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SOME CHARACTERIZATIONS OF SLANT HELICES IN THE EUCLIDEAN SPACE $\mathbf{E^n}$

Ahmad T. Ali* and Melih Turgut†‡

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Abstract

In this work, the notion of a slant helix is extended to the space $\mathbb{E}^n$. First, we introduce the type-2 harmonic curvatures of a regular curve. Thereafter, by using this, we present some necessary and sufficient conditions for a curve to be a slant helix in Euclidean n-space. We also express some integral characterizations of such curves in terms of the curvature functions. Finally, we give some characterizations of slant helices in terms of type-2 harmonic curvatures.

Keywords: Euclidean n-space, Frenet equations, Slant helices, Type-2 Harmonic Curvatures.

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1. Introduction

Curves of constant slope, or so-called general helices (inclined curves), are well-known curves in the classical differential geometry of space curves. They are defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the general helix) [1, 5, 6, 9, 13]. Recently, Izumiya and Takeuchi have introduced the concept of slant helix in Euclidean 3-space $\mathbb{E}^3$ by saying that the normal lines make a constant angle with a fixed direction [10]. They characterize a slant helix by the condition that the function

$$\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)' \tag{1.1}$$

*Present address: King Abdul Aziz University, Faculty of Science, Department of Mathematics, PO Box 80203, Jeddah, 21589, Saudi Arabia.
Permanent address: Mathematics Department, Faculty of Science, Al-Azhar University, Nasr City, 11448, Cairo, Egypt. E-mail: atali71@yahoo.com

†Department of Mathematics, Buca Educational Faculty, Dokuz Eylül University, 35160 Buca, Izmir, Turkey. E-mail: Melih.Turgut@gmail.com

‡Corresponding author.

is constant. In the same space, spherical images, the tangent and the binormal indicatrix and some characterizations of such curves are presented in [11, 12]. Using the notion of a slant helix, similar works are treated by other researchers, see [2, 3, 8, 16, 21, 22].

In this work, we consider the generalization of the concept of a slant helix to Euclidean n-space $\mathbb{E}^n$, and give some characterizations for a non-degenerate slant helix. We also give an example of a slant helix in the space $\mathbb{E}^5$ to illustrate our main results.

2. Preliminaries

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^n$ be an arbitrary curve in $\mathbb{E}^n$. Recall that the curve α is said to be of unit speed (or parameterized by the arc-length function s) if $\langle \alpha'(s), \alpha'(s)\rangle = 1$, where $\langle\, ,\, \rangle$ is the standard scalar product in Euclidean space E^n given by

$$\langle \xi, \zeta \rangle = \sum_{i=1}^{n} \xi_i \, \zeta_i$$

for each $\xi = (\xi_1, \dots, \xi_n)$, $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{E}^n$. In particular, the norm of a vector $\xi \in \mathbb{E}^n$ is given by $\|\xi\| = \sqrt{\langle \xi, \xi \rangle}$.

Let $\{V_1(s), \dots, V_n(s)\}$ be the moving Frenet frame along the unit speed curve α, where the vector V_i, $(i = 1, 2, \dots, n)$ denotes the i-th Frenet vector field. Then the Frenet formulas for α are given by [9]

$$(2.1) \qquad \begin{cases} V_1'(s) = \kappa_1(s) V_2(s), \\ V_i'(s) = -\kappa_{i-1} V_{i-1}(s) + \kappa_i(s) V_{i+1}(s), \quad i = 2, 3, \dots, n-1, \\ V_n'(s) = -\kappa_{n-1}(s) V_{n-1}(s). \end{cases}$$

Recall that the function $\kappa_i(s)$ is called the i-th curvatures of α. If $\kappa_{n-1}(s) = 0$ for any (?? every ??) $s \in I$, then $V_n(s)$ is a constant vector V, and the curve α lies in a $(n-1)$-dimensional affine subspace orthogonal to V, which is isometric to the Euclidean $(n-1)$-space $\mathbb{E}^{n-1}$. If all the curvatures $\kappa_i, (i = 1, 2, \dots, n-1)$ of the curve nowhere vanish in $I \subset \mathbb{R}$, then the curve is called a non-degenerate curve. We will assume throughout this work that the curve is non-degenerate. Here, recall that a regular curve with constant Frenet curvatures is called a W-curve [19]. Also, a curve $\alpha = \alpha(s) : I \to \mathbb{E}^n$ is said to have constant curvature ratios (or be a ccr-curve) if all the quotients $\frac{\kappa_{i+1}}{\kappa_i}$ are constant [14, 18].

