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# A NEW VIEW OF FUZZY GAMMA RINGS

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**Abstract**

The aim of this paper is to define a new kind of fuzzy gamma ring. So the concepts of fuzzy gamma ring, fuzzy ideal, fuzzy quotient gamma ring, and fuzzy gamma homomorphism are introduced.

**Keywords:** Gamma ring, Fuzzy ideal, Fuzzy quotient gamma ring, Canonical gamma homomorphism, Fuzzy binary relation.

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## 1. Introduction

In 1965, L. A. Zadeh introduced the notion of a fuzzy subset of a set as a method for representing uncertainty. Then, in 1971, A. Rosenfeld used the notion of a fuzzy subset of a set to introduce the notion of a fuzzy subgroup of a group. Rosenfeld's paper inspired the development of fuzzy abstract algebra. After these studies, many mathematicians have studied these subject. For more details, see [11].

In [4, 5], M. Demirci introduced the concept of smooth group by using a fuzzy binary-operation and the concept of fuzzy equality, and then this concept was applied to a new kind of fuzzy group based on a fuzzy binary operation by X. Yuan and E. S. Lee [17]. Recently H. Aktaş and N. Çağman [1] considered a type of fuzzy ring based on Yuan and Lee's definition of a fuzzy group.

In [13], N. Nobusawa introduced the notion of a $\Gamma$-ring, which is more general than a ring. W. E. Barnes [2] weakened slightly the conditions in the definition of a $\Gamma$-ring in the sense of Nobusawa. After these two papers were published, many mathematicians obtained interesting results on $\Gamma$-rings in the sense of Barnes and Nobusawa which paralleled results in ring theory.

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In [7], Jun and Lee introduced the concept of fuzzy Γ-ring. After this study several mathematicians worked on this subject, see for instance [8, 9, 14, 15].

In this paper, we define a new kind of fuzzy gamma ring. We obtain the fuzzy quotient gamma ring induced by fuzzy ideals, and present some fuzzy gamma homomorphism theorems.

## 2. Preliminaries

In this section we summarize the preliminary definitions that will be required in this paper. Most of the contents of this section are contained in [2, 7] and [17].

**2.1. Definition.** [2] If $M = \{a, b, c, \dots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \dots\}$ are additive abelian groups and for all $a, b, c \in M$ and all $\alpha, \beta \in \Gamma$, the following conditions are satisfied

(i) $a\alpha b \in M$;
(ii) $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\alpha(b+c) = a\alpha b + a\alpha c$;
(iii) $(a\alpha b)\beta c = a\alpha(b\beta c)$;

then $M$ is called a Γ-ring.

**2.2. Definition.** [2] Let $M$ be a Γ-ring. A subset $U$ of $M$ is a *left* (*right*) *ideal* of $M$ if $U$ is an additive subgroup of $M$ and

$$M\Gamma U = \{a\alpha u \mid a \in M,\ \alpha \in \Gamma,\ u \in U\}\ (U\Gamma M)$$

is contained in $U$. If $U$ is both a left an right ideal, then $U$ is a *two-sided ideal*, or simply an *ideal* of $M$.

Let $M$ and $M'$ be two Γ-rings. A mapping $f : M \to M'$ of Γ-rings is called a Γ-*homomorphism* if $f(x+y) = f(x) + f(y)$ and $f(x\gamma y) = f(x)\gamma f(y)$ for all $x, y \in M$ and all $\gamma \in \Gamma$. If $f$ is one-to-one and onto, we say that $f$ is a Γ-*isomorphism* and that $M$ and $M'$ are Γ-*isomorphic*, denoted by $M \cong M'$.

**2.3. Definition.** [7] Let $M$ be a Γ-ring. A fuzzy subset $\mu$ of a Γ-ring $M$ is called a *fuzzy sub-Γ-ring of M* if

i) $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$,
ii) $\mu(x\gamma y) \geq \max\{\mu(x), \mu(y)\}$,

for all $x, y \in M$ and for all $\gamma \in \Gamma$.

**2.4. Definition.** [7] A fuzzy subset $\mu$ of a Γ-ring $M$ is called a *fuzzy left* (resp. *right*) *ideal of M* if

i) $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$,
ii) $\mu(x\gamma y) \geq \mu(y)$ $(\mu(x\gamma y) \geq \mu(x))$,

for all $x, y \in M$ and for all $\gamma \in \Gamma$.

