Hacettepe Journal of Mathematics and Statistics
Volume 39 (3) (2010), 337 – 349

ON PAIRS OF ℓ-KÖTHE SPACES

Erdal Karapınar*

Received 12 : 06 : 2009 : Accepted 26 : 02 : 2010

Abstract

Let ℓ be a Banach sequence space with a monotone norm $\|\cdot\|_\ell$, in which the canonical system (e_i) is a normalized unconditional basis. Let $a = (a_i),\ a_i \to \infty,\ \lambda = (\lambda_i)$ be sequences of positive numbers. We study the problem on isomorphic classification of pairs

$$F = \Big(K^\ell\Big(\exp\Big(-\frac{1}{p}a_i\Big)\Big), K^\ell\Big(\exp\Big(-\frac{1}{p}a_i + \lambda_i\Big)\Big)\Big).$$

For this purpose, we consider the sequence of so-called m-rectangle characteristics μ_m^F. It is shown that the system of all these characteristics is a complete quasidiagonal invariant on the class of pairs of finite-type ℓ-power series spaces. By using analytic scale and a modification of some invariants (modified compound invariants) it is proven that m-rectangular characteristics are invariant on the class of such pairs. Deriving the characteristic $\widetilde{\beta}$ from the characteristic β, and using the interpolation method of analytic scale, we are able to generalize some results of Chalov, Dragilev, and Zahariuta (*Pair of finite type power series spaces*, Note di Mathematica **17**, 121–142, 1997).

Keywords: m-rectangular characteristic, Power ℓ-Köthe spaces, Linear topological invariants.

2000 AMS Classification: 46 A 45, 46 B 45.

1. Introduction

In this article, the problem of isomorphic classification of pairs of ℓ-Köthe spaces is considered. The first result in this direction was given by V. Zakharyuta [23]. He introduced the so-called *simultaneous diametral dimension* $\Gamma(X, Y)$ for a pair of locally convex spaces (X, Y), and proved that this characteristic is invariant with respect to simultaneous isomorphisms of pairs. This invariant was applied in [23, 21] to give estimates of extendible bases of analytic functions. Later Dragilev ([15, 16]), using those invariants,

*Atılım University, Department of Mathematics, İncek 06836, Ankara, Turkey.
E-mail: `erdalkarapinar@yahoo.com`

gave a complete classification of pairs of imbedded Köthe spaces with common canonical basis which is regular in both spaces.

The invariant $\Gamma(X,Y)$ is inadequate for distinguishing pairs of Köthe spaces if their common canonical bases *are not simultaneously regular*, because it gives quite rough information about pairs of such spaces. In [7] the following class of pairs was studied:

$$(1.1)\quad F = F^{l_1}[\lambda, a] = \Big(K^{l_1}\Big(\exp\Big(-\frac{1}{p}a_i\Big)\Big), K^{l_1}\Big(exp\Big(-\frac{1}{p}a_i+\lambda_i\Big)\Big)\Big)\ a=(a_i),\ \lambda=(\lambda_i)$$

whose canonical bases may not be simultaneously regular if λ_i is oscillating. It was shown there that an appropriate version of compound linear invariants (see, e.g, [25, 26, 9, 10]) is a proper tool in order to classify those pairs. These invariants turn out to be useful in studying the isomorphism of several classes of locally convex spaces of mixed nature depending on two or more sequences; these invariants are used to show that some special m-rectangular characteristics of pairs, describing a mutual distribution of related sequences, are invariant (see, e.g., [2, 11, 26, 9, 10]). In particular, it was proved in [7] that some natural m-rectangular characteristics of the pairs (1.1), expressed through the sequences a and λ, are invariant (for more detail see Section 4 below).

In this manuscript, it is proved that m-rectangular characteristics are invariant on the class

$$(1.2)\quad F = F^{\ell}[\lambda, a] = \Big(K^{\ell}\Big(\exp\Big(-\frac{1}{p}a_i\Big)\Big), K^{\ell}\Big(exp\Big(-\frac{1}{p}a_i+\lambda_i\Big)\Big)\Big),\ a=(a_i),\ \lambda=(\lambda_i),$$

where ℓ is an arbitrary Banach sequence space with a monotone norm in which the canonical system is a normalized unconditional basis. Notice that, the case $\ell=\ell_1$ gives the results of [7]. In this generalization, the S. Krein analytic scale interpolation method plays a crucial role in the proof of the main theorem (see Lemma 3.8 below). Moreover, we need to modify compound invariants from [7], in particular, we use a basic geometric characteristic $\tilde{\beta}(V,U)$ instead of $\beta(V,U)$ (see Section 3 below). With the help of Lemma 3.9, and the basic geometric characteristic $\tilde{\beta}(V,U)$, we get that the m-rectangular characteristics on the class (1.2) are invariant.

2. Preliminaries

Let ℓ be a Banach sequence space in which $\{e_i=(\delta_{i,j})_{j\in\mathbb{N}} : i\in\mathbb{N}\}$ forms an unconditional basis. The norm $\|\cdot\|_\ell$ is called *monotone* [17] if $\|x\|_\ell \le \|y\|_\ell$ whenever $x=(\xi_i)$, $y=(\eta_i)$, $|\xi_i|\le|\eta_i|$, $i\in\mathbb{N}$. By Λ we denote the set of all such spaces ℓ with monotone norm. Such spaces are studied in [1, 22, 19, 18, 14]

Note that the matrix norm on $M_2(\mathbb{R})$ is not monotone. Indeed, take a canonical basis $e_{11}, e_{12}, e_{21}, e_{22}$ in $M_2(\mathbb{R})$ and consider the matrices $A=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}=e_{11}+2e_{12}+2e_{21}+e_{22}$ and $B=\begin{bmatrix}1 & \frac{7}{3}\\ \frac{7}{3} & -1\end{bmatrix}=e_{11}+\frac{7}{3}e_{12}+\frac{7}{3}e_{21}-e_{22}$. Note that $|a_{ij}|\le|b_{ij}|$ for all i,j. But $\|A\|=3>\frac{\sqrt{58}}{3}=\|B\|$. Indeed, $A=A^*$ and so $\det(A-\lambda I)=\begin{vmatrix}1-\lambda & 2\\ 2 & 1-\lambda\end{vmatrix}=(1-\lambda)^2-4=\lambda^2-2\lambda-3$, which implies that $\sigma(A)=\{-1,3\}$ and therefore $\|A\|=\max|\sigma(A)|=3$. On the other hand, $B=B^*$ and so $\det(B-\lambda I)=\begin{vmatrix}1 & \frac{7}{3}\\ \frac{7}{3} & -1\end{vmatrix}=-(1-\lambda)^2-\frac{49}{9}\implies\lambda=\pm\frac{\sqrt{58}}{3}$ and $\sigma(B)=\{-\frac{\sqrt{58}}{3},\frac{\sqrt{58}}{3}\}$, so $\|B\|=\max|\sigma(B)|=\frac{\sqrt{58}}{3}$.