2.1. Definition. A unit speed curve $\alpha : I \to \mathbb{E}^n$ is called a *slant helix* if its unit principal normal V_2 makes a constant angle with a fixed direction U.

3. Some characterizations of slant helices in $\mathbf{E^n}$

Özdamar and Hacisalihoğlu [17] defined the higher order harmonic curvatures of a curve, which they used to characterize general helices, as follows:

3.1. Definition. Let α be a unit curve in $\mathbb{E}^n$. The *harmonic curvatures of* α are defined by $H_i : I \to \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, such that

$$(3.1) \qquad H_i = \begin{cases} 0, & i = 0, \\ \frac{\kappa_1}{\kappa_2}, & i = 1, \\ \dfrac{1}{\kappa_{i+1}} \Big[\kappa_i H_{i-2} + H_{i-1}' \Big], & i = 2, 3, \dots, n-2. \end{cases}$$

We will refer to the functions H_i as the *type-1 harmonic functions* of the curve. In this section we give a characterization of a slant helix in $\mathbb{E}^n$. This will lead to a new class of harmonic functions to which we give the name *type-2 harmonic curvatures* of the curve.

3.2. Theorem. *Let $\alpha : I \subset R \to \mathbb{E}^n$ be a unit speed curve in Euclidean space $\mathbb{E}^n$. Then α is a slant helix if and only if there exist C^2-functions $G_i(s)$, $i = 1, 2, \ldots, n$ satisfying the equalities*

$$(3.2)\qquad G_i = \begin{cases} \int \kappa_1 ds, & i = 1, \\ 1, & i = 2, \\ \dfrac{1}{\kappa_{i-1}}\Big[\kappa_{i-2}G_{i-2} + G'_{i-1}\Big], & i = 3, 4, \ldots, n. \end{cases}$$

and the condition

$$(3.3)\qquad \frac{dG_n}{ds} = -\kappa_{n-1}(s)G_{n-1}(s).$$

Proof. Let α be a unit speed slant helix in $\mathbb{E}^n$. Let U be the direction with which V_2 makes a constant angle θ, $(\cos\theta \neq 0)$, where without loss of generality we may suppose that $\langle U, U\rangle = 1$. Consider the differentiable functions a_i, $1 \leq i \leq n$, satisfying

$$(3.4)\qquad U = \sum_{i=1}^{n} a_i(s)\, V_i(s),\ s \in I,$$

that is,

$$a_i = \langle V_i, U\rangle,\ 1 \leq i \leq n.$$

Then the function $a_2(s) = \langle V_2(s), U\rangle$ is constant with value $\cos\theta$, that is:

$$(3.5)\qquad a_2(s) = \langle V_2, U\rangle = \cos\theta$$

for all s. Because the vector field U is constant, a differentiation in (3.4) together (2.1) gives the following system of ordinary differential equations:

$$(3.6)\qquad \begin{cases} a'_1 - \kappa_1 a_2 = 0, \\ \kappa_1 a_1 - \kappa_2 a_3 = 0, \\ a'_i + \kappa_{i-1}a_{i-1} - \kappa_i a_{i+1} = 0, \quad 3 \leq i \leq n-1 \\ a'_n + \kappa_{n-1}a_{n-1} = 0. \end{cases}$$

Because, $a_2(s) \neq 0$, we can express the functions $a_i(s)$, $(i = 1, \ldots, n)$, in terms of new function $G_i = G_i(s)$, $(i = 1, \ldots, n)$, as follows:

$$(3.7)\qquad a_i(s) = G_i(s)\, a_2(s),\ 1 \leq i \leq n.$$

So, using Equation (3.7) the first $(n-1)$-equations in (3.6) lead to the n-equalities (3.2), while the last equation of (3.6) leads to the condition (3.3) for the function $G_n(s)$.