**2.5. Definition.** [17] Let $G$ be a nonempty set and $R$ a fuzzy subset of $G \times G \times G$. Then $R$ is called a *fuzzy binary operation on G* if

(i) $\forall\, a, b \in G$, $\exists\, c \in G$ such that $R(a, b, c) > \theta$,
(ii) $\forall\, a, b, c_1, c_2 \in G$, $R(a, b, c_1) > \theta$ and $R(a, b, c_2) > \theta$ implies $c_1 = c_2$,

where $\theta \in [0, 1)$ is a fixed number.

Let $R$ be a fuzzy binary operation on $G$. Then we may regard $R$ as a mapping $R : F(G) \times F(G) \to F(G)$, where

$$F(G) = \{A \mid A : G \to [0, 1] \text{ is a mapping}\}$$

and for $A, B \in F(G)$, $R(A,B)$ is defined by

$$R(A,B)(c) = \bigvee_{a,b \in G} (A(a) \wedge B(b) \wedge R(a,b,c)) \tag{2.1}$$

for all $c \in G$.

For $A = \{a\}$ and $B = \{b\}$ we denote $R(A,B)$ by $a \circ b$. Then

$$(a \circ b)(c) = R(a,b,c), \text{ for all } c \in G, \tag{2.2}$$

$$((a \circ b) \circ c)(z) = \bigvee_{d \in G} (R(a,b,d) \wedge R(d,c,z), \text{ for all } z \in G, \tag{2.3}$$

$$(a \circ (b \circ c))(z) = \bigvee_{d \in G} (R(b,c,d) \wedge R(a,d,z)), \text{ for all } z \in G. \tag{2.4}$$

**2.6. Definition.** [17] Let $G$ be a nonempty set and $R$ a fuzzy binary operation on $G$. Then $(G,R)$ is called a *fuzzy group* if the following conditions are true:

(G1) $\forall a,b,c,z_1,z_2 \in G$, $((a \circ b) \circ c)(z_1) > \theta$ and $(a \circ (b \circ c))(z_2) > \theta$ implies $z_1 = z_2$;
(G2) $\exists e_0 \in G$ such that $(e_0 \circ a)(a) > \theta$ and $(a \circ e_0)(a) > \theta$ for any $a \in G$ ($e_0$ is unique and is called the *identity element* of $G$);
(G3) $\forall a \in G$, $\exists b \in G$ such that $(a \circ b)(e_0) > \theta$ ($b$ is unique and is called the *inverse element of* $a$, denoted by $a^{-1}$).

## 3. Results

Let $M$ and $\Gamma$ be nonempty sets, $R_M$ a fuzzy binary operation on $M$ and $R_\Gamma$ on $\Gamma$. Hence, $R_M$ is a fuzzy subset of $M \times M \times M$, and $R_\Gamma$ a fuzzy subset of $\Gamma \times \Gamma \times \Gamma$. We assume throughout that the value of $\theta$ is the same for $R_M$ and $R_\Gamma$.

Let $(M, R_M)$ and $(\Gamma, R_\Gamma)$ be fuzzy groups. We now define a new fuzzy binary operation $S$ on $(M, \Gamma)$ which is a fuzzy subset of $M \times \Gamma \times M \times \Gamma \times M$.

**3.1. Definition.** Let $M$ and $\Gamma$ be two nonempty sets and $S$ a fuzzy subset of $M \times \Gamma \times M \times \Gamma \times M$. Then $S$ is called a *fuzzy binary operation on* $(M, \Gamma)$ if

(i) $\forall a, b \in M$, $\forall \alpha, \beta \in \Gamma$, $\exists c \in M$ such that $S(a, \alpha, b, \beta, c) > \theta$,
(ii) $\forall a, b, c_1, c_2 \in M$, $\forall \gamma \in \Gamma$, $\bigvee_{\beta \in \Gamma} S(a, \gamma, b, \beta, c_1) > \theta$ and $\bigvee_{\beta \in \Gamma} S(a, \gamma, b, \beta, c_2) > \theta$ implies $c_1 = c_2$,

where $\theta \in [0, 1)$ is as above for $R_M$ and $R_\Gamma$.

