}^2$), of class C^2 with respect to the first fundamental form of the surface $\mathbf{X}$ is the operator Δ, defined in [1, 8] as follows:

$$\Delta^{\mathbf{I}}\phi = -\frac{1}{\sqrt{\det \mathbf{I}}}\left[\left(\frac{G\phi_u - F\phi_v}{\sqrt{\det \mathbf{I}}}\right)_u - \left(\frac{F\phi_u - E\phi_v}{\sqrt{\det \mathbf{I}}}\right)_v\right]. \tag{2.5}$$

If $V = (v_1, v_2, v_3)$ is a function of class C^2 then we set

$$\Delta^{\mathbf{I}}V = (\Delta^{\mathbf{I}}v_1, \Delta^{\mathbf{I}}v_2, \Delta^{\mathbf{I}}v_3).$$

3. Minimal surfaces and the Laplace-Beltrami operator

In this section, we study relations between the mean curvature and the Gaussian curvature of the helicoidal and surface of revolution with axis $(0,0,1)$ in $\mathbb{E}^3$, Moreover, we give the minimal harmonic helicoidal surface and surface of revolution via the Laplace-Beltami operator.

3.1. Proposition. *The mean curvature and Gaussian curvature of the helicoidal surface in* (2.3) *are related as follows:*

$$p \cdot H_{\mathbf{H}} + q \cdot K_{\mathbf{H}} = 0, \tag{3.1}$$

in 3-Euclidean space, where $p(u) = 2\big(-a^2 + u^3\varphi'\varphi''\big)$*,* $q(u) = -\big(u^2 + a^2 + u^2\varphi'^2\big)^{1/2} \cdot \big[2a^2\varphi' + u^2\varphi'(1+\varphi'^2) + u(u^2+a^2)\varphi''\big]$*, and* $u \in I \subset \mathbb{R}$*.*

Proof. We consider a helicoidal surface in (2.3). Computing the mean curvature and the Gaussian curvature we obtain

$$H_{\mathbf{H}} = 2^{-1}\Phi(u)\cdot(\det \mathbf{I})^{-3/2} \tag{3.2}$$

and

$$K_{\mathbf{H}} = \Psi(u)\cdot(\det \mathbf{I})^{-2} \tag{3.3}$$

respectively, where $\Phi(u) = 2a^2\varphi' + u^2\varphi'\left(1+\varphi'^2\right) + u(u^2+a^2)\varphi''$, $\Psi(u) = -a^2 + u^3\varphi'\varphi''$, $\det \mathbf{I} = EG - F^2 = u^2 + a^2 + u^2\varphi'^2 > 0$, $\varphi = \varphi(u)$, $\varphi' = \frac{d\varphi}{du}$, and $u \in I, a \in \mathbb{R}\setminus\{0\}$. Hence, we obtain the required results. □

So, we can say that a helicoidal surface is a Weingarten surface (or briefly, a W-surface). For details of W-surfaces see [7].

3.2. Proposition. *If $H_{\mathbf{H}} = 0$, then the function on the profile curve $\gamma(u) = (u, 0, \varphi(u))$ is as follows*

$$\varphi = \int \sqrt{\frac{(u^2+a^2)}{u^2(-1+u^2)}}\,du + c$$

in 3-Euclidean space, where $c \in \mathbb{R}$, $u \in I$ and $a \in \mathbb{R}\setminus\{0\}$.

Proof. We consider the mean curvature of the helicoidal surface in (2.3). If $H_{\mathbf{H}} = 0$ then we have

$$\Phi(u) = 0. \tag{3.4}$$

This equation reduces to an elliptic integral. We can easily find the function φ. □

3.3. Proposition. *If $K_{\mathbf{H}} = 0$ then the function on the profile curve $\gamma(u) = (u, 0, \varphi(u))$ is as follows*

$$\varphi = \mp\int \frac{\sqrt{cu^2 - a^2}}{u}\,du + d$$

in 3-Euclidean space, where $c, d \in \mathbb{R}$, $u \in I \subset \mathbb{R}$ and $a \in \mathbb{R}\setminus\{0\}$.