For the converse, let α be a unit speed curve for which the functions G_i exist satisfying (3.3) and (3.2). Define the unit vector U by

$$U = \cos\theta\Bigg[\sum_{i=1}^{n} G_i\, V_i\Bigg],$$

where θ is a constant angle satisfying $\cos\theta \neq 0$. By taking into account (3.3) and (3.2), a differentiation of U gives that $\dfrac{dU}{ds} = 0$, which it means that U is a constant vector field. On the other hand, the scalar product between the unit tangent vector field V_2 with U is

$$\langle V_2(s), U\rangle = \cos\theta.$$

Thus, α is a slant helix in the space $\mathbb{E}^n$. This result completes the proof. □

We are now in a position to define the type-2 harmonic curvatures of a curve.

3.3. Definition. Let α be a unit curve in $\mathbb{E}^n$. The *type-2 harmonic curvatures* of α are the functions $G_i : I \to \mathbb{R}$, $i = 1, 2, \ldots, n$ given by (3.2), that is

$$G_i = \begin{cases} \int \kappa_1 ds, & i = 1, \\ 1, & i = 2, \\ \dfrac{1}{\kappa_{i-1}}\Big[\kappa_{i-2}G_{i-2} + G'_{i-1}\Big], & i = 3, 4, \ldots, n. \end{cases}$$

3.4. Corollary. *A unit speed curve α in $\mathbb{E}^n$ is a slant helix if and only if the type-2 harmonic curvatures G_n and G_{n-1} satisfy* (3.3)*, that is*

$$\frac{dG_n}{ds} = -\kappa_{n-1}(s)G_{n-1}(s).$$

To obtain a new characterization we make the following change of variables:

$$t(s) = \int^s \kappa_{n-1}(u)du, \quad \frac{dt}{ds} = \kappa_{n-1}(s).$$

In particular, and from Equation (3.3), we have

$$G'_{n-1}(t) = G_n(t) - \left(\frac{\kappa_{n-2}(t)}{\kappa_{n-1}(t)}\right)G_{n-2}(t).$$

As a consequence, if α is a slant helix, substituting the last equation of (3.2) into the last equation, we obtain

$$G''_n(t) + G_n(t) = \frac{\kappa_{n-2}(t)G_{n-2}(t)}{\kappa_{n-1}(t)}.$$

By the method of variation of parameters, the general solution of this equation is obtained as

$$\begin{aligned} G_n(t) = &\left(A - \int \frac{\kappa_{n-2}(t)G_{n-2}(t)}{\kappa_{n-1}(t)} \sin t\, dt\right) \cos t \\ &+ \left(B + \int \frac{\kappa_{n-2}(t)G_{n-2}(t)}{\kappa_{n-1}(t)} \cos t\, dt\right) \sin t, \end{aligned} \tag{3.8}$$

where A and B are arbitrary constants. Then (3.8) takes the following form

$$\begin{aligned} G_n(s) = &\left(A - \int \Big[\kappa_{n-2}(s)G_{n-2}(s) \sin \int \kappa_{n-1}(s)\, ds\Big]\, ds\right) \cos \int \kappa_{n-1}(s)\, ds \\ &+ \left(B + \int \Big[\kappa_{n-2}(s)G_{n-2}(s) \cos \int \kappa_{n-1}(s) ds\Big]\, ds\right) \sin \int \kappa_{n-1}(s)\, ds. \end{aligned} \tag{3.9}$$

From the last equation of (3.2), the function G_{n-1} is given by

$$\begin{aligned} G_{n-1}(s) = &\left(A - \int \Big[\kappa_{n-2}(s)G_{n-2}(s) \sin \int \kappa_{n-1}(s) ds\Big]\, ds\right) \sin \int \kappa_{n-1}(s)\, ds \\ &- \left(B + \int \Big[\kappa_{n-2}(s)G_{n-2}(s) \cos \int \kappa_{n-1}(s)\, ds\Big]\, ds\right) \cos \int \kappa_{n-1}(s)\, ds. \end{aligned} \tag{3.10}$$

From the above discussion, we give an integral characterization of a slant helix which is a consequence of Corollary 3.4.