Let $S$ be a fuzzy binary operation on $(M, \Gamma)$. Then we may regard $S$ as the mapping

$$S : F(M) \times F(\Gamma) \times F(M) \to F(M), \ (A, G, B) \mapsto S(A, G, B),$$

where

$$F(M) = \{A \mid A : M \to [0,1] \text{ is a mapping}\},$$
$$F(\Gamma) = \{G \mid G : \Gamma \to [0,1] \text{ is a mapping}\},$$

and

$$S(A,G,B)(c) = \bigvee_{\substack{a,b \in M \\ \alpha,\beta \in \Gamma}} (A(a) \wedge G(\alpha) \wedge B(b) \wedge S(a,\alpha,b,\beta,c)), \ \forall c \in M. \tag{3.1}$$

Let $A = \{a\}$, $B = \{b\}$, $G = \{\alpha\}$ and $G' = \{\alpha'\}$. Let $R_M(A,B)$, $R_\Gamma(G,G')$ and $S(A,G,B)$ be denoted by $a \circ b$, $\alpha \circ \alpha'$ and $a * \alpha * b$, respectively. We will use the following notation to simplify the calculations:

$$(a*\alpha*b)(c)=\bigvee_{\alpha'\in\Gamma} S(a,\alpha,b,\alpha',c) \text{ for all } c\in M, \tag{3.2}$$

$$((a*\alpha*b)*\beta*c)(z)=\bigvee_{\substack{d\in M\\ \alpha',\beta'\in\Gamma}} (S(a,\alpha,b,\alpha',d)\wedge S(d,\beta,c,\beta',z)), \tag{3.3}$$

$$(a*\alpha*(b*\beta*c))(z)=\bigvee_{\substack{d\in M\\ \alpha',\beta'\in\Gamma}} (S(b,\beta,c,\alpha',d)\wedge S(a,\alpha,d,\beta',z)), \tag{3.4}$$

$$(a*\alpha*(b\circ c))(z)=\bigvee_{\substack{d\in M\\ \alpha'\in\Gamma}} (R_M(b,c,d)\wedge S(a,\alpha,d,\alpha',z)), \tag{3.5}$$

$$((a*\alpha*b)\circ(a*\alpha*c))(z)=\bigvee_{\substack{d,e\in M\\ \alpha',\beta'\in\Gamma}} (S(a,\alpha,b,\alpha',d)\wedge S(a,\alpha,c,\beta',e)\wedge R_M(d,e,z)), \tag{3.6}$$

$$(a*(\alpha\circ\beta)*b)(c)=\bigvee_{\gamma,\alpha'\in\Gamma} (R_\Gamma(\alpha,\beta,\gamma)\wedge S(a,\gamma,b,\alpha',c)), \tag{3.7}$$

$$((a*\alpha*b)\circ(a*\beta*b))(c)=\bigvee_{\substack{d,e\in M\\ \alpha',\beta'\in\Gamma}} (S(a,\alpha,b,\alpha',d)\wedge S(a,\beta,b,\beta',e)\wedge R_M(d,e,c)), \tag{3.8}$$

$$((a\circ b)*\alpha*c)(z)=\bigvee_{\substack{d\in M\\ \alpha'\in\Gamma}} (R_M(a,b,d)\wedge S(d,\alpha,c,\alpha',z)), \tag{3.9}$$

$$((a*\alpha*c)\circ(b*\alpha*c))(z)=\bigvee_{\substack{d,e\in M\\ \alpha',\beta'\in\Gamma}} (S(a,\alpha,c,\alpha',d)\wedge S(b,\alpha,c,\beta',e)\wedge R_M(d,e,z)), \tag{3.10}$$

**3.2. Definition.** Let $M$ and $\Gamma$ be nonempty sets, $R_M$, $R_\Gamma$ and $S$ fuzzy binary operations on $M$, $\Gamma$ and $(M,\Gamma)$, respectively, all with the same value of $\theta$. To simplify the notation, from now on we denote both $R_M$ and $R_\Gamma$ by $R$. Then $(M,\Gamma,R,S)$ is called a *fuzzy gamma ring* if the following conditions hold.