For a given $\ell\in\Lambda$ and a Köthe matrix $A=(a_{i,n})_{i,n\in\mathbb{N}}$, the ℓ-Köthe space $X=K^\ell(A)$ [19] is defined as a Fréchet space of scalar sequences $x=(\xi_i)$ such that $(\xi_i a_{i,n})\in\ell$, for each n, with the topology generated by the system of seminorms $\{|(\xi_i)|_n := \|(\xi_i a_{i,n})\|_\ell, n \in \mathbb{N}\}$.

Set

$$\omega_+ := \{a = (a_i)_{i\in\mathbb{N}} : a_i > 1,\ i \in \mathbb{N}\}.$$

For $a \in \omega_+$ and $\lambda_n \nearrow \alpha$, $-\infty < \alpha \le \infty$, the ℓ-Köthe space

$$E^\ell_\alpha(a) := K^\ell(\exp(\lambda_n a_i)),$$

is called the *ℓ-power series space of finite (infinite) type* if $\alpha < \infty$ $(\alpha = \infty)$.

We consider pairs (X, Y) of locally convex spaces X and Y with a linear continuous injection $Y \hookrightarrow X$. Two pairs (X, Y) and $(\tilde{X}, \tilde{Y})$ are called *isomorphic* if there exists an isomorphism $T : X \to \tilde{X}$ whose restriction on Y is also an isomorphism from Y to $\tilde{Y}$. For pairs of ℓ-power series spaces of finite type, we use the notation

$$F^\ell[\lambda, a] := \Big(K^\ell\Big(\exp\Big(-\frac{1}{p}a_i)),\ K^\ell\Big(\exp(-\frac{1}{p}a_i + \lambda_i\Big)\Big)\Big),$$

where $a = (a_i),\ a_i \to \infty,\ \lambda = (\lambda_i)$ are sequences of positive numbers. Throughout this study, we consider $a_i, \lambda_i > 1,\ i \in \mathbb{N}$.

3. Invariants for pairs of ℓ-Köthe spaces

Let

$$(X, Y) := F^\ell[\lambda, a] = \Big(K^\ell(\exp\Big(-\frac{1}{p}a_i)\Big),\ K^\ell\Big(\exp(-\frac{1}{p}a_i + \lambda_i\Big)\Big)\Big),$$
$$(\tilde{X}, \tilde{Y}) := F^\ell[\tilde{\lambda}, \tilde{a}] = \Big(K^\ell\Big(\exp\Big(-\frac{1}{p}\tilde{a}_i\Big)\Big), K^\ell\Big(\exp\Big(-\frac{1}{p}\tilde{a}_i + \tilde{\lambda}_i\Big)\Big)\Big)$$

be pairs of ℓ-Köthe spaces.

An isomorphism $T : (X, Y) \to (\tilde{X}, \tilde{Y})$ is called *quasi-diagonal* if there exists a bijection $\varphi : \mathbb{N} \to \mathbb{N}$, and constants $t_i,\ i \in \mathbb{N}$, such that $Te_i := t_i e_{\varphi(i)},\ i \in \mathbb{N}$. In this case we write $(X, Y) \overset{qd}{\simeq} (\tilde{X}, \tilde{Y})$, and the pairs (X, Y) and $(\tilde{X}, \tilde{Y})$ are called *quasi-diagonally isomorphic*.

Similarly, an isomorphism T is *permutable*, (respectively, *equivalent*) if it is quasidiagonal and $t_i \equiv 1,\ \forall i$ (respectively, if it is quasidiagonal, $t_i \equiv 1$ and $\varphi(i) = i$). Analogously, we write $(X, Y) \overset{p}{\simeq} (\tilde{X}, \tilde{Y})$ (respectively, $(X, Y) \overset{e}{\simeq} (\tilde{X}, \tilde{Y})$) if there is a permutable (respectively, equivalent) isomorphism.

Let $\mathcal{R}$ be a set with an equivalence relation $\sim$ and $\mathcal{P}$ a class of pairs of locally convex spaces. We consider that $\gamma : \mathcal{P} \to \mathcal{R}$ is a *linear topological invariant* if $(X, Y) \simeq (\tilde{X}, \tilde{Y}) \implies (\gamma(X, Y)) \sim (\gamma(\tilde{X}, \tilde{Y}))$, where $(X, Y), (\tilde{X}, \tilde{Y}) \in \mathcal{P}$. For more details about linear topological invariants we refer the reader to [26].

Suppose (X, Y) is a pair of linear spaces, U and V are absolutely convex sets in Y and $\mathcal{Y}_V$ is the set of all finite dimensional subspaces of Y that are spanned on the elements of V. Set

$$\mathcal{L}(V, U) := \{L \in \mathcal{Y}_V : \exists q := q(L) < 1,\ L \cap U \subset qV\}.$$

For Banach sequence spaces with monotone norm, it is convenient to consider the characteristic

$$\tilde{\beta}(V, U) := \sup\{\dim\ L : L \in \mathcal{L}(V, U)\},$$

which is a modification of the characteristic $\beta(V, U)$ (see, e.g. [24]).

3.1. Lemma. *Let E be a linear space, U, V, U_1 and V_1 absolutely convex sets in E, and T an injective linear operator on E.*

(3.1) $\widetilde{\beta}(V_1, U_1) \leq \widetilde{\beta}(V, U)$ *if* $V_1 \subset V,\ U \subset U_1$,

(3.2) $\widetilde{\beta}(CV, U) = \widetilde{\beta}(V, \frac{1}{C}U)$ *if* $C > 0$, *and*

(3.3) $\widetilde{\beta}(T(V), T(U)) = \widetilde{\beta}(V, U)$.

Proof. Obvious and is omitted. □

For $a \in \omega_+$ we introduce the weighted ℓ-space as

$$\ell(a) := \{x = (\xi_i) : \|x\|_{\ell(a)} := \|(\xi_i\, a_i)\|_\ell < \infty\}.$$

Let E be a vector sequence space containing the system $\{e_i\}_{i\in\mathbb{N}}$. Given $a \in \omega_+$, we define the weighted ball by $B^\ell(a) = \{x \in E \cap \ell(a) : \|x\|_{\ell(a)} \leq 1\}$.

3.2. Definition. Let S be a set in $K^\ell(a_{i,n})$. Then S is *bounded* if there exists a sequence $m = (m_i)$ of non-decreasing natural numbers and a constant $C > 0$ such that $S \subset B^\ell((a_{i,m_i}))$.

For any $a^{(j)} = (a_i^{(k)}) \in \omega_+,\ j = 1, 2, \ldots, r \in \mathbb{N}$, we set

$$\bigwedge_{j=1}^{r} a^{(j)} = \big(\min\{a_i^{(1)}, a_i^{(2)}, \ldots, a_i^{(r)}\}\big)$$

and

$$\bigvee_{j=1}^{r} a^{(j)} = \big(\max\{a_i^{(1)}, a_i^{(2)}, \ldots, a_i^{(r)}\}\big).$$

The following result is an extension of [26, Proposition 5].

3.3. Lemma. *Let $a^{(1)}, a^{(2)}, \ldots, a^{(r)} \in \omega_+,\ r \in \mathbb{N}$. Then*

(i) $\frac{1}{r}B^\ell\big(\bigwedge_{j=1}^r a^{(j)}\big) \subset \operatorname{conv}\big(\bigcup_{j=1}^r B^\ell(a^{(j)})\big) \subset B^\ell\big(\bigwedge_{j=1}^r a^{(j)}\big)$;

(ii) $B^\ell\big(\bigvee_{j=1}^r a^{(j)}\big) \subset \bigcap_{j=1}^r B^\ell\big(a^{(j)}\big) \subset rB^\ell\big(\bigvee_{j=1}^r a^{(j)}\big)$. □

The notation $|S|$ will be used to show the number of elements in S if it is finite, and ∞ if S is infinite.