3.5. Theorem. *Let $\alpha : I \subset R \to \mathbb{E}^n$ be a unit speed curve in Euclidean space $\mathbb{E}^n$. Then α is a slant helix if and only if the following condition is satisfied*

$$\begin{aligned} G_{n-1}(s) =& \left(A - \int \left[\kappa_{n-2}G_{n-2} \sin \int \kappa_{n-1}\, ds\right] ds\right) \sin \int^s \kappa_{n-1}(u)\, du \\ & - \left(B + \int \left[\kappa_{n-2}G_{n-2} \cos \int \kappa_{n-1}\, ds\right] ds\right) \cos \int^s \kappa_{n-1}(u)\, du. \end{aligned} \tag{3.11}$$

for some constants A and B.

Proof. Suppose that α is a slant helix. Let us define $m(s)$ and $n(s)$ by

$$\begin{aligned} m(s) &= G_n(s) \cos \phi + G_{n-1}(s) \sin \phi + \int \kappa_{n-2}G_{n-2} \sin \phi\, ds, \\ n(s) &= G_n(s) \sin \phi - G_{n-1}(s) \cos \phi - \int \kappa_{n-2}G_{n-2} \cos \phi\, ds. \end{aligned} \tag{3.12}$$

where

$$\phi(s) = \int^s \kappa_{n-1}(u)\, du,$$

and the functions G_i, $i = n-2, n-1, n$ are as in Theorem 3.2. If we differentiate equations (3.12) with respect to s and take (3.11) and (3.3) into account, we obtain $\dfrac{dm}{ds} = 0$ and $\dfrac{dn}{ds} = 0$. Therefore, there exist constants A and B such that $m(s) = A$ and $n(s) = B$. By substituting these into (3.12), and solving the equations obtained for $G_{n-1}(s)$, we get

$$G_{n-1}(s) = \left(A - \int \kappa_{n-2}G_{n-2} \sin \phi\, ds\right) \sin \phi - \left(B + \int \kappa_{n-2}G_{n-2} \cos \phi\, ds\right) \cos \phi.$$

Conversely, let G_i, $i = 1, \ldots, n-1$, satisfy (3.2) and suppose that (3.11) holds. In order to apply Theorem (3.2), we define $G_n(s)$ by

$$G_n(s) = \left(A - \int \kappa_{n-2}G_{n-2} \sin \phi\, ds\right) \cos \phi + \left(B + \int \kappa_{n-2}G_{n-2} \cos \phi\, ds\right) \sin \phi,$$

where $\phi(s) = \int^s \kappa_{n-1}(u)\, du$. A direct differentiation of (3.11) gives

$$G'_{n-1} = \kappa_{n-1}G_n - \kappa_{n-2}G_{n-2}.$$

This verifies (3.2) for $i = n$. Moreover, a straightforward computation leads to $G'_n(s) = -\kappa_{n-1}G_{n-1}$, which finishes the proof. □

4. Type-2 harmonic curvatures and slant helices

In this section we give some more characterizations for slant helices in terms of the type-2 harmonic curvatures of the curve. First, we express the following important theorem.

4.1. Theorem. *Let $\alpha : I \to \mathbb{E}^n$ be a unit speed curve in n-dimensional Euclidean space $\mathbb{E}^n$. Let $\{V_1, V_2, \ldots, V_n\}$, $\{G_1, G_2, \ldots, G_n\}$ denote the Frenet frame and the higher ordered type-2 harmonic curvatures of the curve, respectively. Then, if α is a slant helix the following condition is satisfied*

$$\sum_{i=1}^{n} G_i^2 = C, \tag{4.1}$$

where C is a non-zero constant. Moreover, the constant $C = \sec^2 \theta$, θ being the angle that V_2 makes with the fixed direction U that determines α.

Proof. Obvious from Theorem 3.2 and from the fact that the vector U is unitary. $\square$

As a direct consequence of the proof, we generalize Theorem 4.1 to Minkowski space for timelike curves, and give an another characterization for slant helices with constant curvatures.

4.2. Theorem. *Let $\mathbb{E}_1^n$ be Minkowski n-dimensional space, and let $\alpha : I \to \mathbb{E}_1^n$ be a unit speed timelike curve. Then, if α is a slant helix the following condition is satisfied: $\sum_{i=i}^n G_i^2$ is constant, where the functions G_i are defined as in (3.2).*

Proof. The proof follows the same steps as above, and we omit the details. We only point out that the fact that α is timelike means that $V_1(s) = \alpha'(s)$ is a timelike vector field. The other V_i in the Frenet frame, $2 \le i \le n$, are unit spacelike vectors and so, the second of the Frenet equations changes to $V_2' = \kappa_1 V_1 + \kappa_2 V_3$ (for details of Frenet equations see [7]). $\square$

Besides, from the definition of the type-2 harmonic curvature functions, one can easily prove the following result.