$(M,\Gamma)_1$ $(M,R)$ and $(\Gamma,R)$ are abelian fuzzy groups,
$(M,\Gamma)_2$ $\forall\, a,b,c,z_1,z_2\in M$, $\forall\,\gamma,\beta\in\Gamma$, $((a*\gamma*b)*\beta*c)(z_1)>\theta$ and $(a*\gamma*(b*\beta*c))(z_2)>\theta$ implies $z_1=z_2$,
$(M,\Gamma)_3$ $\forall\, a,b,c,z_1,z_2\in M$, $\forall\,\gamma,\beta\in\Gamma$,
(i) $(a*\gamma*(b\circ c))(z_1)>\theta$ and $((a*\gamma*b)\circ(a*\gamma*c))(z_2)>\theta$ implies $z_1=z_2$,
(ii) $(a*(\gamma\circ\beta)*b)(z_1)>\theta$ and $((a*\gamma*b)\circ(a*\beta*b))(z_2)>\theta$ implies $z_1=z_2$,
(iii) $((a\circ b)*\gamma*c)(z_1)>\theta$ and $((a*\gamma*c)\circ(b*\gamma*c))(z_2)>\theta$ implies $z_1=z_2$.

The identity element of the fuzzy group $(M,R)$ is called the *zero element* of $(M,\Gamma,R,S)$, and is denoted by $e_0$.

**3.3. Definition.** A fuzzy gamma ring $(M,\Gamma,R,S)$ is called *commutative* if

$$(a*\gamma*b)(z)>\theta \iff (b*\gamma*a)(z)>\theta.$$

for all $a,b,z\in M$ and for all $\gamma\in\Gamma$.

For a fuzzy gamma ring $(M,\Gamma,R,S)$,

$$C(M,\Gamma,R,S)=\{a\in M \mid (a*\gamma*b)(z)>\theta \iff (b*\gamma*a)(z)>\theta \text{ for all } b,z\in M \text{ and for all } \gamma\in\Gamma\}$$

is called the *center* of $(M, \Gamma, R, S)$. It follows that $(M, \Gamma, R, S)$ is commutative if and only if $M = C(M, \Gamma, R, S)$.

We now prove some elementary properties of fuzzy gamma rings.

**3.4. Theorem.** *Let $(M, \Gamma, R, S)$ be a fuzzy gamma ring, $a, b, c \in M$ and $\gamma \in \Gamma$. Then*

(1) i) *$(a * \gamma * b)(b) > \theta$ and $(a * \gamma * b)(e_0) > \theta$ implies $b = e_0$, and*
ii) *$(b * \gamma * a)(a) > \theta$ and $(b * \gamma * a)(e_0) > \theta$ implies $a = e_0$.*

(2) *Let $b^{-1}$ be the inverse of $b$ in $(M, R)$. Then*
i) *$(a * \gamma * b^{-1})(v) > \theta$ and $(a * \gamma * b)(w) > \theta$ implies $v = w^{-1}$,*
ii) *$(a^{-1} * \gamma * b)(u) > \theta$ and $(a * \gamma * b)(s) > \theta$ implies $u = s^{-1}$,*
iii) *$(a^{-1} * \gamma * b^{-1})(t) > \theta$ and $(a * \gamma * b)(r) > \theta$ implies $t = r$,*

(3) i) *$(a * \gamma * (b \circ c^{-1}))(z_1) > \theta$ and $((a * \gamma * b) \circ (a * \gamma * c^{-1}))(z_2) > \theta$ implies $z_1 = z_2$,*
ii) *$((a \circ b^{-1}) * \gamma * c)(z_1) > \theta$ and $((a * \gamma * c) \circ (b^{-1} * \gamma * c))(z_2) > \theta$ implies $z_1 = z_2$,*
iii) *$(a * (\gamma \circ \beta^{-1}) * b)(z_1) > \theta$ and $((a * \gamma * b) \circ (a * \beta^{-1} * b))(z_2) > \theta$ implies $z_1 = z_2$.*

*Proof.* (1)(i) Let $(a * \gamma * b)(b) > \theta$ and $(a * \gamma * b)(e_0) > \theta$. Thus, we have for all $a, b \in M$ and for all $\gamma, \alpha \in \Gamma$ that $(a * \gamma * b)(b) = \bigvee_{\beta \in \Gamma} S(a, \gamma, b, \beta, b) > \theta$ and $(a * \gamma * b)(e_0) = \bigvee_{\beta \in \Gamma} S(a, \gamma, b, \beta, e_0) > \theta$ from (3.2), and so $b = e_0$ by Definition 3.1 (ii).