3.4. Lemma. $\widetilde{\beta}(B^\ell(a), B^\ell(b)) = \big|\{i : \frac{a_i}{b_i} < 1\}\big|$. □

The proofs are straightforward, (see e.g.[18]).

Let $(X, Y) = F[\lambda, a]$ be a pair of Fréchet spaces with $Y \hookrightarrow X$ such that $(U_p)_{p\in\mathbb{N}}$ (respectively, $(V_p)_{p\in\mathbb{N}}$) is a sequence of neighborhoods of 0 in X (respectively, Y) which defines the topology in X (respectively, Y). Here, we consider the following family of functions: $\beta_{(X,Y)} := (\beta(tV_q, U_p))_{p,q\in\mathbb{N}}$ where $t > 0$.

In an analogous way, we consider $(\widetilde{X}, \widetilde{Y}) = F[\tilde{\lambda}, \tilde{a}]$ with the corresponding systems of neighborhoods $(\widetilde{U}_p)_{p\in\mathbb{N}}$, $(\widetilde{V}_p)_{p\in\mathbb{N}}$, which define the topology in $\tilde{X}$ and $\tilde{Y}$, respectively. $F[\lambda, a]$ and $F[\tilde{\lambda}, \tilde{a}]$ are said to be equivalent, $\widetilde{\beta}_{(X,Y)} \sim \widetilde{\beta}_{(\tilde{X},\tilde{Y})}$, if $\forall\, p\, \exists\, p'$ and $\forall\, q'\, \exists\, q \exists\, C$ so that

$$\widetilde{\beta}(tV_q, U_p) \leq \widetilde{\beta}(Ct\widetilde{V}_{q'}, \widetilde{U}_{p'}),\ \widetilde{\beta}(\widetilde{V}_q, \widetilde{U}_p) \leq \widetilde{\beta}(CtV_{q'}, U_{p'}).$$

The single space invariant $\widetilde{\beta}_X := \beta(X, X)$, where $V_p = U_p$ and $\widetilde{V}_p = \widetilde{U}_p,\ p \in \mathbb{N}$.

3.5. Proposition. *If $(X, Y) \simeq (\widetilde{X}, \widetilde{Y})$ then, $\widetilde{\beta}_{(X,Y)} \sim \widetilde{\beta}_{(\tilde{X},\tilde{Y})}$.* □

A pair (X,Y) of ℓ-Köthe spaces $X=K^\ell(a_{im})$, $Y=K^\ell(b_{im})$ is called *regular coherently*, denoted by $(X,Y)\in\mathcal{R}_{\mathcal{C}}$, if $\frac{a_{im}}{a_{in}}\downarrow 0$, $\frac{b_{im}}{b_{in}}\downarrow 0$ for $m<n$.

3.6. Theorem. (cf.[7]) *Let $(X,Y),(\widetilde{X},\widetilde{Y})\in\mathcal{R}_{\mathcal{C}}$. Then the following are equivalent:*

(i) $(X,Y)\simeq(\widetilde{X},\widetilde{Y})$,
(ii) $(X,Y)\overset{qd}{\simeq}(\widetilde{X},\widetilde{Y})$,
(iii) $\widetilde{\beta}_X\sim\widetilde{\beta}_{\widetilde{X}}$ *and* $\widetilde{\beta}_{(X,Y)}\sim\widetilde{\beta}_{(\widetilde{X},\widetilde{Y})}$. □

3.7. Corollary. *Let $a,\tilde{a}\in\omega_+$ and $\lambda=(\lambda_i)$, $\tilde{\lambda}=(\tilde{\lambda}_i)$ be positive and tend monotonically to ∞. Then the following are equivalent:*

(i) $F^\ell[\lambda,a]\simeq F^\ell[\tilde{\lambda},\tilde{a}]$,
(ii) $F^\ell[\lambda,a]\overset{qd}{\simeq}F^\ell[\tilde{\lambda},\tilde{a}]$,
(iii) $\exists C:\ \frac{1}{C}a_i\le\tilde{a}_i\le Ca_i$ *and* $\frac{\lambda_i-\tilde{\lambda}_i}{a_i}\to 0$. □

Let us recall the definition of an analytic scale of Banach spaces [20]. Suppose that M is a normed linear space with a family of linear operators $T(z)$ satisfying the following conditions:

(i) For every $x\in M$ the function $T(z)x$ is an entire function of the complex variable z;
(ii) The function $\|T(z)x\|_M$ is bounded on every straight line parallel to the imaginary axis;
(iii) $T(0)x=x$;
(iv) $\sup\limits_{\mu,\nu}\|T(\alpha+i\mu)T(\beta+i\nu)x\|_M\le\sup\limits_{\tau}\|T(\alpha+\beta+i\tau)x\|_M$;
(v) $T(i\mu)\frac{T(z+\Delta z)x-T(z)x}{\Delta z}\to T(i\mu)(T(z)x)'$
uniformly in μ as $\Delta z\to 0$.

Define the norms

$$\|x\|_\alpha:=\sup_{-\infty<\tau<\infty}\|T(\alpha+i\tau)x\|_M$$

on the space M, and consider the completions E_α of M by these norms. The family E_α $(-\infty<\alpha<\infty)$ of Banach spaces will be called an *analytic scale of Banach spaces.*

Now we construct an analytic scale which is generated by the Banach spaces $\ell(a)$ and $\ell(b)$:

3.8. Lemma. *Let $\ell\in\Lambda$ and $a,b\in\omega_+$. Then $E_\alpha=\ell(a^{1-\alpha}b^\alpha)$ is an analytic scale such that $E_0=\ell(a)$ and $E_1=\ell(b)$.* □

Applying the interpolation theorem for analytic scales [20, Chapter IV, Theorem 1.10] to the above scale we obtain the following

3.9. Lemma. (see [19, 18]) *Suppose E and $\tilde{E}$ are ℓ-Köthe spaces, (e_i) and $(\tilde{e_i})$ are their canonical bases, and $T:E\to\tilde{E}$ a linear operator. If $a,\tilde{a},b,\tilde{b}\in\omega_+$ and*

$$T(B^\ell(a))\subset B^\ell(\tilde{a}),\ T(B^\ell(b))\subset B^\ell(\tilde{b}),$$

then for any $\alpha\in(0,1)$ we have

$$T((B^\ell(a))^{1-\alpha}(B^\ell(b))^\alpha)\subset(B^\ell(\tilde{a}))^{1-\alpha}(B^\ell(\tilde{b}))^\alpha.$$ □

4. m-rectangular Characteristics Invariants

As in [7], m-rectangular characteristics of pairs are considered on analogy with m-rectangular characteristics [6]. Notice that m-rectangular characteristics have been used to classify some kinds of Köthe spaces (see e.g. [5, 4, 13, 3]).