Proof. We consider the Gaussian curvature of the helicoidal surface (2.3). If $K_{\mathbf{H}} = 0$ then we obtain

$$\varphi'\varphi'' = a^2/u^3. \tag{3.5}$$

Hence, if we solve $\varphi'^2 = c - a^2/u^2$, then we have complex solutions as follows:

$$\varphi = \mp\left[\sqrt{cu^2 - a^2} + \frac{a^2}{\sqrt{-a^2}}\log\left(\frac{-2a^2 + 2\sqrt{-a^2}\sqrt{cu^2-a^2}}{u}\right)\right] + d,$$

where $c, d \in \mathbb{R}$, $u \in I$, $a \in \mathbb{R}\setminus\{0\}$. □

3.4. Theorem. *The helicoidal surface* (2.3) *and the surface of revolution*

$$\mathbf{R}(u,v) = \begin{pmatrix} \sqrt{u^2+a^2}\cos\left(v + \int \frac{a\varphi'}{u^2+a^2}\,du\right) \\ \sqrt{u^2+a^2}\sin\left(v + \int \frac{a\varphi'}{u^2+a^2}\,du\right) \\ \int \sqrt{\frac{a^2+u^2\varphi'^2}{u^2+a^2}}\,du \end{pmatrix} \tag{3.6}$$

are isometric surfaces by Bour's theorem and have the same Gauss map, and the surfaces are harmonic in 3-Euclidean space, where $\varphi = \varphi(u)$, $a \in \mathbb{R}\setminus\{0\}$, $u \in I \subset \mathbb{R}$, $0 \le v < 2\pi$.

Proof. The coefficients of the first and the second fundamental forms are

$$E_{\mathbf{H}} = 1 + \varphi'^2,\ F_{\mathbf{H}} = a\varphi',\ G_{\mathbf{H}} = u^2 + a^2 \tag{3.7}$$

and

$$L_{\mathbf{H}} = \frac{u\varphi''}{(\det \mathbf{I})^{1/2}},\ M_{\mathbf{H}} = \frac{-a}{(\det \mathbf{I})^{1/2}},\ N_{\mathbf{H}} = \frac{u^2\varphi'}{(\det \mathbf{I})^{1/2}}, \tag{3.8}$$

where $\det \mathbf{I} = u^2 + a^2 + u^2\varphi'^2$, $u \in I$, $a \in \mathbb{R}\setminus\{0\}$. The Gauss map of $\mathbf{H}(u,v)$ is given in (2.4). The Gauss map of $\mathbf{R}(u,v)$ is

$$e_{\mathbf{R}} = \frac{1}{(\det \mathbf{I})^{1/2}} \cdot \begin{pmatrix} -\sqrt{a^2+u^2\varphi'^2}\cos(v + \int \frac{a\varphi'}{u^2+a^2}du) \\ -\sqrt{a^2+u^2\varphi'^2}\sin(v + \int \frac{a\varphi'}{u^2+a^2}du) \\ u \end{pmatrix}$$

When $e_{\mathbf{H}} = e_{\mathbf{R}}$, then $\Phi(u) = 0$ in [6]. Using (2.3), (2.4), (2.5) and (3.4), we get

$$\Delta^{\mathbf{I}}\mathbf{H}(u,v) \equiv \frac{-\Phi(u)}{(\det \mathbf{I})^{3/2}} \cdot \mathbf{e}_{\mathbf{H}}. \tag{3.9}$$

Using similar methods, we can easily see the relation between $\Delta^{\mathbf{I}}$ and $\mathbf{R}$. So, for the helicoidal surface in (2.3), we have the following:

$$\begin{aligned} e_{\mathbf{H}} = e_{\mathbf{R}} &\iff \Phi(u) = 0 \\ &\iff H_{\mathbf{H}} = 0 \\ &\iff \Delta^{\mathbf{I}}\mathbf{H}(u,v) \equiv \frac{-\Phi(u)}{(\det \mathbf{I})^{3/2}} \cdot e_{\mathbf{H}} \\ &\iff \mathbf{H}(u,v) \text{ is harmonic.} \end{aligned}$$

□

4. Bour's theorem on the Gauss map

In this section we study Bour's theorem on the Gauss map (2.4) of the helicoidal surface (2.3).

