CERTAIN APPLICATIONS OF SUBORDINATION ASSOCIATED WITH NEIGHBORHOODS

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Abstract
In this paper we introduce the classes $T_p^n(\lambda, A, B)$ and $K_p^n(\lambda, \mu, A, B)$, and derive coefficient bounds and distortion inequalities for functions belonging to the class $T_p^n(\lambda, A, B)$. Further, we make use of the $(n, \delta)$-neighborhoods of functions in both classes $T_p^n(\lambda, A, B)$ and $K_p^n(\lambda, \mu, A, B)$.

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1. Introduction and definitions
Let $T_p^n$ denote the class of functions $f(z)$ normalized by

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, \ (a_k \geq 0, \ n, p \in \mathbb{N} = \{1, 2, 3, \ldots\})$$

which are analytic and multivalent in the unit disk

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Given two functions $f$ and $g$, which are analytic in $U$, the function $f$ is said to be subordinate to $g$, written

$$f \prec g \text{ and } f(z) \prec g(z)$$

if there exists a Schwarz function $w$ analytic in $U$, with

$$w(0) = 0 \text{ and } |w(z)| < 1 \ (z \in U)$$

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and such that
\[(1.3) \quad f(z) = g(w(z)) \quad (z \in U).\]

Earlier investigations have been carried out by Goodman [7] and Rucheweyh [9] (see also [1, 2, 3, 4, 5] and [8]). We define the \((n, \delta)\)-neighborhoods of functions \(f \in T^p_n\) by
\[(1.4) \quad N_{n, \delta}(f; g) = \left\{ g \in T^p_n : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \quad \text{and} \quad \sum_{k=n+p}^{\infty} k|a_k - b_k| \leq \delta \right\},\]
so that, obviously,
\[(1.5) \quad N_{n, \delta}(h; g) = \left\{ g \in T^p_n : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \quad \text{and} \quad \sum_{k=n+p}^{\infty} k|b_k| \leq \delta \right\},\]
where
\[(1.6) \quad h(z) = z^p \quad (p \in \mathbb{N}, \quad q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).\]

We also let \(T^p_n(\lambda, A, B)\) denote the subclass of \(T^p_n\) consisting of functions \(f(z)\) which satisfy the following relation
\[(1.7) \quad \frac{zf'(z) + \lambda z^2 f''(z)}{\lambda z f'(z) + (1 - \lambda)f(z)} \leq \frac{1 + Az}{1 + Bz},\]
where \(0 \leq \lambda \leq 1, \quad p \in \mathbb{N}, \quad -1 \leq A < B \leq 1\). The classes \(T^p_n(0, A, B)\) and \(T^p_n(1, A, B)\) are studied in [6].

Finally, let \(K^p_n(\lambda, \mu, A, B)\) denote the subclass of the general class \(T^p_n\) consisting of functions \(f \in T^p_n\) which satisfy the following non-homogeneous Cauchy-Euler differential equation:
\[(1.8) \quad z^p \frac{d^2w}{dz^2} + (2(p + 1))z \frac{dw}{dz} + (\mu(p + 1))w = (p + \mu)(p + \mu + 1)g,\]
where \(w = f(z) \in T^p_n, \quad g = g(z) \in T^p_n(\lambda, A, B)\) and \(\mu > -p\).

In this paper, we obtain coefficient bounds, distortion inequalities and \((n, \delta)\)-neighborhoods of functions \(f \in T^p_n\) in both the classes \(T^p_n(\lambda, A, B)\) and \(K^p_n(\lambda, \mu, A, B)\).

2. Coefficient bounds and distortion inequalities

We begin with the following lemmas.

2.1. Lemma. Let the function \(f \in T^p_n\) be defined by (1.1). Then \(f(z)\) is in the class \(T^p_n(\lambda, A, B)\) if and only if
\[(2.1) \quad \sum_{k=n+p}^{\infty} (k - p - pA + kB)(\lambda k + 1 - \lambda)a_k \leq p(B - A)(\lambda p + 1 - \lambda)\]
\[(0 \leq \lambda \leq 1, \quad p \in \mathbb{N}, \quad -1 \leq A < B \leq 1).\]

The result is sharp for the function \(f(z)\) given by
\[(2.2) \quad f(z) = z^p - \frac{p(B - A)(\lambda p + 1 - \lambda)}{(n + p)(1 + B) - p(1 + A)[\lambda(n + p) + 1 - \lambda]} z^{n+p}.\]