(1)(ii) Similarly, it may be shown that $a = e_0$.

(2)(i) Let $c \in M$ such that $R(v, w, c) > \theta$. Then

$$((a * \gamma * b^{-1}) \circ (a * \gamma * b))(c) \geq S(a, \gamma, b^{-1}, \beta, v) \wedge S(a, \gamma, b, \alpha, w) \wedge R(v, w, c) > \theta$$

and

$$(a * \gamma * (b^{-1} \circ b))(e_0) \geq R(b^{-1}, b, e_0) \wedge S(a, \gamma, e_0, \alpha', e_0) > \theta.$$

Thus we get that $c = e_0$ from $(M, \Gamma)_3(i)$, and so $R(v, w, e_0) > \theta$.

Let $c \in M$ be such that $R(w, v, c) > \theta$. Then

$$((a * \gamma * b) \circ (a * \gamma * b^{-1}))(c) \geq S(a, \gamma, b, \alpha, w) \wedge S(a, \gamma, b^{-1}, \beta, v) \wedge R(w, v, c) > \theta$$

and

$$(a * \gamma * (b \circ b^{-1}))(e_0) \geq R(b, b^{-1}, e_0) \wedge S(a, \gamma, e_0, \beta, e_0) > \theta.$$

Thus we get that $c = e_0$ from $(M, \Gamma)_3\,(i)$, and so $R(w, v, e_0) > \theta$. Hence we obtain $v = w^{-1}$ from (G3).

(2)(ii) Similarly, it may be shown that $u = s^{-1}$.

(2)(iii) Let $(a^{-1} * \gamma * b^{-1})(t) > \theta$. In this case, $(a^{-1} * \gamma * b)(t^{-1}) > \theta$ by (2)(i) and $S(e_0, \gamma, b, \alpha, e_0) > \theta$ by (1). If $k \in M$ is such that $R(r, t^{-1}, k) > \theta$, then

$$((a*\gamma*b)\circ(a^{-1}*\gamma*b))(k) \geq S(a, \gamma, b, \alpha', r) \wedge S(a^{-1}, \gamma, b, \beta, t^{-1}) \wedge R(r, t^{-1}, k) > \theta$$

and

$$((a \circ a^{-1}) * \gamma * b)(e_0) \geq R(a, a^{-1}, e_0) \wedge S(e_0, \gamma, b, \alpha, e_0) > \theta.$$

It follows that $k = e_0$ from $(M, \Gamma)_3\,(iii)$ and $R(r, t^{-1}, e_0) > \theta$. Also, similarly $R(t^{-1}, r, e_0) > \theta$. Consequently, $t = r$ by $(G3)$.

(3)(i) Let $(b \circ c^{-1})(z_1) > \theta$ and $(b \circ w)(z_1) > \theta$. In this case, we have $c^{-1} = w$ by [17, Proposition 2.1 (3)]. If $a \neq e_0$, then $(a * \gamma * (b \circ c^{-1}))(z_1) > \theta$, and we have that $(a * \gamma * (b \circ w))(z_1) > \theta$, where $\gamma \in \Gamma$. Let $k \in M$ be such that $R(u, v, k) > \theta$. Then

$$((a * \gamma * b) \circ (a * \gamma * w))(k) \geq S(a, \gamma, b, \alpha, u) \wedge S(a, \gamma, w, \alpha', v) \wedge R(u, v, k) > \theta,$$

and so we get that $z_1 = k$ from $(M,\Gamma)_3\,(i)$. Since $((a*\gamma*b)\circ(a*\gamma*c^{-1}))(z_2) > \theta$, we have $((a*\gamma*b)\circ(a*\gamma*w))(z_2) > \theta$, and so $z_1 = z_2$ from $(M,\Gamma)_3\,(i)$.