For a given $m \in \mathbb{N}$ we consider the *m-rectangular characteristic of the space* $F = F^{\ell}[\lambda, a]$ (with a basis) as being the function that evaluates how many points (λ_i, a_i) are contained in the union of m-rectangles:

$$\mu_m^F(\delta, \varepsilon; \tau, t) := \left| \bigcup_{k=1}^{m} \{ i : \delta_k \le \lambda_i \le \varepsilon_k,\ \tau_k \le a_i \le t_k \} \right|, \tag{4.1}$$

where $\delta = (\delta_k), \varepsilon = (\varepsilon_k),\ \tau = (\tau_k)$ and $t = (t_k),\ 1 \le \tau_k \le t_k,\ k = 1, 2, \dots, m$.

4.1. Definition. Let $F = F^{\ell}[\lambda, a]$, $\tilde{F} = F^{\ell}[\tilde{\lambda}, \tilde{a}]$ and let m be a fixed natural number. The functions μ_m^F and $\mu_m^{\tilde{F}}$ are *equivalent* (denoted by $\mu_m^F \approx \mu_m^{\tilde{F}}$) if there exists a strictly decreasing function $\varphi : (0, \infty) \to (0, \infty)$, such that $\varphi(t) \to 0$ as $t \to \infty$, and a constant $\Delta > 1$ such that the following inequalities

$$\mu_m^F(\delta, \varepsilon; \tau, t) \le \mu_m^{\tilde{F}}\big(\delta - \varphi(\tau)t, \varepsilon + \varphi(\tau)t; \tfrac{\tau}{\Delta}, \Delta t\big), \tag{4.2}$$

$$\mu_m^{\tilde{F}}(\delta, \varepsilon; \tau, t) \le \mu_m^F\big(\delta - \varphi(\tau)t, \varepsilon + \varphi(\tau)t; \tfrac{\tau}{\Delta}, \Delta t\big), \tag{4.3}$$

hold with $\varphi(\tau) = (\varphi(\tau_k)),\ \varphi(\tau)t = (\varphi(\tau_k)t_k),\ \frac{\tau}{\Delta} = \left(\frac{\tau_k}{\Delta}\right),\ \Delta t = (\Delta t_k)$, for all collections of parameters $\delta, \varepsilon, \tau, t$ defined above.

Note that the function φ and the constant Δ depends on m. The systems of characteristics $\left(\mu_m^F\right)_{m\in\mathbb{N}}$ and $\left(\mu_m^{\tilde{F}}\right)_{m\in\mathbb{N}}$ are equivalent (denoted by $\left(\mu_m^F\right) \approx \left(\mu_m^{\tilde{F}}\right)$), if φ and Δ are independent of m, that is, they can be chosen in such a way that (4.2) and (4.3) hold for all $m \in \mathbb{N}$.

We note that (μ_m^F) is a complete quasidiagonal invariant on the class of pairs $F = F^{\ell}[\lambda, a]$:

4.2. Theorem. *Let $F = F^{\ell}[\lambda, a]$ and $\widetilde{F} = F^{\ell}[\tilde{\lambda}, \tilde{a}]$. The following statements are equivalent:*

- (a) $F \overset{p}{\simeq} \widetilde{F}$,
- (b) $F \overset{qd}{\simeq} \widetilde{F}$,
- (c) $\left(\mu_m^F\right) \approx \left(\mu_m^{\tilde{F}}\right)$. □

The proof is omitted because of the analogy with the proof of [7, Theorem 15].

5. Main Results

In this section we prove that the m-rectangular characteristics are an invariant on the class (1.2). The case $\ell = \ell_1$ implies the main results of [7].

5.1. Theorem. *Let $F = F^{\ell}[\lambda, a]$ and $\tilde{F} = F^{\ell}[\tilde{\lambda}, \tilde{a}]$. If $F \simeq \tilde{F}$, then $\mu_m^F \approx \mu_m^{\tilde{F}}$ for each m.*

Proof. Since (4.3) is obtained analogously, we restrict ourselves to proving (4.2). Take an arbitrary $m \in \mathbb{N}$. Let $T : \widetilde{F} \to F$ be an isomorphism. Consider two unconditional bases of the space X: the canonical basis $e = (e_i)_{i\in\mathbb{N}}$, and the T-image of the basis of $\widetilde{X}$: $\tilde{e} = (\tilde{e}_i)_{i\in\mathbb{N}}, \tilde{e}_i = Te_i$, for each i. In terms of these bases we have two representations, $x = \sum_{i=1}^{\infty} \xi_i e_i = \sum_{i=1}^{\infty} \eta_i \tilde{e}_i$, for each $x \in X$.

For simplicity, we define weights $\alpha_p = (\alpha_{ip})$, $\gamma_p = (\gamma_{ip})$, $\tilde{\alpha}_p = (\tilde{\alpha}_{ip})$, and $\tilde{\gamma}_p = (\tilde{\gamma}_{ip})$ as follows:

$$\alpha_{ip} := \exp\Big(-\frac{1}{p}a_i\Big), \quad \gamma_{ip} := \exp\Big(-\frac{1}{p}a_i + \lambda_i\Big),$$
$$\tilde{\alpha}_{ip} := \exp\Big(-\frac{1}{p}\tilde{a}_i\Big), \quad \tilde{\gamma}_{ip} := \exp\Big(-\frac{1}{p}\tilde{a}_i + \tilde{\lambda}_i\Big).$$

For each $x \in X$, the system of norms $\big\{\|x\|_p = \|(\eta_i\tilde{\alpha}_{ip})\| : p \in \mathbb{N}\big\}$ (respectively, $\big\{\|x\|_p = \|(\eta_i\tilde{\gamma}_{ip})\| : p \in \mathbb{N}\big\}$), and the original system of norms in X, $\big\{|x|_p = \|(\xi_i\alpha_{ip})\| \ : p \in \mathbb{N}\big\}$ (respectively, $\big\{|x|_p = \|(\xi_i\gamma_{ip})\| \ : p \in \mathbb{N}\big\}$) are equivalent. Also we use the weighted balls

(5.1) $\quad B^\ell(\alpha_p)B^\ell(\gamma_p)$ with respect to the basis e.

(5.2) $\quad B^\ell(\tilde{\alpha}_p)B^\ell(\tilde{\gamma}_p)$ with respect to the basis $\tilde{e}$.

to build two pairs of synthetic neighborhoods U, V and $\widetilde{U}, \widetilde{V}$. By by using Lemma 3.1 and Lemma 3.9, the geometrical and interpolational construction of these pairs of neighborhood will provide the assertion of the theorem.

Construction of synthetic neighborhoods: Since $a_i \to \infty$, and in view of the equivalence of the systems of norms, we can choose an infinite chain of positive integers $(k = 1, 2, \dots, m-1)$:

$$r < q < s < r_i^{k+1} < q_i^{k+1} < s_i^{k+1} < r_i^k;\ s_i^{(k)}/\tilde{a}_i \to 0,\ i \in \mathbb{N}; \tag{5.3}$$

in such a way that each consequence number is four times larger than the preceding one, and that the following inclusions

$$\begin{aligned} C^{-1}B^\ell(\tilde{\alpha}_{is}) \subset B^\ell(\alpha_{iq}) \subset CB^\ell(\tilde{\alpha}_{ir}),\\ C^{-1}B^\ell(\tilde{\alpha}_{is_i}) \subset B^\ell(\alpha_{iq_i}) \subset CB^\ell(\tilde{\alpha}_{ir_i}), \end{aligned} \tag{5.4}$$

hold, together with the inclusions which are obtained by replacing the role of α with γ.

