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ON ONE SIDED STRONGLY PRIME IDEALS

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Abstract

The notion of strongly prime right ideal is analogous to that of completely prime ideal in a commutative ring. We prove that the intersection of all strongly prime right ideals of a ring R coincides with the Levitzki radical of this ring. We also give various conditions on a noncommutative ring R so that R is 2-primal.

Keywords: Prime right ideal, Strongly prime right ideal, 2-primal ring, AC-ring, Regular ring.

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1. Introduction

Throughout this article R denotes an associative ring and $I\,(\neq R)$ a right ideal of R. In [6], a right ideal I in R is called a *prime right ideal* if $AB \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$ for any right ideals A, B of R. In [3], the right ideal I was defined to be *strongly prime* if for each x and y in R, $xIy \subseteq I$ and $xy \in I$ imply that either $x \in I$ or $y \in I$. Let $\mathrm{m}(R)$ (resp., $\mathrm{p}(R), \mathrm{sp}(R)$) be the set of maximal right ideals (resp., prime right ideals, strongly prime right ideals) of R. Clearly, any strongly prime right ideal is prime. But the converse need not be true. For example, the zero ideal in the ring of all $n \times n$ matrices over a division ring is a prime right ideal but not strongly prime.

Recall that a two-sided ideal P of R is *completely prime* (*completely semiprime*) if $ab \in P$ implies $a \in P$ or $b \in P$ (if $a^2 \in P$ implies $a \in P$) for $a, b \in R$. Note that any ideal (two-sided) of a ring is strongly prime if and only if it is completely prime.

The goal of this paper is to prove that the intersection $\mathrm{rad}_r(R)$ of all strongly prime right ideals coincides with the largest locally nilpotent ideal of the ring R. Also, we give some characterizations of rings through strongly prime right ideals.

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2. The right radical

Recall that a ring R is *right* (resp., *left*) *s-unital* (see [10] and [11]) if $a \in aR$ (resp., $a \in Ra$) for every a in R, and R is s-unital, if it is both right and left s-unital. It is well known that if R is an s-unital ring and F a finite subset of R, then there exist $u \in R$ such that $u = ux = xu$ for all $x \in F$ (see [8]) We shall now show that $\mathrm{m}(R) \subseteq \mathrm{sp}(R)$ even when R is s-unital and does not have an identity.

2.1. Lemma. *Let R be an s-unital ring. Then* $\mathrm{m}(R) \subseteq \mathrm{sp}(R)$.

Proof. Let I be a maximal right ideal of R and suppose that $xIy \subseteq I$ and $xy \in I$ for some $x, y \in R$. If x is not in I and y is not in I, then $R = I + xR = I + yR$. Choose $u \in R$ such that $xu = x$ and $uy = y$. Then $u = a + ys$ for some $a \in I$, $s \in R$. So, $x = xu = xa + xys \in xI + I$, which implies that $xR \subseteq xI + I$. Thus, $R = I + xR \subseteq I + xI$, that is $R = I + xI$. Now, $u = b + xc$ for some $b, c \in I$, and hence, $y = uy = by + xcy \in I$, a contradiction. □

2.2. Example. The ring of all 2×2 matrices over an integral domain with a maximal right ideal, shows that a strongly prime right ideal need not be a maximal right ideal. Indeed, the right ideal of all 2×2 matrices with zero entries in the second rows is strongly prime, but not maximal.

2.3. Corollary. *If R is an s-unital ring, then* $\mathrm{m}(R) \subseteq \mathrm{sp}(R) \subseteq \mathrm{p}(R)$. □

As is well known, the intersection of all prime right ideals of a ring is the prime radical of the ring (see [6]), and the intersection of all maximal right ideals of a ring with unity is the Jacobson radical. So one can ask the following.

2.4. Question. Is the intersection of all strongly prime right ideals of a ring an ideal in the ring?

Rosenberg in [9] defined a strongly prime right ideal as follows. Let I be a right ideal of a ring R with identity. If I has the property that for every $x \in R \setminus I$, there exists a finite subset V of R such that $(I : xV) = \{r \in R; xVr \subseteq I\} \subseteq I$, then I is called *strongly prime.*

Let $\mathrm{sr}(R)$ be the set of all strongly prime right ideals in the sense of Rosenberg. It is clear that a strongly prime right ideal in the sense of Rosenberg is a strongly prime right ideal. But the above example shows that the converse is not true.