Proof. Let \(f \in T^p_n(\lambda, A, B)\) and
\[(2.3) \quad F(z) = \lambda zf'(z) + (1 - \lambda)f(z)\]
we find from (1.7) that
\[(2.4) \quad \frac{zF'(z)}{F(z)} < p \frac{1 + Az}{1 + Bz}.\]
and
\[
\left| \frac{zF'(z)}{F(z)} - p \right| - \left| B \frac{zF'(z)}{F(z)} - Ap \right| < 1, \ (z \in U).
\] (2.5)

Since \(|\Re(z)| \leq |z|\) for all \(z\), we have
\[
\Re \left( \frac{\sum_{k=n+1}^{\infty} (k-p)\lambda k + 1 - \lambda)ak}{p(B - A)(\lambda p + 1 - \lambda)z^p + \sum_{k=n+1}^{\infty} (pA - kB)(\lambda k + 1 - \lambda)ak} \right) < 1.
\] (2.6)

By letting \(z \to 1^+\) along the real axis, we get
\[
\sum_{k=n+1}^{\infty} (k-p-pA+kB)ak \leq p(B - A)(\lambda k + 1 - \lambda).
\] (2.7)

Conversely, let \(|z| = 1\) in (2.5). Then
\[
\left| \frac{zF'(z)}{F(z)} - p \right| - \left| B \frac{zF'(z)}{F(z)} - Ap \right| = \left| \sum_{k=n+1}^{\infty} (k-p)\lambda k + 1 - \lambda)ak \right|
\]
\[
\quad - \left| p(B - A)(\lambda p + 1 - \lambda) + \sum_{k=n+1}^{\infty} (pA - kB)(\lambda k + 1 - \lambda)ak \right|
\]
\[
\leq \sum_{k=n+1}^{\infty} (k-p-pA+kB)(\lambda k + 1 - \lambda)ak - p(B - A)(\lambda p + 1 - \lambda)
\]
\[
\leq 0.
\]

Hence, by the principle of maximum modulus, we have \(f \in T_p^0(\lambda, A, B)\), which completes the proof of Lemma 2.1. \(\square\)

2.2. Lemma. Let the function \(f(z) \in T_p^0\) defined by (1.1) be in the class \(T_p^0(\lambda, A, B)\). Then
\[
\sum_{k=n+1}^{\infty} ak \leq \frac{p(B - A)(\lambda p + 1 - \lambda)}{[(n + p)(1 + B) - p(1 + A)][\lambda(n + p) + 1 - \lambda]}.
\] (2.8)

and
\[
\sum_{k=n+1}^{\infty} k\lambda ak \leq \frac{p(B - A)(\lambda p + 1 - \lambda)(n + p)}{[(n + p)(1 + B) - p(1 + A)][\lambda(n + p) + 1 - \lambda]}.
\] (2.9)

Proof. By using Lemma 2.1, we find from (2.1) that
\[
\frac{[(n + p)(1 + B) - p(1 + A)][\lambda(n + p) + 1 - \lambda]}{\sum_{k=n+1}^{\infty} ak}
\]
\[
\leq \sum_{k=n+1}^{\infty} (k-p-pA+kB)(\lambda k + 1 - \lambda)ak
\]
\[
\leq p(B - A)(\lambda p + 1 - \lambda),
\]
which immediately yields the first assertion (2.8) of Lemma 2.2.
Next, by appealing to (2.1), we also have
\[
(\lambda(n + p) + 1 - \lambda) \left[ \sum_{k=n+p}^{\infty} (1 + B)k\lambda - p(1 + A) \sum_{k=n+p}^{\infty} a_k \right] 
\leq p(B - A)(\lambda p + 1 - \lambda),
\]
or
\[
(1 + B) \sum_{k=n+p}^{\infty} k\lambda \leq \frac{p(B - A)(\lambda p + 1 - \lambda)}{\lambda(n + p) + 1 - \lambda} + p(1 + A) \sum_{k=n+p}^{\infty} a_k.
\]
Thus, in the light of (2.8), the above inequality immediately yields the second assertion (2.9) of Lemma 2.2.