4.3. Lemma. *Let $\alpha(s)$ be a unit speed slant helix in n-dimensional Euclidean space $\mathbb{E}^n$. Let $\{V_1, V_2, \ldots, V_n\}$, $\{G_1, G_2, \ldots, G_n\}$ denote the Frenet frame and the higher ordered type-2 harmonic curvatures of the curve, respectively. Then, the following equation holds:*

$$\big\langle V_i,\, U\big\rangle = G_i\big\langle V_2,\, U\big\rangle,\ 1 \le i \le n, \tag{4.2}$$

where U is an axis of the slant helix α.

By using the above lemma, we have the following corollary:

4.4. Corollary. *If U is an axis of the slant helix α, then we can write*

$$U = \sum_{i=1}^n \lambda_i V_i.$$

From Lemma 4.3 we get

$$\lambda_i = \big\langle U,\, V_i\big\rangle = G_i\big\langle V_2,\, U\big\rangle,$$

where $\big\langle V_2,\, U\big\rangle = \cos\theta = \text{constant}$. By the definition of the type-2 harmonic curvatures of the curve, we obtain

$$U = \cos\theta\bigg(\sum_{i=1}^n G_i V_i\bigg).$$

Also

$$D = \sum_{i=1}^n G_i V_i$$

is an axis of the slant helix α.

4.5. Definition. Let $\alpha(s)$ be a non-degenerate unit speed curve in n-dimensional Euclidean space $\mathbb{E}^n$. Let $\{V_1, V_2, \ldots, V_n\}$, $\{G_1, G_2, \ldots, G_n\}$ denote the Frenet frame and the higher ordered type-2 harmonic curvatures of the curve, respectively. The vector

$$D = \sum_{i=1}^n G_i V_i, \tag{4.3}$$

is called the *type-2 Darboux vector* of the curve α.

4.6. Lemma. *Let $\alpha(s)$ be a unit speed curve in n-dimensional Euclidean space $\mathbb{E}^n$. Let $\{V_1, V_2, \ldots, V_n\}$, $\{G_1, G_2, \ldots, G_n\}$ denote the Frenet frame and the higher ordered type-2 harmonic curvatures of the curve, respectively. Then α is a slant helix if and only if D is a constant vector*

Proof. Let α be a slant helix in $\mathbb{E}^n$. From Corollary 4.4 we get

$$U = \cos\theta\bigg(\sum_{i=1}^{n} G_i V_i\bigg).$$

Since $\cos\theta$ is a constant and we can easily see that D is constant.

Conversely, if D is constant vector, then we can see that $\langle D, V_2\rangle = 1$. Thus we get $\cos\theta = \frac{1}{\|D\|}$, where θ is the constant angle between D and V_1. In this case, we can define a unique axis of the slant helix as follows:

$$U = (\cos\theta)D,$$

where $\langle U, V_1\rangle = \cos\theta =$ constant. Therefore U is a constant. So, this complete the proof. □

4.7. Corollary. *In three-dimensional Euclidean space, from equation (4.3), we can write the axis of a non-degenerate curve as:*

$$D = \bigg(\int \kappa_1 ds\bigg)V_1 + V_2 + \bigg(\frac{\kappa_1}{\kappa_2}\int \kappa_1\, ds\bigg)V_3,$$

where κ_1 and κ_2 are curvatures of the curve.

If we take derivative of D along the curve, we get

$$D' = \bigg[\kappa_2 + \bigg(\frac{\kappa_1}{\kappa_2}\int \kappa_1\, ds\bigg)'\bigg]V_3. \tag{4.4}$$

Thus, from the above equation, if the curve is a slant helix, then from Lemma 4.6, we have $D' = 0$, so we get

$$\kappa_2 + \bigg(\frac{\kappa_1}{\kappa_2}\int \kappa_1\, ds\bigg)' = 0.$$

Multiplying the above equation by $\frac{\kappa_1}{\kappa_2}\int \kappa_1\, ds$, and integrating the result we have

$$\bigg(1 + \frac{\kappa_1^2}{\kappa_2^2}\bigg)\bigg(\int \kappa_1\, ds\bigg)^2 = C^2, \tag{4.5}$$

where C is a constant of integration. It is easy to prove that Equation (4.5) is equivalent to

$$\frac{\kappa_1^2}{(\kappa_1^2 + \kappa_2^2)^{3/2}}\bigg(\frac{\kappa_2}{\kappa_1}\bigg)' = \frac{1}{C},$$

when $C \neq 0$, which is Equation (1.1).