Now we introduce a k-system, and employ it to give another condition for a right ideal to be strongly prime.

2.5. Definition. Let S be a nonempty subset of a ring R. We say that S is a *k-system* if $a, b \in S$ implies that either $ab \in S$, or there exist s is in $R \setminus I$ such that $asb \in S$.

Evidently, if I is an ideal of a ring, then $C(I)$ is a multiplicative system if and only if it is k-system. It is known that an ideal P in a ring R is a prime ideal if and only if $C(P)$ is an m-system. One can easily show that a right ideal $I\,(\neq R)$ is a prime right ideal if and only if $C(I)$ is an m-system.

For a subset S of a right s-unital ring, we have the following.

$C(S)$ is a multiplicative system implies $C(S)$ is a k-system implies $C(S)$ is a m-system.

2.6. Lemma. *Let R be a ring and $I\,(\neq R)$ a right ideal of R. Then I is strongly prime if and only if $C(I)$ is a k-system.*

Proof. $\Longrightarrow$ Let $I \in \mathrm{sp}(R)$. If $x, y \in C(I)$, then x, y is not in I, hence either xIy is not in I or xy is not in I. In the former case, there exists $a \in I$ with xay is not in I, that is, $xay \in C(I)$. In the latter case, $xy \in C(I)$. So $C(I)$ is a k-system.

$\Longleftarrow$ Let $aIb \subseteq I$ and $ab \in I$ with a, b not in I. Then $a, b \in C(I)$, so either $ab \in C(I)$ or $aub \in C(I)$ for some $u \in I$. In the former case $ab \in I \cap C(I) = \emptyset$. This is a contradiction. In the latter case, $aub \in C(I) \cap I = \emptyset$. This is also contradiction. Thus, $I \in \mathrm{sp}(R)$. □

Recall that a ring is *locally nilpotent* if every finite subset X generates a nilpotent subring. This means that there exists a positive natural number m depending on X such that the product of any m elements of X is zero.

The largest locally nilpotent ideal $L(R)$ is called the *Levitzki radical* of R.

2.7. Lemma. *The following are equivalent for any right ideal J in the ring R.*

(a) *Any k-system S such that $S \cap J \neq \emptyset$ contains $\{0\}$.*
(b) *Any multiplicative subset S of R such that the intersection of S and J is nonempty contains $\{0\}$.*
(c) *The ideal J is locally nilpotent.*

Proof. (a) $\Longrightarrow$ (b) Since any multiplicative subset S of R is an k-system, we have $S \cap J$ contains $\{0\}$.

(b) $\Longrightarrow$ (c) Since $\{0\}$ belongs to the multiplicative system generated by t, there exists a fixed positive integer n such that $t^n = 0$ for every $t \in J$. Thus, J is locally nilpotent (see [2, Theorem 53]).

(c) $\Longrightarrow$ (a) Let S be a k-system and $t \in S \cap J$. By definition, there exists $a \in R \setminus S$ such that either $tat \in S$ or $t^2 \in S$. If $tat \in S$, then either $tat^2 \in S$ or $tata_1t \in S$ for some $a_1 \in R \setminus S$. If we continue like this, $\{0\}$ belongs to S since J is a locally nilpotent right ideal. If $t^2 \in S$, it can be shown easily that $t^l = 0$ for some l. So, $S \cap J$ contains $\{0\}$. □

2.8. Theorem. *The intersection $\mathrm{rad}_r(R)$ of all strongly prime right ideals coincides with the largest locally nilpotent ideal (Levitzki radical) of this ring.*