4.1. Proposition. *The Gauss map (2.4) of the helicoidal surface (2.3), which is a surface of revolution, is as follows:*

$$\mathbf{e}_{\mathbf{H}} = \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{-u\varphi'}{(\det \mathbf{I})^{1/2}} \\ \frac{-a}{(\det \mathbf{I})^{1/2}} \\ \frac{u}{(\det \mathbf{I})^{1/2}} \end{pmatrix}, \tag{4.1}$$

where $\det \mathbf{I} = u^2 + a^2 + u^2\varphi'^2$, $a \in \mathbb{R}\setminus\{0\}$, $u \in I \subset \mathbb{R}$ *and* $0 \le v < 2\pi$.

When we assume $a = 0$, for the sake of clearness, the profile curve is

$$\gamma(u) = \left(\frac{-\varphi'}{(1+\varphi'^2)^{1/2}}, 0, \frac{1}{(1+\varphi'^2)^{1/2}} \right).$$

So this surface is just a surface of revolution. Therefore, if we add pitch $h \in \mathbb{R}\setminus\{0\}$ to the surface of revolution, we obtain the *helicoidal surface of the Gauss map* as follows

$$\Im(u,v) = \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{-\varphi'}{(1+\varphi'^2)^{1/2}} \\ 0 \\ \frac{1}{(1+\varphi'^2)^{1/2}} \end{pmatrix} + hv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4.2}$$

4.2. Theorem. *A helicoidal surface of the Gauss map in (4.2) is isometric to a surface of revolution*

$$\Re(u,v)=\begin{pmatrix} -\sqrt{\frac{h^2+\varphi'^2+h^2\varphi'^2}{1+\varphi'^2}}\cos\left(v-\int\frac{h\varphi'\varphi''}{\left(h^2+\varphi'^2+h^2\varphi'^2\right)\sqrt{1+\varphi'^2}}\,du\right)\\ -\sqrt{\frac{h^2+\varphi'^2+h^2\varphi'^2}{1+\varphi'^2}}\sin\left(v-\int\frac{h\varphi'\varphi''}{\left(h^2+\varphi'^2+h^2\varphi'^2\right)\sqrt{1+\varphi'^2}}\,du\right)\\ \varphi_{\Re}\end{pmatrix}$$

by Bour's theorem, where

$$\varphi_{\Re}=\int\left[\tan\left(\int\sqrt{\frac{(h^2+\varphi'^2)\,\varphi''^2}{(1+\varphi'^2)^2\,(h^2+\varphi'^2+h^2\varphi'^2)}}\,du\right)\right]du+c$$

and $\varphi=\varphi(u)$, $h\in\mathbb{R}\setminus\{0\}$, $c\in\mathbb{R}$, $u\in I\subset\mathbb{R}$, $0\le v<2\pi$.

Proof. We assume that the profile curve is on the x_1x_3-plane. Since a generalized helicoid is given by rotating the profile curve around the axis and simultaneously displacing it parallel to the axis, so that the speed of displacement is proportional to the speed of rotation, from (4.2), we have the following representation of a generalized helicoid

$$\Im(u_\Im,v_\Im)=\left(-\frac{\varphi'_\Im}{\sqrt{1+\varphi'^2_\Im}}\cos(v_\Im),-\frac{\varphi'_\Im}{\sqrt{1+\varphi'^2_\Im}}\sin(v_\Im),\frac{1}{\sqrt{1+\varphi'^2_\Im}}+hv_\Im\right), \tag{4.3}$$

where h is a constant.