2.3. Theorem. If the function \( f \in T^2_p \) is in the class \( T^2_p(\lambda, A, B) \), then
\[
|f(z)| \leq |z|^p + \frac{p(B - A)(\lambda p + 1 - \lambda)}{[(n + p)(1 + B) - p(1 + A)][\lambda(n + p) + 1 - \lambda]} |z|^{n+p}
\]
and
\[
|f(z)| \geq |z|^p - \frac{p(B - A)(\lambda p + 1 - \lambda)}{[(n + p)(1 + B) - p(1 + A)][\lambda(n + p) + 1 - \lambda]} |z|^{n+p}.
\]
Also,
\[
|f'(z)| \leq p|z|^{p-1} + \frac{p(B - A)(\lambda p + 1 - \lambda)(n + p)}{[(n + p)(1 + B) - p(1 + A)][\lambda(n + p) + 1 - \lambda]} |z|^{n+p-1},
\]
and
\[
|f'(z)| \geq |z|^{p-1} - \frac{p(B - A)(\lambda p + 1 - \lambda)(n + p)}{[(n + p)(1 + B) - p(1 + A)][\lambda(n + p) + 1 - \lambda]} |z|^{n+p-1}.
\]

Proof. Suppose that \( f \in T^2_p \) is in the class \( T^2_p(\lambda, A, B) \). Then, from (1.1) we have
\[
|f(z)| \leq |z|^p + \sum_{k=n+p}^{\infty} a_k z^k \leq |z|^p + |z|^{n+p} \sum_{k=n+p}^{\infty} a_k
\]
and
\[
|f(z)| \geq |z|^p - |z|^{n+p} \sum_{k=n+p}^{\infty} a_k.
\]
Using (2.8), the first assertion of Lemma 2.2, in (2.14) and (2.15), we get (2.10) and (2.11).

Similarly, using (2.8) in the following inequality
\[
|f'(z) - p|z|^{p-1}| \leq (n + p) \sum_{k=n+p}^{\infty} a_k |z|^{n+p-1}
\]
we have (2.12) and (2.13). □

By setting \( \lambda = 0, n = 1 \) in Lemma 2.1 we get following result.

2.4. Corollary. (See Goel at al. [6, Theorem 1]) If \( f(z) \in T^2_p(0, A, B) \) then
\[
\sum_{n=1}^{\infty} [(1 + B)n + p(B - A)]a_{n+p} \leq p(B - A).
\]

By setting \( \lambda = 0, n = 1 \) in Theorem 2.3, we have the following result.
2.5. Corollary. (See Goel et al. [6, Theorem 3]) If \( f(z) \in T^p_n(0, A, B) \) then
\[
|z|^p - \frac{p(B - A)}{1 + B + p(B - A)} |z|^{p+1} \leq |f(z)| \leq |z|^p + \frac{p(B - A)}{1 + B + p(B - A)} |z|^{p+1}.
\]

Similarly, letting \( \lambda = 0, n = 1 \) in Theorem—2.3, we get:

2.6. Corollary. (See Goel et al. [6, Theorem 3]) If \( f(z) \in T^p_n(0, A, B) \) then
\[
p|z|^{p-1} - \frac{p(B - A)(1 + p)}{1 + B + p(B - A)} |z|^p \leq |f'(z)| \leq p|z|^{p-1} + \frac{p(B - A)p}{1 + B + p(B - A)} |z|^p.
\]

2.7. Corollary. If \( f(z) \in T^p_n(1, A, B) \) then
\[
|z|^p - \frac{p^2(B - A)}{[(n + p)(1 + B) - p(1 + A)](n + p)} |z|^{n+p}
\]
\[
\leq |f(z)| \leq |z|^p + \frac{p^2(B - A)}{[(n + p)(1 + B) - p(1 + A)](n + p)} |z|^{n+p},
\]
and
\[
p|z|^{p-1} - \frac{p^2(B - A)}{[(n + p)(1 + B) - p(1 + A)](n + p)} |z|^{n+p-1}
\]
\[
\leq |f'(z)| \leq p|z|^{p-1} + \frac{p^2(B - A)}{[(n + p)(1 + B) - p(1 + A)](n + p)} |z|^{n+p-1}.
\]

The distortion inequalities for functions in the class \( K^p_n(\lambda, \mu, A, B) \) are given in Theorem 2.8 below.