Proof. Since every strongly prime right ideal in the sense of Rosenberg is strongly prime, we have

$$\mathrm{rad}_r(R) = \bigcap_{I_\alpha \in \mathrm{sp}(R)} I_\alpha \subseteq \bigcap_{I_\alpha \in \mathrm{sr}(R)} I_\alpha = L(R)$$

by [9, Theorem 7.1]. It remains to verify the inverse inclusion. Let J be a right ideal in R such that J does not belong to $\mathrm{rad}_r(R)$. Take a strongly prime right ideal I such that the right ideal J is not in I. This means that $C(I)$ is a k-system and $J \cap C(I) \neq \emptyset$. If J were locally nilpotent, then this would imply that the set $R \setminus I$ contains $\{0\}$ by Lemma 2.7 and Lemma 2.6. But this is impossible, and so J is not in $L(R)$. Therefore $L(R)$ is a subset of $\mathrm{rad}_r(R)$. □

3. Characterizations of rings through strongly prime right ideals

Let f be a homomorphism from a ring R onto a ring S. We set

$$\mathrm{sp}_f(R) = \{I \in \mathrm{sp}(R) : I \supseteq \mathrm{Ker} f\}.$$

The following theorem shows that there is a one to one correspondence between $\mathrm{sp}_f(R)$ and $\mathrm{sp}(S)$.

3.1. Theorem. *Let f be a homomorphism from a ring R onto a ring S. Then*

1) *$f(I) \in \mathrm{sp}(S)$ for any $I \in \mathrm{sp}_f(R)$.*

2) *$f^{-1}(I) \in \mathrm{sp}_f(R)$ for any $I \in \mathrm{sp}(S)$.*
3) *The mapping $I \to f(I)$ defines a one-to-one correspondence between $\mathrm{sp}_f(R)$ and $\mathrm{sp}(S)$.*

Proof. Straightforward. □

Let $E(R)$ be the endomorphism ring of the additive group of R. If $a \in R$, define $a_r : R \to R$ by $a_r(x) = xa$ and $a_l(x) = ax$. For any $a \in R$, the maps a_r, a_l are in $E(R)$. Let $B(R)$ be the subring of $E(R)$ generated by all the a_r and a_l for $a \in R$. The centroid of R is the set of elements in $E(R)$ which commute element-wise with $B(R)$.

In Theorem 3.1, if we take a group endomorphism in the centroid of the ring instead of a ring homomorphism, we have the following theorem.

3.2. Theorem. *Let R be a ring with identity and $f : (R,+) \to (R,+)$ an onto group endomorphism of R. If f is in the centroid of R, then 1), 2), and 3) in Theorem 3.1 are satisfied.*

Proof. 1) Let $I \in sp(R)$ and $u \in I$. Since

$$f(u)f(r) = (f(r)_r f)(u) = (f(f(r))_r)(u) = f(uf(r)) \in f(I),$$

$f(I)$ is a right ideal. Suppose that $f(I) = R$. Then $If(1) = R$, which implies $R = If(1) \subseteq I$. This is a contradiction. So $f(I)$ is proper. Now we will show that $f(I)$ is strongly prime.

Let $f(a)f(I)f(b) \subseteq f(I)$ and $f(a)f(b) \in f(I)$. We have $f(a)f(I)Rf(b) \subseteq I$. Since a strongly prime right ideal is a prime right ideal, we have either $f(a)f(I) \subseteq I$ or $f(b) \in I$. If $f(a)f(I) \subseteq I$, then $f(a)If(b) \subseteq I$ and $f(a)f(b) \in I$. But $I \in \mathrm{sp}(R)$, so either $f(a) \in I$ or $f(b) \in I$. If $f(a) \in I$, then $aRf(1) \subseteq I$. This implies that either $a \in I$ or $f(1) \in I$. If $a \in I$, then $f(a) \in f(I)$. If $f(1) \in I$, then $f(R) \subseteq I$. This contradicts the fact that f is onto. Similarly, $f(b) \in f(I)$.

2) It is clear that the right ideal $f^{-1}(K)$ is proper. Let $af^{-1}(K)b \subseteq f^{-1}(K)$, $ab \in f^{-1}(K)$ for some $a, b \in R$. Then $aKb \subseteq K$ and $f(a)b \in K$. But $f(K) \subseteq K$. This implies that $f(a)Kf(b) \subseteq K$ and $f(a)f(b) \in K$. Since K is a strongly prime right ideal, $f(a) \in K$ or $f(b) \in K$. Thus $a \in f^{-1}(K)$ or $b \in f^{-1}(K)$. Therefore $f^{-1}(K)$ is a strongly prime right ideal.