Making use of (5.3) and (5.4), the substructure of the sets $U, V, \widetilde{U}, \widetilde{V}$ may be defined in the following way:

The first couple of sets U, V is built with blocks $(k = 1, \dots, m,\ \ l = 1, 2, 3, 4)$:

$$(B^\ell((w_{i,l}^{(k)})) \quad B^\ell((\bar{w}_{i,l}^{(k)})),$$

where each weighted sequence will be responsible for a certain inequality for (λ_i) and (a_i) in (4.1). We start with the following blocks $(k = 1, \dots, m)$:

$$\begin{aligned}
\bar{w}_{i,1}^{(k)} &= (\alpha_{i,q_i^{(k)}})^{1/2}(\gamma_{i,q_i^{(k)}})^{1/2}, & w_{i,1}^{(k)} &= \Big(\alpha_{i,q_i^{(k)}}\Big)^{1/2}(\gamma_{i,q_i^{(k)}})^{1/2},\\
\bar{w}_{i,2}^{(k)} &= \exp\Big(\frac{\tau_k}{2q}\Big)(\alpha_{i,q})^{1/2}(\gamma_{i,q})^{1/2}, & w_{i,2}^{(k)} &= \exp\Big(\frac{t_k}{q}\Big)(\alpha_{i,q})^{1/2}(\gamma_{i,q})^{1/2},\\
\bar{w}_{i,3}^{(k)} &= \exp\Big(\frac{\delta_k}{2}\Big)\alpha_{i,q_i^{(k)}}, & w_{i,3}^{(k)} &= \exp\Big(-\frac{\delta_k}{2}\Big)\gamma_{i,q_i^{(k)}},\\
\bar{w}_{i,4}^{(k)} &= \exp(-\frac{\varepsilon_k}{2})\gamma_{i,q_i^{(k)}}, & w_{i,4}^{(k)} &= \exp\Big(\frac{\varepsilon_k}{2}\Big)\alpha_{i,q_i^{(k)}}.
\end{aligned}$$

To construct the second couple of sets $\widetilde{U},\ \widetilde{V}$, the corresponding series of blocks (which are balls with respect to the T-image basis $\tilde{e}$) will be used $(k = 1, \dots, m,\ l = 1, 2, 3, 4)$:

$$B^\ell((\tilde{w}_{i,l}^{(k)})) \quad B^\ell((\tilde{\bar{w}}_{i,l}^{(k)})),$$

where

$$\tilde{\bar{w}}_{i,1}^{(k)} = \big(\tilde{\alpha}_{i,s_i(k)}\big)^{1/2}\big(\tilde{\gamma}_{i,s_i(k)}\big)^{1/2}, \qquad \tilde{w}_{i,1}^{(k)} = \big(\tilde{\alpha}_{i,r_i(k)}\big)^{1/2}\big(\tilde{\gamma}_{i,r_i(k)}\big)^{1/2},$$
$$\tilde{\bar{w}}_{i,2}^{(k)} = \exp\Big(\frac{\tau_k}{2s}\Big)\big(\tilde{\alpha}_{i,s}\big)^{1/2}\big(\tilde{\gamma}_{i,s}\big)^{1/2}, \qquad \tilde{w}_{i,2}^{(k)} = \exp\Big(\frac{t_k}{r}\Big)\big(\tilde{\alpha}_{i,r}\big)^{1/2}\big(\tilde{\gamma}_{i,r}\big)^{1/2},$$
$$\tilde{\bar{w}}_{i,3}^{(k)} = \exp\Big(\frac{\delta_k}{2}\Big)\tilde{\alpha}_{i,s_i(k)}, \qquad \tilde{w}_{i,3}^{(k)} = \exp\Big(-\frac{\delta_k}{2}\Big)\tilde{\gamma}_{i,r_i(k)},$$
$$\tilde{\bar{w}}_{i,4}^{(k)} = \exp\Big(-\frac{\varepsilon_k}{2}\Big)\tilde{\gamma}_{i,s_i(k)}, \qquad \tilde{w}_{i,4}^{(k)} = \exp\Big(\frac{\varepsilon_k}{2}\Big)\tilde{\alpha}_{i,r_i(k)}.$$

After these preparations, the sets $U,\ V,\ \widetilde{U},\ \widetilde{V}$ are constructed as follows:

$$U = \bigcap_{k=1}^{m} \operatorname{conv}\Big(\bigcup_{l=1}^{4} B^{(\ell)}\big(w_{i,l}^{(k)}\big)\Big), \qquad \widetilde{U} = \bigcap_{k=1}^{m} \operatorname{conv}\Big(\bigcup_{l=1}^{4} B^{(\ell)}\big(\tilde{w}_{i,l}^{(k)}\big)\Big),$$
$$V = \operatorname{conv}\Big(\bigcup_{k=1}^{m}\bigcap_{k=1}^{4} B^{(\ell)}\big(\bar{w}_{i,l}^{(k)}\big)\Big), \qquad \widetilde{V} = \operatorname{conv}\Big(\bigcup_{k=1}^{m}\bigcap_{k=1}^{4} B^{(\ell)}\big(\tilde{\bar{w}}_{i,l}^{(k)}\big)\Big).$$

On account of Lemma 3.1, Lemma 3.9 and the inclusions (5.4),

$$B^{(\ell)}(w_{i,l}^{(k)})) \supset B^{(\ell)}(\tilde{w}_{i,l}^{(k)})), \quad B^{(\ell)}(\bar{w}_{i,l}^{(k)})) \subset B^{(\ell)}(\tilde{\bar{w}}_{i,l}^{(k)})),$$

are satisfied for $k = 1, \dots, m,\ l = 1, 2, 3, 4$, and these imply the inclusions $V \subset \widetilde{V}$ and $\widetilde{U} \subset U$.

Our difficulty here is that the sets $U,\ V,\ \widetilde{U}$ and $\widetilde{V}$ are not weighted balls. Thus, we cannot use these inclusions $V \subset \widetilde{V}$ and $\widetilde{U} \subset U$ to evaluate $\widetilde{\beta}_{(X,Y)}(V,U)$ and $\widetilde{\beta}_{(X,Y)}(\widetilde{V},\widetilde{U})$ by Lemma 3.4. On the other hand, these characteristics can be approximated by applying Lemma 3.3 and Lemma 3.9 as follows:

$$\frac{1}{m}B^{\ell}\Big(\Big(\wedge_{k=1}^{m}\bigvee_{j=1}^{4}\bar{w}_{i,j}^{(k)}\Big)\Big) \subset V \text{ and } U \subset mB^{\ell}\Big(\Big(\bigvee_{k=1}^{m}\bigwedge_{j=1}^{4} w_{i,j}^{(k)}\Big)\Big), \tag{5.5}$$

$$\widetilde{V} \subset 4B^{\ell}\Big(\Big(\bigwedge_{k=1}^{m}\vee_{j=1}^{4}\tilde{\bar{w}}_{i,j}^{(k)}\Big)\Big) \text{ and } \frac{1}{4}B^{\ell}\Big(\Big(\bigvee_{k=1}^{m}\bigwedge_{j=1}^{4}\tilde{w}_{i,j}^{(k)}\Big)\Big) \subset \widetilde{U}. \tag{5.6}$$