The coefficients of the first fundamental form and the line element of the generalized helicoid (4.3) are given by

$$E_\Im=\frac{\varphi''^2_\Im}{(1+\varphi'^2_\Im)^2},\; F_\Im=\frac{-h\varphi'_\Im\varphi''_\Im}{(1+\varphi'^2_\Im)^{3/2}},\; G_\Im=\frac{\varphi'^2_\Im}{1+\varphi'^2_\Im}+h^2,$$

$$ds^2_\Im=\frac{\varphi''^2_\Im}{(1+\varphi'^2_\Im)^2}\,du^2_\Im+2\frac{-h\varphi'_\Im\varphi''_\Im}{(1+\varphi'^2_\Im)^{3/2}}\,du_\Im\,dv_\Im+\left(\frac{\varphi'^2_\Im}{1+\varphi'^2_\Im}+h^2\right)dv^2_\Im.$$

Helices in $\Im(u_\Im,v_\Im)$ are curves defined by $u_\Im=$ constant, so curves in $\Im(u_\Im,v_\Im)$ that are orthogonal to helices satisfy the the following condition of orthogonality

$$\frac{-h\varphi'_\Im\varphi''_\Im}{(1+\varphi'^2_\Im)^{3/2}}\,du_\Im+\left(\frac{\varphi'^2_\Im}{1+\varphi'^2_\Im}+h^2\right)dv_\Im=0.$$

Thus we obtain

$$v_\Im=\int\frac{h\varphi'_\Im\varphi''_\Im}{(h^2+\varphi'^2_\Im+h^2\varphi'^2_\Im)\,(1+\varphi'^2_\Im)^{1/2}}du_\Im+c,$$

where c is constant. Hence if we put

$$\bar v_\Im=v_\Im-\int\frac{h\varphi'_\Im\varphi''_\Im}{(h^2+\varphi'^2_\Im+h^2\varphi'^2_\Im)\,(1+\varphi'^2_\Im)^{1/2}}\,du_\Im,$$

then the curves that are orthogonal to the helices are given by $\bar v_\Im=$ constant. Substituting the equation

$$dv_\Im=d\bar v_\Im+\frac{h\varphi'_\Im\varphi''_\Im}{(h^2+\varphi'^2_\Im+h^2\varphi'^2_\Im)\,(1+\varphi'^2_\Im)^{1/2}}\,du_\Im$$

into the line element, we have

$$ds^2_\Im=\frac{\left(h^2+\varphi'^2\right)\varphi''^2_\Im}{(1+\varphi'^2_\Im)^2\,(h^2+\varphi'^2_\Im+h^2\varphi'^2_\Im)}\,du^2_\Im+\frac{h^2+\varphi'^2_\Im+h^2\varphi'^2_\Im}{1+\varphi'^2_\Im}\,d\bar v^2_\Im. \tag{4.4}$$

By putting

$$\bar{u}_{\Im} = \int \sqrt{\frac{(h^2+\varphi'^2)\,\varphi_{\Im}''^2}{(1+\varphi_{\Im}'^2)^2\,(h^2+\varphi_{\Im}'^2+h^2\varphi_{\Im}'^2)}}\,du_{\Im}, \quad f_{\Im}(\bar{u}_{\Im}) = -\sqrt{\frac{h^2+\varphi_{\Im}'^2+h^2\varphi_{\Im}'^2}{1+\varphi_{\Im}'^2}}$$

then (4.4) reduces to

$$ds_{\Im}^2 = d\bar{u}_{\Im}^2 + f_{\Im}^2(\bar{u}_{\Im})d\bar{v}_{\Im}^2. \tag{4.5}$$

On the other hand, the surface of revolution

$$\left(-\frac{\varphi_{\Re}'}{\sqrt{1+\varphi_{\Re}'^2}}\cos(v_{\Re}), -\frac{\varphi_{\Re}'}{\sqrt{1+\varphi_{\Re}'^2}}\sin(v_{\Re}), \frac{1}{\sqrt{1+\varphi_{\Re}'^2}}\right) \tag{4.6}$$

has the line element

$$ds_{\Re}^2 = \frac{\varphi_{\Re}''^2}{(1+\varphi_{\Re}'^2)^2}\,du_{\Re}^2 + \frac{\varphi_{\Re}'^2}{1+\varphi_{\Re}'^2}d\bar{v}_{\Re}^2. \tag{4.7}$$