2.8. Theorem. If the function \( f \in T^p_n \) is in the class \( K^p_n(\lambda, \mu, A, B) \), then
\[
|f(z)| \leq |z|^p + \frac{p(B - A)(\lambda p + 1 - \lambda)(\mu + 1)(p + 1)}{[(n + p)(1 + B) - p(1 + A)](n + p) + 1 - \lambda[(n + p) + 1]} |z|^{n+p}
\]
and
\[
|f(z)| \geq |z|^p - \frac{p(B - A)(\lambda p + 1 - \lambda)(\mu + 1)(p + 1)}{[(n + p)(1 + B) - p(1 + A)](n + p) + 1 - \lambda[(n + p) + 1]} |z|^{n+p}.
\]

Proof. Suppose that the function \( f(z) \in T^p_n(\lambda, \mu, A, B) \) occurring in the non-homegenous differential equation (1.8) is given as in the definitions (1.4) and (1.5), with of course
\[ b_k \geq 0, \quad (k = n + p, n + p + 1, n + p + 2, \ldots). \]

Then we readily find from (1.8) that
\[
\alpha_k = \frac{(p + \mu)(p + \mu + 1)}{(k + \mu)(k + \mu + 1)} b_k, \quad (k = n + p, n + p + 1, n + p + 2, \ldots),
\]
so that
\[
f(z) = z^p - \sum_{k=n+p}^{\infty} \alpha_k z^k = z^p - \sum_{k=n+p}^{\infty} \frac{(p + \mu)(p + \mu + 1)}{(k + \mu)(k + \mu + 1)} b_k z^k, \quad (z \in \Omega)
\]
and
\[
|f(z)| \leq |z|^p + |z|^{n+p} \sum_{k=n+p}^{\infty} \frac{(p + \mu)(p + \mu + 1)}{(k + \mu)(k + \mu + 1)} b_k.
\]
Since \( g(z) \in T^n_\alpha(\lambda, A, B) \), using the first assertion (2.8) of Lemma 2.2 we have the following inequality:

\[
(2.25) \quad b_k \leq \frac{p(B-A)(\lambda p + 1 - \lambda)}{[(n+p)(1+B) - p(1+A)][\lambda(n+p) + 1 - \lambda]}
\]

Together with (2.24) and (2.25), this yields that

\[
|f(z)| \leq |z|^p + \frac{p(B-A)(\lambda p + 1 - \lambda)}{[(n+p)(1+B) - p(1+A)][\lambda(n+p) + 1 - \lambda]} \times \sum_{k=n+p}^{\infty} \frac{(p+\mu)(p+\mu+1)}{(k+\mu)(k+\mu+1)} |z|^{n+p},
\]

by using the following identity

\[
(2.27) \quad \sum_{k=n+p}^{\infty} \frac{1}{(k+\mu)(k+\mu+1)} = \sum_{k=n+p}^{\infty} \frac{1}{k+\mu} - \frac{1}{k+\mu+1} = \frac{1}{n+p+\mu},
\]

where \( \mu \in \mathbb{R} \setminus \{-n-p, -n-p-1, \ldots\} \).

The assertion (2.21) of Theorem 2.8 follows at once from (2.26) together with (2.27). The assertion (2.22) of Theorem 2.8 can be proven similarly. \( \square \)

2.9. Corollary. If \( f(z) \in K^n_\alpha(\lambda, \mu, -1, 1) \), then

\[
(2.28) \quad |f(z)| \leq |z|^p + \frac{p(\lambda p + 1 - \lambda)(p+\mu)(p+\mu+1)}{(n+p)[\lambda(n+p) + 1 - \lambda]} |z|^{n+p},
\]

and

\[
(2.29) \quad |f(z)| \leq |z|^p - \frac{p(\lambda p + 1 - \lambda)(p+\mu)(p+\mu+1)}{(n+p)[\lambda(n+p) + 1 - \lambda]} |z|^{n+p}.
\]

By setting \( A = -1, B = 1 \) in Theorem 2.8, we have the result [3, Theorem 1] of Altıntaş et al. on letting \( \alpha = 0 \).

3. Neighborhoods for the classes \( T^n_{\alpha}(\lambda, A, B) \) and \( K^n_{\alpha}(\lambda, \mu, A, B) \)

In this section, we find inclusion relations for the classes \( T^n_{\alpha}(\lambda, A, B) \) and \( K^n_{\alpha}(\lambda, \mu, A, B) \) involving the \((n, \delta)\)-neighborhoods defined by (1.4) and (1.5).