3) Now, define $\Phi : \mathrm{sp}_f(R) \to \mathrm{sp}(S)$ by $\Phi(T) = f(T)$ and $\Psi : \mathrm{sp}(S) \to \mathrm{sp}_f(R)$ by $\Psi(K) = f^{-1}(K)$. Then it is easy to show that $\Phi\Psi = 1$ and $\Psi\Phi = 1$. □

If I is a right ideal of a ring R, then we observe that the subring $N(I)$ is the set $N(I) = \{x \in R : xI \subseteq I\}$, and is called the normalizer of I in R. We set $(I : x) = \{r \in R : xr \in I\}$ for any $x \in R$.

We now characterize the rings in which every right ideal is strongly prime.

3.3. Theorem. *Let R be a right s-unital ring. The following are equivalent:*

(a) *Every proper right ideal of R is in $\mathrm{sp}(R)$.*
(b) *R is simple, and $I = (I : a)$ for any proper right ideal I of R and $a \in N(I) \setminus I$.*

Proof. (a) $\Longrightarrow$ (b) Since $\mathrm{sp}(R) \subseteq p(R)$, the ring R is simple by [6, Theorem 4.2] as this result remains true for any right s-unital ring. Let I be any proper right ideal of R. For each $a \in N(I) \setminus I$, we have $I \subseteq (I : a)$. If $x \in (I : a)$, then $ax \in I$ and $aIx \subseteq I$. Since I is strongly prime and a is not in I, we obtain $x \in I$ and $I = (I : a)$.

(b) $\Longrightarrow$ (a) Let I be a proper right ideal of R. Again by [6, Theorem 4.2], I is prime. We are going to show that $I \in \mathrm{sp}(R)$. If there were elements x, y in $R \setminus I$ such that $xIy \subseteq I$

and $xy \in I$, we would get $I \subseteq (I : x)$. Further, $xy \in I$ would yield that $y \in (I : x)$. Thus, I would be a proper subset of $(I : x)$, a contradiction. So $I \in \mathrm{sp}(R)$. □

A ring R is said to be *almost commutative* (an AC-ring) if for any prime right ideal $P(\neq R)$ of R and a is not in P, there exists an element a' such that aa' is central and aa' is not in P (see [12]).

3.4. Proposition. *Let R be an s-unital AC-ring and $I\,(\neq R)$ a right ideal of R. Then I is a prime right ideal if and only if it is strongly prime right ideal.*

Proof. It suffices to show that every prime right ideal is strongly prime, as every strongly prime right ideal is a prime right ideal in a s-unital ring. Let I be a prime right ideal of R and suppose that $ab \in I$, $aIb \subseteq I$ and b is not in I for some $a, b \in R$. Then there exists $r \in R$ such that br is central and br is not in I. Hence $abrR = aRbr \subseteq I$ implies that $a \in I$. Thus I is a strongly prime right ideal of R. □

We use rad$R$ and $N(R)$ to represent the prime radical and the set of all nilpotent elements of R, respectively. A ring R is called *2-primal* if its prime radical rad$R$ coincides with the set $N(R)$. Note that commutative rings and reduced rings (i.e., rings without nonzero nilpotent elements) are 2-primal. Thus, 2-primal rings provide a sort of bridge between commutative and noncommutative ring theory.

3.5. Proposition. *Let R be a ring and suppose that every prime right ideal of R is strongly prime. Then R is 2-primal.*

Proof. It suffices to show that $P(R)$ contains all the nilpotent elements of R, because any element of $P(R)$ is nilpotent. Suppose $x^n = 0$ for some positive integer n. If x is not in $P(R)$, then there exists a prime right ideal I such that $x \notin I$. Since

$$I^* = \{r \in R : Rr \subseteq I\}$$

is the largest two sided ideal of R which is contained in I, the element $x \notin I^*$. Since I^* is prime right, I^* is strongly prime right by hypothesis. So the ring R/I^* has no nonzero divisor of zero since any ideal of a ring is a strongly prime right ideal if and only if it is completely prime. Hence $\bar{x}^n = 0$ implies that $\bar{x} = 0$, a contradiction. □

3.6. Corollary. *Let R be an s-unital AC-ring. Then R is 2-primal.* □

3.7. Theorem. *Let R be a reduced ring with identity. Then R is regular if and only if every strongly prime right ideal is a maximal right ideal.*

Proof. Let I be a strongly prime right ideal of R. Since every one-sided ideal in a reduced and regular ring is a two-sided ideal, I is a two-sided ideal of R. Hence, I is a completely prime ideal and R/I is a domain, and hence it is a division ring. Thus, I is a maximal right ideal.