Hence, if we put

$$\bar{u}_{\Re} = \int \frac{\varphi_{\Re}''}{1+\varphi_{\Re}'^2}\,du_{\Re},\ f_{\Re}(\bar{u}_{\Re}) = -\sqrt{\frac{\varphi_{\Re}'^2}{1+\varphi_{\Re}'^2}},\ \bar{v}_{\Re} = v_{\Re},$$

then (4.7) reduces to

$$ds_{\Re}^2 = d\bar{u}_{\Re}^2 + f_{\Re}^2(\bar{u}_{\Re})\,d\bar{v}_{\Re}^2. \tag{4.8}$$

Comparing (4.5) with (4.8), if

$$\bar{u}_{\Im} = \bar{u}_{\Re},\ \bar{v}_{\Im} = \bar{v}_{\Re}, \text{ and } f_{\Im}(\bar{u}_{\Im}) = f_{\Re}(\bar{u}_{\Re}),$$

then we have an isometry between $\Im(u_{\Im}, v_{\Im})$ and $\Re(u_{\Re}, v_{\Re})$.

Therefore it follows that

$$\int \sqrt{\frac{(h^2+\varphi'^2)\,\varphi_{\Im}''^2}{(1+\varphi_{\Im}'^2)^2\,(h^2+\varphi_{\Im}'^2+h^2\varphi_{\Im}'^2)}}\,du_{\Im} = \int \frac{\varphi_{\Re}''}{1+\varphi_{\Re}'^2}\,du_{\Re}$$

and we have

$$\arctan\left(\varphi_{\Re}'\right) = \int \sqrt{\frac{(h^2+\varphi'^2)\,\varphi''^2}{(1+\varphi'^2)^2\,(h^2+\varphi'^2+h^2\varphi'^2)}}\,du + c,$$

where $c \in \mathbb{R}$. □

4.3. Proposition. *The mean curvature and the Gaussian curvature of the helicoidal surface of the Gauss map in (4.3) are related by*

$$H_{\Im}^2 = \Omega \cdot K_{\Im}, \tag{4.9}$$

where $\Omega = \Omega(u) = \frac{\left[\left(1+\varphi_{\Im}'^2\right)\left(-\varphi_{\Im}'^3+h^2\varphi_{\Im}'^4-2h^2\varphi_{\Im}'\right)+\varphi_{\Im}'^6\right]^2}{4\left(h^2+\varphi_{\Im}'^2\right)\left(\varphi_{\Im}'^4-h^2\right)\left(1+\varphi_{\Im}'^2\right)^2}$, $\varphi_{\Im} = \varphi_{\Im}(u)$ *is on the profile curve,* $\varphi_{\Im}' := \frac{d\varphi_{\Im}}{du}$, *and* $h, u \in \mathbb{R} \setminus \{0\}$.

Proof. First we consider the helicoid (4.3). Using the coefficients of the first and second fundamental forms we have

$$E_{\Im}G_{\Im} - F_{\Im}^2 = \frac{\left(h^2+\varphi_{\Im}'^2\right)\varphi_{\Im}''^2}{(1+\varphi_{\Im}'^2)^3}$$

and

$$L_{\Im}N_{\Im} - M_{\Im}^2 = \frac{\left(\varphi_{\Im}'^6+\varphi_{\Im}'^4-h^2\varphi_{\Im}'^2-h^2\right)\varphi_{\Im}''^2}{(h^2+\varphi_{\Im}'^2)\,(1+\varphi_{\Im}'^2)^4}.$$

Hence, the mean curvature and the Gaussian curvature of the helicoidal surface of the Gauss map are given, respectively, by

$$H_{\Im} = \frac{\left(1+\varphi_{\Im}'^{2}\right)\left(-\varphi_{\Im}'^{3}+h^{2}\varphi_{\Im}'^{4}-2h^{2}\varphi_{\Im}'\right)+\varphi_{\Im}'^{6}}{2\left(h^{2}+\varphi_{\Im}'^{2}\right)^{3/2}\left(1+\varphi_{\Im}'^{2}\right)} \tag{4.10}$$

and

$$K_{\Im} = \frac{\varphi_{\Im}'^{4}-h^{2}}{\left(\varphi_{\Im}'^{2}+h^{2}\right)^{2}}, \tag{4.11}$$

where $h, u \in \mathbb{R} \setminus \{0\}$. Therefore, we obtain the required results. $\square$

If $\Omega(u) = 1$, then $H_{\Im}^{2} = K_{\Im}$. This means the surface has an umbilical point. See [7] for details.