3.1. Theorem. If \( f \in T^n_{\alpha} \) is in the class \( T^n_{\alpha}(\lambda, A, B) \), then

\[
(3.1) \quad T^n_{\alpha}(\lambda, A, B) \subset N_n, \delta(h; f),
\]

where \( h(z) \) is given by (1.6) and

\[
(3.2) \quad \delta = \frac{p(B-A)(\lambda p + 1 - \lambda)(n+p)}{[(n+p)(1+B) - p(1+A)][\lambda(n+p) + 1 - \lambda]}.
\]

Proof. The relation (3.1) asserted by Theorem 3.1 follows easily from the definition (1.5) of \( N_n, \delta(h; f) \) with \( g(z) \) replaced by \( f(z) \), and the second assertion (2.9) of Lemma 2.2. \( \square \)

3.2. Theorem. If \( f \in T^n_{\alpha} \) is in the class \( K^n_{\alpha}(\lambda, \mu, A, B) \) then

\[
(3.3) \quad K^n_{\alpha}(\lambda, \mu, A, B) \subset N_n, \delta(g; f),
\]

where \( g(z) \) is given by (1.8) and

\[
(3.4) \quad \delta = \frac{p(B-A)(\lambda p + 1 - \lambda)(n+p+(p+\mu)(p+\mu+2))}{[(n+p)(1+B) - p(1+A)][\lambda(n+p) + 1 - \lambda]}.
\]
Proof. Suppose that \( f(z) \in K^p_\delta(\lambda, \mu, A, B) \). Then, upon substituting from (2.23) into the following coefficient inequality:

\[
(3.4) \quad \sum_{k=n+p}^{\infty} k|b_k - a_k| \leq \sum_{k=n+p}^{\infty} kb_k + \sum_{k=n+p}^{\infty} ka_k, \quad (a_k \geq 0, \ b_k \geq 0)
\]

we easily obtain

\[
(3.5) \quad \sum_{k=n+p}^{\infty} k|b_k - a_k| \leq \sum_{k=n+p}^{\infty} kb_k + \sum_{k=n+p}^{\infty} \frac{(p + \mu)(p + \mu + 1)}{(k + \mu)(k + \mu + 1)} kb_k.
\]

Since \( g(z) \in T^p_\delta(\lambda, A, B) \), the second assertion (2.9) of Lemma 2.2 yields

\[
(3.6) \quad kb_k \leq \frac{p(B - A)(\lambda p + 1 - \lambda)(n + p)}{[(n + p)(1 + B) - p(1 + A)][\lambda(n + p) + 1 - \lambda]}.
\]

By making use of (2.9) as well as (3.6) on the right hand side of (3.5), we find that

\[
\sum_{k=n+p}^{\infty} k|b_k - a_k| \leq \frac{p(B - A)(\lambda p + 1 - \lambda)(n + p)}{[(n + p)(1 + B) - p(1 + A)][\lambda(n + p) + 1 - \lambda]}
\times \left(1 + \sum_{k=n+p}^{\infty} \frac{(p + \mu)(p + \mu + 1)}{(k + \mu)(k + \mu + 1)}\right),
\]

which, by virtue of the teloscopic sum (2.27), immediately yields

\[
\sum_{k=n+p}^{\infty} k|b_k - a_k| \leq \frac{p(B - A)(\lambda p + 1 - \lambda)[n + (p + \mu)(p + \mu + 2)](n + p)}{[(n + p)(1 + B) - p(1 + A)][\lambda(n + p) + 1 - \lambda][(n + p + \mu)]}
= \delta.
\]

So, in definition (1.4) with \( g(z) \) interchanged by \( f(z) \), we conclude that

\( f \in N_{n, \delta}(g; f) \).

This completes the proof of Theorem 3.2. \( \square \)

If we let \( A = -1, B = 1 \) in Theorem 3.2 we have the following corollary.

3.3. Corollary. If \( f(z) \in K^p_\delta(\lambda, \mu, -1, 1) \), then

\[
K^p_\delta(\lambda, \mu, -1, 1) \subset N_{n, \delta}(g; f),
\]

where \( g(z) \) is given by (1.8) and

\[
\delta = \frac{p(\lambda p + 1 - \lambda)[n + (p + \mu)(p + \mu + 2)]}{[\lambda(n + p) + 1 - \lambda][(n + p + \mu)]}.
\]

This result was given in Altintas et al. [3, Theorem 3] for \( \alpha = 0 \).

References