Conversely, since every completely prime ideal of R is strongly prime and hence maximal right by hypothesis, R is strongly regular by [1]. □

3.8. Theorem. *Let R be a ring with unity. Then the following are equivalent.*

(a) *R is 2-primal and* $\mathrm{sp}(R) \subseteq \mathrm{m}(R)$.
(b) $R/\mathrm{rad}R$ *is strongly regular.*

Proof. (a) $\Longrightarrow$ (b) Let $\overline{R} = R/\mathrm{rad}R$. Then $\overline{R}$ has no nonzero nilpotent elements. Since $\mathrm{sp}(R) \subseteq \mathrm{m}(R)$, we have $\mathrm{sp}(\overline{R}) \subseteq \mathrm{m}(\overline{R})$. Thus, $\overline{R}$ is strongly regular.

(b) $\Longrightarrow$ (a) It is clear that $\mathrm{sp}(\overline{R}) \subseteq \mathrm{m}(\overline{R})$. By [6, Corollary 2.2], and since any strongly prime right ideal is prime right, we have that rad$R$ is contained in every element of $\mathrm{sp}(R)$. Therefore, $\mathrm{rad}R = N(R)$. □

In [4], we proved the following:

3.9. Theorem. [4, Corollary 2] *Let R be a regular ring. Then R is reduced if and only if* $\mathrm{p}(R) \subseteq \mathrm{sp}(R)$. □

3.10. Proposition. *Let R be a regular ring. If R is 2-primal, then* $\mathrm{p}(R) \subseteq \mathrm{sp}(R)$.

Proof. By [5, Proposition 3], a regular 2-primal ring is reduced. By Theorem 3.9, every prime right ideal is strongly prime. □

We ask now the following question: If R is a 2-primal ring is it true that $\mathrm{p}(R) \subseteq \mathrm{sp}(R)$?

3.11. Proposition. *Let R be a regular ring. If I is a strongly prime right ideal, then I is modular if and only if $N(I) \setminus I \neq \emptyset$.*

Proof. Let $a \in N(I) \setminus I$. Then $aI \subseteq I$, so $I \subseteq (I : a)$. Take any b in $(I : a)$, then by the definition of $(I : a)$, we have $ab \in I$. Therefore, $aIb \subseteq I$ and $ab \in I$. Since I is a strongly prime right ideal and a is not in I, we get $(I : a) = I$. Since R is regular, there is $x \in R$ such that $axa = a$. Then $axar = ar$ for every $r \in R$, which implies $a(xar - r) = 0$. Hence $xar - r \in (I : a) = I$. So, I is modular.

Conversely, let I be modular. So there exist $e \in R$ such that $er - r \in I$ for all $r \in R$, hence $eb - b \in I$ for all $b \in I$, so $eI \subseteq I$. This means that $e \in I$, but e is not in I. Thus $N(I) \setminus I \neq \emptyset$. □

3.12. Corollary. [7] *If I is a maximal right ideal of a regular ring R, then I is modular if and only if $N(I) \setminus I \neq \emptyset$.* □

3.13. Proposition. *Let R be a right s-unital ring and I an essential strongly prime right ideal. If $L = \{x \in R : xI = 0\}$, then $L^2 = 0$.*

Proof. If $x \neq 0,\ y \neq 0$ are elements in L, then $I \cap yR \neq 0$ and $x(yr) = 0$ for some $r \in R$ such that $0 \neq yr \in I$. Suppose that $xy \neq 0$. As $yr \in I$ and $yIr = 0$, we obtain either $y \in I$ or $r \in I$. Hence y is not in I. So $r \in I$. This contradicts the fact that $yr \neq 0$. Therefore $L^2 = 0$. □

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