Considering Lemma 3.1 the following basic estimation is satisfied,

$$\widetilde{\beta}_{(X,Y)}\Big(B^{\ell}\Big(\Big(\bigwedge_{k=1}^{m}\bigvee_{j=1}^{4}\bar{w}_{i,j}^{(k)}\Big)\Big), B^{\ell}\Big(\Big(\bigvee_{k=1}^{m}\bigwedge_{j=1}^{4} w_{i,j}^{(k)}\Big)\Big)\Big) \tag{5.7}$$

$$\le \widetilde{\beta}_{(X,Y)}\Big(mV, \frac{1}{m}U\Big) \le \widetilde{\beta}_{(X,Y)}\big(C^2m^2\widetilde{V}, \widetilde{U}\big) \tag{5.8}$$

$$\le \widetilde{\beta}_{(X,Y)}\Big(16C^2m^2B^{\ell}\Big(\Big(\bigwedge_{k=1}^{m}\bigvee_{j=1}^{4}\tilde{\bar{w}}_{i,j}^{(k)}\Big)\Big), B^{\ell}\Big(\Big(\bigvee_{k=1}^{m}\bigwedge_{j=1}^{4}\tilde{v}_{i,j}^{(k)}\Big)\Big)\Big). \tag{5.9}$$

First, (5.7) is estimated from below as follows:

$$\widetilde{\beta}_{(X,Y)}\left(B^{\ell}\left(\left(\bigwedge_{k=1}^{m}\bigvee_{j=1}^{4}\bar{w}_{i,j}^{(k)}\right)\right),B^{\ell}\left(\left(\bigvee_{k=1}^{m}\bigwedge_{j=1}^{4}w_{i,j}^{(k)}\right)\right)\right)$$

$$= \left|\bigcup_{k=1}^{m}\bigcup_{\nu=1}^{m}\left\{i:\left(\bigvee_{j=1}^{4}\bar{w}_{i,j}^{(k)}\right)\le\left(\bigwedge_{j=1}^{4}w_{i,j}^{(\nu)}\right)\right\}\right|$$

$$\ge \left|\bigcup_{k=1}^{m}\left\{i:\left(\bigvee_{j=1}^{4}\bar{w}_{i,j}^{(k)}\right)\le\left(\bigwedge_{j=1}^{4}w_{i,j}^{(k)}\right)\right\}\right|$$

$$= \left|\bigcup_{k=1}^{m}\{i:\max_{1\le j\le 4}\bar{w}_{i,j}^{(k)}\le\min_{1\le j\le 4}w_{i,j}^{(k)}\}\right| \tag{5.10}$$

On account of the fact that $\bar{w}_{i,1}^{(k)}=w_{i,1}^{(k)}$, the expression inside the set in (5.10) can be written in the following way:

$$\bigcap_{j=2}^{4}\left(\{i:\bar{w}_{i,j}^{(k)}\le w_{i,1}^{(k)}\}\cap\{i:w_{i,1}^{(k)}\le\bar{w}_{i,j}^{(k)}\}\right) \tag{5.11}$$

Analyzing the inequality $\bar{w}_{i,2}^{(k)}\le w_{i,1}^{(k)}$ in (5.11) implies that

$$\tau_k\le 2q\left(\frac{1}{q}-\frac{1}{q_i^{(k)}}\right)a_i. \tag{5.12}$$

Applying (5.3), we observe $2q\left(\frac{1}{q}-\frac{1}{q_i^{(k)}}\right)\ge 1$, which yields that

$$\{i:\tau_k\le a_i\}\subset\left\{i:\tau_k\le 2q\left(\frac{1}{q}-\frac{1}{q_i^{(k)}}\right)a_i\right\}=\left\{i:\bar{w}_{i,2}^{(k)}\le w_{i,1}^{(k)}\right\}. \tag{5.13}$$

Analogously, analyzing the rest of the inequalities in (5.11) indicates the following inclusions:

$$\{i:\delta_k\le\lambda_i\}\subset\left\{i:\bar{w}_{i,3}^{(k)}\le w_{i,1}^{(k)}\right\}, \tag{5.14}$$

$$\{i:\lambda_i\le\varepsilon_k\}\subset\left\{i:\bar{w}_{i,4}^{(k)}\le w_{i,1}^{(k)}\right\}, \tag{5.15}$$

$$\{i:a_i\le t_k\}\subset\left\{i:\bar{w}_{i,1}^{(k)}\le w_{i,2}^{(k)}\right\}, \tag{5.16}$$

$$\{i:\delta_k\le\lambda_i\}\subset\left\{i:\bar{w}_{i,1}^{(k)}\le w_{i,3}^{(k)}\right\}, \tag{5.17}$$

$$\{i:\lambda_i\le\varepsilon_k\}\subset\{i:\bar{w}_{i,1}^{(k)}\le w_{i,4}^{(k)}\Big\}. \tag{5.18}$$

Combining the inclusions (5.13)-(5.18) one can obtain

$$\{i:\delta_k\le\lambda_i\le\varepsilon_k;\tau_k\le a_i\le t_k\}\subset\bigcap_{j=2}^{4}\left(\left\{i:\bar{w}_{i,j}^{(k)}\le w_{i,1}^{(k)}\right\}\cap\left\{i:w_{i,1}^{(k)}\le\bar{w}_{i,j}^{(k)}\right\}\right) \tag{5.19}$$

from which we conclude that the m-rectangular characteristic is an estimation for the expression (5.7) from below, that is,

$$\mu_m^F(\delta,\varepsilon;\tau,t)\le\widetilde{\beta}_{(X,Y)}\left(B^{\ell}\left(\left(\bigwedge_{k=1}^{m}\bigvee_{j=1}^{4}\bar{w}_{i,j}^{(k)}\right)\right),B^{\ell}\left(\left(\bigvee_{k=1}^{m}\bigwedge_{j=1}^{4}w_{i,j}^{(k)}\right)\right)\right). \tag{5.20}$$

Now, to complete the proof, it is sufficient to obtain an estimation for the expression (5.9) from above. Due to Lemma 3.4, the expression (5.9) is equal to

$$(5.21)\qquad \left|\bigcup_{k,\nu=1}^{m}\left\{i:\max_{1\le j\le 4}\tilde{\tilde{w}}_{i,j}^{(k)}\le 16m^2C^2\min_{1\le j\le 4}\tilde{w}_{i,j}^{(\nu)}\right\}\right|$$

Clearly, the set in (5.21) can be written as a subset of

$$(5.22)\qquad \begin{aligned}&\bigcap_{j=2}^{4}\left(\left\{i:\tilde{\tilde{w}}_{i,j}^{(k)}\le 16m^2C^2\tilde{w}_{i,1}^{(\nu)}\right\}\cap\left\{i:\tilde{w}_{i,1}^{(k)}\le 16m^2C^2\tilde{\tilde{w}}_{i,j}^{(\nu)}\right\}\right)\\ &\qquad\qquad\cap\left\{i:\tilde{\tilde{w}}_{i,1}^{(k)}\le 16m^2C^2\tilde{w}_{i,1}^{(\nu)}\right\}.\end{aligned}$$