5. Examples

In this section, we give some special examples of surfaces of revolution, helicoidal surfaces and the Gauss maps of these surfaces. We draw these surfaces with profile curve $\gamma(u) = (u, 0, \varphi(u))$ and axis z, where $-1 < u < 1$, $0 \leq v < 2\pi$, via the Maple programme.

Figure 1. Surface of revolution, $\varphi(u) = u$

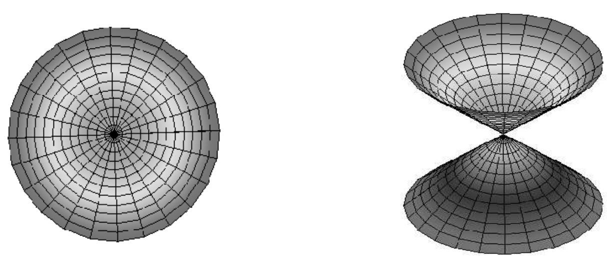


Figure 2. Helicoidal surface, $\varphi(u) = u$

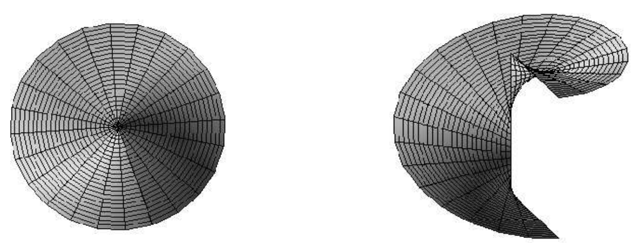


Figure 3. Gauss map of helicoidal surface, $\varphi(u) = u$

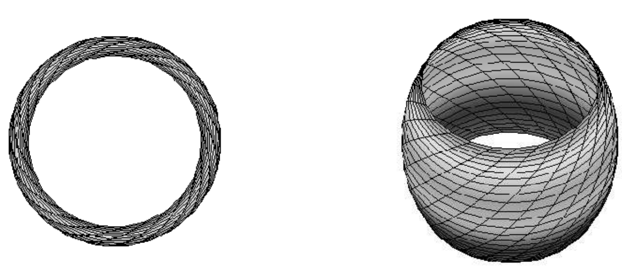


Figure 4. Surface of revolution, $\varphi(u) = u^2$

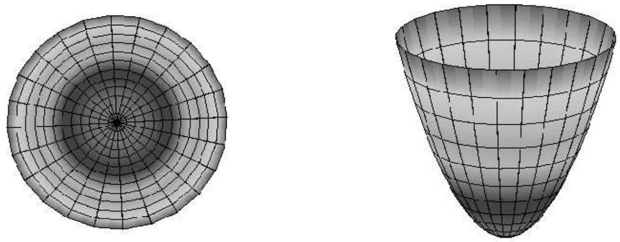


Figure 5. Helicoidal surface, $\varphi(u) = u^2$

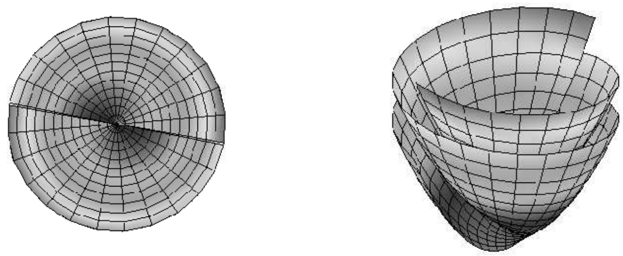