4.8. Lemma. *There are no slant helices with non-zero constant curvatures (i.e., W-slant helices) in the space $\mathbb{E}^4$.*

Proof. In four-dimensional Euclidean space, from Equation (4.3), we get the axis of a non-degenerate curve as:

$$D = \bigg(\int \kappa_1\, ds\bigg)V_1 + V_2 + \bigg(\frac{\kappa_1}{\kappa_2}\int \kappa_1\, ds\bigg)V_3 + \frac{1}{\kappa_3}\bigg[\kappa_2 + \bigg(\frac{\kappa_1}{\kappa_2}\int \kappa_1\, ds\bigg)'\bigg]V_4,$$

where κ_1, κ_2 and κ_3 are curvatures of the curve. If all the curvatures of the curve are non-zero constants, i.e., the curve is a W-curve, then we get

$$D = (\kappa_1\, s + c)V_1 + V_2 + \frac{\kappa_1}{\kappa_2}(\kappa_1\, s + c)V_3 + \left(\frac{\kappa_1^2 + \kappa_2^2}{\kappa_2 \kappa_3}\right)V_4. \tag{4.6}$$

If we take derivative of equation (4.6) along the curve, we obtain

$$D' = \frac{\kappa_1 \kappa_3}{\kappa_2}(\kappa_1\, s + c)V_4. \tag{4.7}$$

So, we can easily see that D' is not equal to zero, then D is not constant vector. In this case, according to Lemma (4.6), the curve is not slant helix. □

4.9. Lemma. *There are no slant helices with non-zero constant curvature ratios (i.e., ccr-slant helices) in the space* $\mathbb{E}^4$.

Proof. The proof of this lemma is the same as the proof of Lemma (4.8). □

5. An example

In Euclidean space $\mathbb{E}^3$, circular helices are simple examples of general helices with constant curvatures, while Salkowski curves, anti-Salkowski curves and curves of constant precession are interesting examples of slant helices (see, for example, [4, 15, 20]). In Euclidean space $\mathbb{E}^4$, Monterde [14], Öztürk *et. al.* [18] and Camcı [6] *et. al.* introduced some examples of curves with constant curvatures (W-curves), and curves with constant curvature ratios (ccr-curves). It is worth noting that the ccr-curves are not general helices or slant helices in higher space $n \geq 4$. There are no papers introducing examples of general helices or slant helices in the Euclidean space $\mathbb{E}^n$, where $n \geq 4$. Therefore, we will introduce an example for a general helix and a slant helix in the Euclidean space $\mathbb{E}^5$ as follows:

Let $\alpha = \alpha(s) = \Big(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\Big)$ be a curve in Euclidean space $\mathbb{E}^5$ such that:

$$\begin{cases} \alpha_1 = \dfrac{-2a\cos[as]\sin[s] + (1+a^2)\sin[as]\cos[s]}{(a^2-1)r}, \\ \alpha_2 = \dfrac{(1+a^2)\cos[as]\cos[s] + 2a\sin[as]\sin[s]}{(a^2-1)r}, \\ \alpha_3 = \dfrac{-2b\cos[bs]\sin[s] + (1+b^2)\sin[bs]\cos[s]}{(b^2-1)r}, \\ \alpha_4 = \dfrac{(1+b^2)\cos[bs]\cos[s] + 2b\sin[bs]\sin[s]}{(b^2-1)r}, \\ \alpha_5 = -\sqrt{\dfrac{r^2-2}{r^2}}\cos[s]. \end{cases} \tag{5.1}$$

where $r = \sqrt{a^2+b^2}$, a and b are constants, and s is the arc-length parameter of the curve α. By differentiating the above equation with respect to s, we have