Analyzing the inequality $\tilde{\tilde{w}}_{i,j}^{(k)}\le 16m^2C^2\tilde{w}_{i,1}^{(\nu)}$ in (5.22) implies that

$$(5.23)\qquad \left(\frac{1}{s_i^{(\nu)}}-\frac{1}{r_i^{(k)}}\right)\tilde{a}_i\le\ln(16m^2C^2).$$

The case $\nu\le k$ in inequality (5.23) is a trivial case, that is,

$$\left\{i:\tilde{\tilde{w}}_{i,j}^{(k)}\le 16m^2C^2\tilde{w}_{i,1}^{(\nu)}\right\}=\mathbb{N}.$$

Consider the case $\nu>k$ in inequality (5.23): By the construction of the indices (5.3), we observe that $\frac{\tilde{a}_i}{2s_i^{(\nu)}}\le\ln(16m^2C^2)$, which is weaker than the inequality (5.23). Thus we conclude that

$$(5.24)\qquad \left\{i:\tilde{\tilde{w}}_{i,1}^{(k)}\le 16m^2C^2\tilde{w}_{i,1}^{(\nu)}\right\}\subset\{i:\tilde{a}_i\le M\},$$

where $M:=\max\left\{\tilde{a}_i:a_i\le 2s_i^{(\nu)}\ln(16m^2C^2)\right\}$.

Analogously, by the definition of the weights, the inequality $\tilde{\tilde{w}}_{i,2}^{(k)}\le 16m^2C^2\tilde{w}_{i,1}^{(\nu)}$ yields the weaker inequality

$$\frac{\tau_k}{2q}-\ln(16m^2C^2)\le\frac{1}{r}\tilde{a}_i.$$

Thus, by the construction of the indices (5.3), we obtain that

$$(5.25)\qquad \left\{i:\tilde{\tilde{w}}_{i,2}^{(k)}\le 16m^2C^2\tilde{w}_{i,1}^{(\nu)}\right\}\subset\left\{i:\frac{\tau_k}{4q\ln(16m^2C^2)}\le\tilde{a}_i\right\}.$$

In a similar way,

$$(5.26)\qquad \left\{i:\tilde{\tilde{w}}_{i,2}^{(k)}\le 16m^2C^2\tilde{w}_{i,1}^{(\nu)}\right\}\subset\left\{i:\tilde{a}_i\le 4rt_\nu\ln(16m^2C^2)\right\}.$$

Combination of these three estimates (5.24), (5.25) and (5.26) can be considered as giving the following set

$$\{i:\tilde{\tilde{w}}_{i,1}^{(k)}\le 16m^2C^2\tilde{w}_{i,1}^{(\nu)},\tilde{\tilde{w}}_{i,2}^{(k)}\le 16m^2C^2\tilde{w}_{i,1}^{(\nu)},\tilde{\tilde{w}}_{i,2}^{(k)}\le 16m^2C^2\tilde{w}_{i,1}^{(\nu)}\}$$

which is a subset of the set

$$(5.27)\qquad \begin{cases}\left\{i:\frac{\tau_k}{\Delta}\le\tilde{a}_i\le\Delta t_k\right\} & \text{if } \nu>k,\\ \left\{i:\frac{\tau_\nu}{\Delta}\le\tilde{a}_i\le\Delta t_\nu\right\} & \text{if } \nu\le k,\end{cases}$$

where $\Delta:=\max\{M,4q\ln(16m^2C^2)\}$.

On the other hand, analyzing the rest of the inequalities in (5.23) yields that

$$\begin{aligned}\big\{i : \tilde{\tilde{w}}_{i,3}^{(k)} \le 16m^2C^2\tilde{w}_{i,1}^{(\nu)}\big\} \cap \big\{i : \tilde{\tilde{w}}_{i,4}^{(k)} \le 16m^2C^2\tilde{w}_{i,1}^{(\nu)}\big\}\\ \subset \big\{i : \delta_k - L\tilde{a}_i \le \tilde{\lambda}_i \le \varepsilon_k + L\tilde{a}_i\big\},\end{aligned} \tag{5.28}$$

$$\begin{aligned}\big\{i : \tilde{\tilde{w}}_{i,1}^{(k)} \le 16m^2C^2\tilde{w}_{i,4}^{(\nu)}\big\} \cap \big\{i : \tilde{\tilde{w}}_{i,1}^{(k)} \le 16m^2C^2\tilde{w}_{i,3}^{(\nu)}\big\}\\ \subset \big\{i : \delta_\nu - L\tilde{a}_i \le \tilde{\lambda}_i \le \varepsilon_\nu + L\tilde{a}_i\big\},\end{aligned} \tag{5.29}$$

where $L = L(i) := 2\Big(\frac{\ln(16m^2C^2)}{\tilde{a}_i} + \frac{1}{r_i^{(m)}}\Big)$.

Combining the observed inclusions (5.28), (5.29) and (5.22), one can conclude that the set involved in (5.21) is included in

$$R_{k,\nu} \cap \big\{i : \max\{\delta_\nu, \delta_k\} - L\tilde{a}_i < \tilde{\lambda}_i \le \max\{\varepsilon_\nu, \varepsilon_k\} + L\tilde{a}_i\big\}. \tag{5.30}$$

It is now time to clarify how to choose the function φ. Let $\psi : [0,\infty) \to [0,\infty)$ be any non-decreasing function such that $r_i^{(m)} \ge \psi(a_i)$. Any decreasing function $\varphi : [0,\infty) \to [0,\infty)$ which satisfies

$$2\delta\bigg(\frac{\delta\ln(16m^2C^2)}{\tau} + \frac{1}{\psi(\tau)}\bigg) \le \varphi(\tau), \tag{5.31}$$

and for which $\varphi(t) \to 0$ as $t \to \infty$, can be considered.

Since φ satisfies (5.31), and using (5.30), one can obtain that the set involved in (5.21) is contained in

$$\begin{aligned}\Big\{i : \delta_\nu\varphi(\tau_\nu)t_\nu < \tilde{\lambda}_i \le \varepsilon_\nu, \delta_\nu\varphi(\tau_\nu)t_\nu; \frac{\tau_\nu}{\Delta} \le \tilde{a}_i\Delta t_\nu\Big\}\\ \cup \Big\{i : \delta_k\varphi(\tau_k)t_k < \tilde{\lambda}_i \le \varepsilon_k + \delta_k\varphi(\tau_k)t_k; \frac{\tau_k}{\Delta} \le \tilde{a}_i\Delta t_k\big\}\Big\}\end{aligned} \tag{5.32}$$

for any $k, \nu = 1, \dots, m$).