Figure 6. Gauss map of helicoidal surface, $\varphi(u) = u^2$

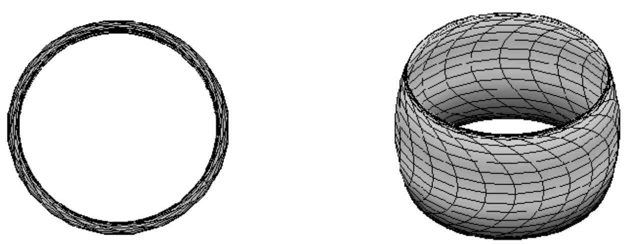


Figure 7. Surface of revolution, $\varphi(u) = \log u$

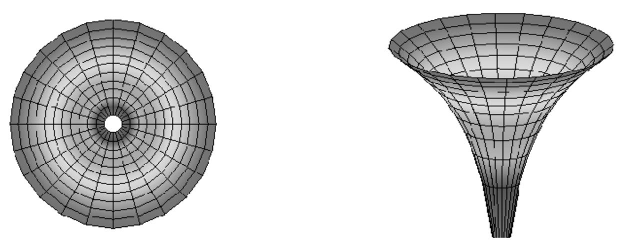


Figure 8. Helicoidal surface, $\varphi(u) = \log u$.

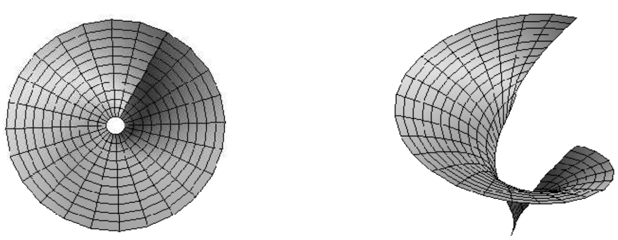


Figure 9. Gauss map of helicoidal surface, $\varphi(u) = \log u$

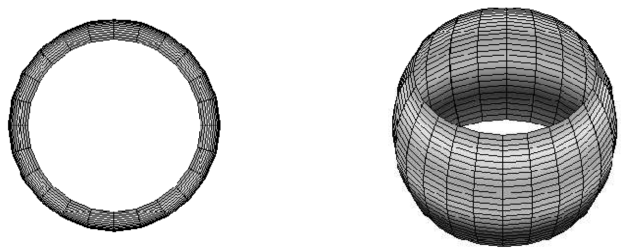


Figure 10. Surface of revolution, $\varphi(u) = e^u$

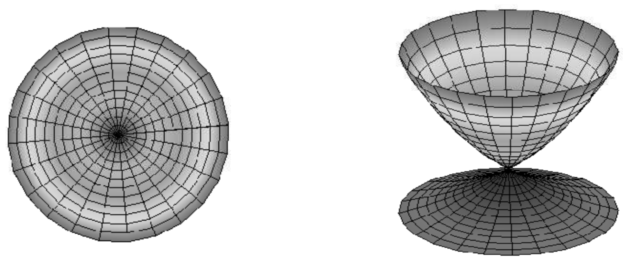


Figure 11. Helicoidal surface, $\varphi(u) = e^u$

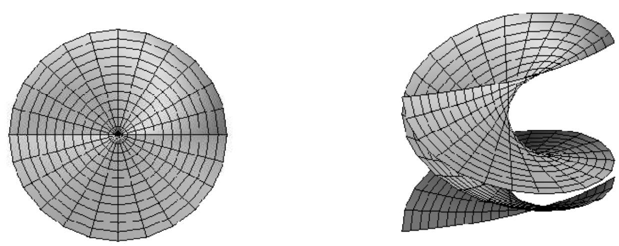
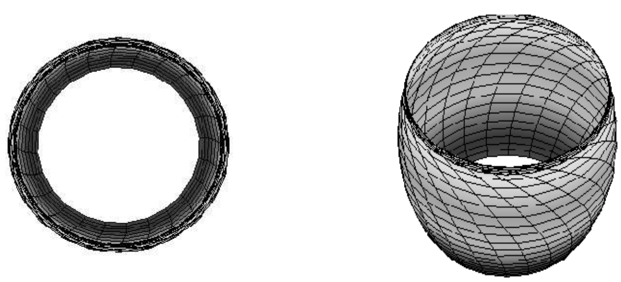


Figure 12. Gauss map of helicoidal surface, $\varphi(u) = e^u$



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