$$\begin{aligned} V_1(s) = \frac{1}{r}\big(&a\cos[as]\cos[s] + \sin[as]\sin[s],\ \cos[as]\sin[s] - a\sin[as]\cos[s], \\ &b\cos[bs]\cos[s] + \sin[bs]\sin[s],\ \cos[bs]\sin[s] - b\sin[bs]\cos[s], \\ &\sqrt{r^2-2}\sin[s]\big), \end{aligned} \tag{5.2}$$

$$\begin{aligned} V_2(s) = \frac{1}{\sqrt{a^2(a^2-1)+b^2(b^2-1)}}\big(&(1-a^2)\sin[as],\ (1-a^2)\cos[as], \\ &(1-b^2)\sin[bs],\ (1-b^2)\cos[bs],\ \sqrt{r^2-2}\big). \end{aligned} \tag{5.3}$$

Thus the vector V_2 makes a constant angle with $(0,0,0,0,1)$ and so α is a V_2-slant helix.

It is worth noting that the spherical indicatrix of the V_2-slant helix $\beta(s) = \alpha'(s) = V_1(s)$ is a general helix. On other hand, we can say that if $\beta(s)$ is a spherical V_1-helix (general helix), then $\alpha(s) = \int \beta(s)\,ds$ is a V_2-slant helix. Therefore we can write the following important lemma:

5.1. Lemma. *The unit speed curve $\alpha(s)$ is a V_2-slant helix if and only if its tangent indicatrix is a generalized helix.*

If we put $a = \sqrt{3}$ and $b = \sqrt{2}$, and use the *Mathematica Program*, we can obtain the curvatures κ_i, $(i = 1, 2, 3, 4)$ and the type-2 harmonic curvatures G_i, $(i = 1, 2, 3, 4, 5)$ of the regular curve α as follows:

$$\begin{aligned} &\kappa_1 = \sqrt{\frac{8}{5}}\cos[s], && \kappa_2 = \sqrt{\frac{19}{20} - \frac{4}{5}\cos[2s]}, \\ &\kappa_3 = \frac{\sqrt{240\cos[2s] - 705}}{38 - 32\cos[2s]}, && \kappa_4 = \frac{6\sin[s]\sqrt{32\cos[2s] - 38}}{47 - 16\cos[2s]}, \end{aligned} \tag{5.4}$$

$$\begin{aligned} &G_1 = \sqrt{\frac{8}{5}}\sin[s] && G_2 = 1, \\ &G_3 = \frac{8\sin[2s]}{\sqrt{95 - 80\cos[2s]}}, && G_4 = \frac{-21}{\sqrt{47 - 16\cos[2s]}\sqrt{57 - 48\cos[2s]}}, \\ &G_5 = \frac{2\sqrt{2}\cos[s]\sqrt{16\cos[2s] - 19}}{\sqrt{47 - 16\cos[2s]}\sqrt{57 - 48\cos[2s]}}. \end{aligned} \tag{5.5}$$

It is easy to prove that:

(1) $G_5' = \kappa_4 G_4$, Theorem 3.2.
(2) $\sum_{i=1}^{5} G_i^2 = \dfrac{8}{3}$, Theorem 4.1.
(3) $D = \sum_{i=1}^{5} G_i\, V_i = \left(0, 0, 0, 0, \dfrac{2\sqrt{2}}{\sqrt{3}}\right)$, Lemma 4.6.

6. Conclusion and further remarks

In this paper we have extended the notion of a slant helix to the space $\mathbb{E}^n$. First, we introduced the type-2 harmonic curvatures of a regular curve. Using this, some necessary and sufficient conditions for a curve to be a slant helix in Euclidean n-space are presented. Some further integral characterizations of such curves in terms of the curvature functions are also expressed. Additionally, we give some characterizations for slant helices by using the type-2 harmonic curvatures.

We can generalize the concepts of slant helices, and so the definition of type-2 harmonic curvatures of the curve in the Euclidean space $\mathbb{E}^n$. We can define a slant helix of type-k as a curve whose unit normal vector V_k makes a constant angle with a fixed direction U. Also, we can define generalized type-k harmonic functions of a curve in n-dimensional Euclidean space $\mathbb{E}^n$ as functions G_i^k. This idea may be of interest, and we may treat it in the future.

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