As a conclusion, we get that the expression (5.9) is estimated from above, that is,

$$\begin{aligned}\widetilde{\beta}_{(X,Y)}\bigg(16C^2m^2B^\ell\bigg(\bigg(\bigwedge_{k=1}^{m}\bigvee_{j=1}^{4}\tilde{\tilde{w}}_{i,j}^{(k)}\bigg)\bigg), B^\ell\bigg(\bigg(\bigvee_{k=1}^{m}\bigwedge_{j=1}^{4}\tilde{v}_{i,j}^{(k)}\bigg)\bigg)\bigg)\\ \le \mu_m^{\tilde{F}}\Big(\delta - \varphi(\tau)t, \varepsilon + \varphi(\tau)t; \frac{\tau}{\Delta}, \Delta t\Big)\end{aligned} \tag{5.33}$$

Thus, (5.20) and (5.33) imply the inequality (4.2), which concludes the proof. ☐

5.2. Theorem. *For each m there exist $F = F^\ell[\lambda, a]$ and $\tilde{F} = F^\ell[\tilde{\lambda}, \tilde{a}]$ such that*

(i) $\mu_l^F \approx \mu_l^{\tilde{F}}$, $l = 1, 2, \dots, m$
(ii) $\mu_{m+1}^F \not\approx \mu_{m+1}^{\tilde{F}}$. ☐

5.3. Theorem. *There exist $F = F^\ell[\lambda, a]$ and $\tilde{F} = F^\ell[\tilde{\lambda}, \tilde{a}]$ such that*

(i) $\mu_m^F \approx \mu_m^{\tilde{F}}$, *for each* $m \in \mathbb{N}$,
(ii) $(\mu_m^F) \not\approx (\mu_m^{\tilde{F}})$. ☐

In [7], for $\ell = \ell_1$, the construction of $F^\ell[\lambda, a]$ and $\tilde{F}^\ell[\tilde{\lambda}, \tilde{a}]$ are considered. Theorem 5.2 and Theorem 5.3 are proved in [12] for the case $\ell = \ell_1$. In view of the analogy, the proofs of these theorems are omitted.

6. Acknowledgment

The author is deeply indebted to Prof. Dr. V. P. Zakharyuta, who suggested some important references (in Russian) and helped me to understand the main ideas of those papers. He would like to thank him also for all his support, interest and valuable hints.

References

[1] Abdeljawad, T. and Yurdakul, M. *The property of smallness up to a complemented Banach subspace*, Publ. Math. Debrecen **64** (3-4), 415–425, 2004.
[2] Chalov, P. A. *Quasiequivalence of bases for families of Hilbert spaces*, Aktualnye Voprosy Matem. Analiza, Rostov-on-Don, 167–173, 1978.
[3] Chalov, P. A. *Comparison of invariants for triples of Hilbert spaces*, Turkish J. Math. **20** (4), 493–501, 1996.
[4] Chalov, P. A. *Multirectangular characteristics for Köthe power spaces of the second kind*, Mat. Sb. **199** (3) (in Russian), 143–159, 2008 (Translation in Sb. Math. **199** (3-4), 459–475, 2008).
[5] Chalov, P. A. *Tensor products of Köthe power spaces of different types*, Mat. Zametki **83** (4) (in Russian), 629–635, 2008 (Translation in Math. Notes **83** (3-4), 573–578, 2008).
[6] Chalov, P. A., Djakov, P. B., Terzioğlu, T. and Zahariuta, V. P. *On Cartesian products of locally convex spaces. Linear topological spaces and complex analysis* (Ankara, 1995) Linear Topol. Spaces Complex Anal. **2**, 9–33, 1995.
[7] Chalov, P. A., Dragilev, M. M. and Zahariuta, V. P. *Pair of finite type power series spaces*, Note di Mathematica **17**, 121–142, 1997.
[8] Chalov, P. A., Terzioğlu, T. and Zahariuta, V. P. *First type power Köthe spaces and m-rectangular invariants*, Linear Topological Spaces and Complex Analysis **3**, METU - TÜBİTAK, Ankara, 30–44, 1997.
[9] Chalov, P. A., Terzioglu, T. and Zahariuta, V. P. *Compound invariants and mixed F-, DF-power spaces*, Canad. J. Math. **50** (6), 1138–1162, 1998.
[10] Chalov, P. A., Terzioğlu, T. and Zahariuta, V. P. *Multirectangular invariants for power Köthe spaces*, J. Math. Anal. Appl. **297** (2), 673–695, 2004.
[11] Chalov, P. A. and Zakharyuta, V. P. *Invariants for the class of families of Hilbert spaces*, School on the theory of operators in functional spaces (abstract reports (in Russian)), Chelyabinsk **43**, 1986.
[12] Chalov, P. A. and Zahariuta, V. P. *On quasidiagonal isomorphism of generalized power spaces*, Linear Topological Spaces and Complex Analysis **2**, METU - TÜBİTAK, Ankara, 35–44, 1995.
[13] Chalov, P. A. and Zakharyuta, V. P. *Finite families of l_p-spaces, and multirectangular characteristics*, Sibirsk. Mat. Zh **42** (3) (in Russian), 538–549, 2001 (Translation in Siberian Math. J. **42** (3), 455–464, 2001).
[14] Djakov, P., Terziouglu, T., Yurdakul, M. and Zahariuta, V. *Bounded operators and complemented subspaces of Cartesian products*, Math. Nachr. (to appear).
[15] Dragilev, M. M. *Extendable bases of certain Köthe spaces*, Sibirsk. Mat. Z. **14** (in Russian), 878–882, 1973.
[16] Dragilev, M. M. *Compatibly regular bases of Köthe spaces*, Mat. Zametki **19** (1) (in Russian), 115–122, 1976.
[17] Dragilev, M. M. *Basis in Köthe spaces* (Rostov State University, Rostov-on-Don, 1983, 2003) (in Russian).
[18] Karapınar, E. *Multirectangular characteristic invariants for power ℓ-Köthe spaces of first type*, J. Math. Anal. Appl. **335** (1), 79–92, 2007.
[19] Karapınar, E., Yurdakul, M. and Zahariuta, V. P. *Isomorphisms of Cartesian products of ℓ-power series spaces*, Bull. Polish Acad. Sci. Math. **54** (2), 103–111, 2006.
[20] Krein, S. G, Petunin, J. I and Semenov, E. M . *Interpolation of linear operators* (American Mathematical Society, Providence, Rhode Island, 1978).
[21] Nguyen, T. V. *Bases de Schauder dans certains espaces de fonctions holomorphes*, Ann. Inst. Fourier (Grenoble) **22** (2) (in French), 169–253, 1972.
[22] Terzioğlu, T. *Diametral dimension and Köthe spaces*, Turkish J. Math. **32** (2), 213–218, 2008.
[23] Zakharyuta, V. P. *Continuable bases in spaces of analytic functions of one and several variables*, Sibirsk. Mat. Z. **8** (in Russian), 277–292, 1967.

[24] Zakharyuta, V. P. *Generalized Mityagin invariants and a continuum of mutually nonisomorphic spaces of analytic functions*, Funktsional Anal. i Prilozhen. **11** (in Russian), 24–30, 1977.

[25] Zakharyuta, V. P. *Synthetic diameters and linear topological invariant*, School on the theory of operators in functional spaces (abstracts report (in Russian)), Minsk, 51–52, 1978.

[26] Zakharyuta, V. P. *Linear topologic invariants and their applications to isomorphic classification of generalized power spaces*, Turkish J. Math. **20**, 237–289, 1996.