(3)(ii) and (3)(iii) may be shown in a similar way. □

**3.5. Definition.** Let $(M,\Gamma,R,S)$ be a fuzzy gamma ring.

(i) $(M,\Gamma,R,S)$ is called a *ring with identity* if there is an element $e_*$ in $(M,\Gamma,R,S)$ such that $(e_**\gamma*a)(a) > \theta$ and $(a*\gamma*e_*)(a) > \theta$ for all $a \in M$, and all $\gamma \in \Gamma$.

(ii) Let $(M,\Gamma,R,S)$ be a fuzzy gamma ring with identity. If $(a*\gamma*b)(e_*) > \theta$ and $(b*\gamma*a)(e_*) > \theta$ for all $a,b \in M$, and all $\gamma \in \Gamma$, then $b$ is called an *invertible* (or *unit*) element of $a$, and is denoted by $a_*^{-1}$.

**3.6. Theorem.** *If $(M,\Gamma,R,S)$ is a fuzzy gamma ring with identity, then $e_*$ is unique.*

*Proof.* Let $e'_*, e''_*$ be identity elements of $(M,\Gamma,R,S)$. In this case, $(e'_**\gamma*e''_*)(e'_*) > \theta$ and $(e'_**\gamma*e''_*)(e''_*) > \theta$, where $\gamma \in \Gamma$. Thus $\bigvee_{\beta\in\Gamma} S(e'_*,\gamma,e''_*,\beta,e'_*) > \theta$ and $\bigvee_{\beta\in\Gamma} S(e'_*,\gamma,e''_*,\beta,e''_*) > \theta$. So we get $e'_* = e''_*$ by Definition 3.1. □

**3.7. Definition.** A nonzero element $a$ in a fuzzy gamma ring $(M,\Gamma,R,S)$ is called a *zero divisor* if there exists $b$ in $(M,\Gamma,R,S)$ such that $b \neq e_0$ and either $(a*\gamma*b)(e_0) > \theta$ or $(b*\gamma*a)(e_0) > \theta$, where $\gamma \in \Gamma$.

The following theorem establishes a relation between zero divisors and the cancellation property of a fuzzy gamma ring.

**3.8. Theorem.** *A fuzzy gamma ring $(M,\Gamma,R,S)$ has no zero divisor if and only if for all $a,b,c,v \in M$ with $a \neq e_0$ and all $\gamma \in \Gamma$, $(a*\gamma*b)(v) > \theta$ and $(a*\gamma*c)(v) > \theta$ implies $b = c$* ( left cancellation law) *or $(b*\gamma*a)(v) > \theta$ and $(c*\gamma*a)(v) > \theta$ implies $b = c$* (right cancellation law).

*Proof.* $\Longrightarrow$ Suppose that $(M,\Gamma,R,S)$ has no zero divisor. If $(a*\gamma*c)(v) > \theta$, then $(a*\gamma*c^{-1})(v^{-1}) > \theta$ by Theorem 3.4 (2). Let $k,m \in M$ be such that $R(a,c^{-1},k) > \theta$ and $S(a,\gamma,k,\alpha,m) > \theta$, for all $a \neq e_0, b, c \in M$ and all $\gamma,\beta,\alpha \in \Gamma$. Then

$$((a*\gamma*b)\circ(a*\gamma*c^{-1}))(e_0) \geq S(a,\gamma,b,\beta,v) \wedge S(a,\gamma,c^{-1},\beta',v^{-1}) \wedge R(v,v^{-1},e_0) > \theta$$

and

$$(a*\gamma*(b\circ c^{-1}))(m) \geq R(b,c^{-1},k) \wedge S(a,\gamma,k,\alpha,m) > \theta.$$

Thus $m = e_0$ by $(M,\Gamma)_3\,(i)$, and so $S(\alpha,\gamma,k,\alpha,e_0) > \theta$. Since $a \neq e_0$ and $(M,\Gamma,R,S)$ has no zero divisor, we get $k = e_0$ and so

$$R(a,c^{-1},e_0) > \theta. \tag{3.11}$$

On the other hand, if $(a*\gamma*b)(v) > \theta$, then $(a*\gamma*b^{-1})(v^{-1}) > \theta$ by Theorem 3.4 (2). Let $t,n \in M$ be such that $R(c,b^{-1},t) > \theta$ and $S(a,\gamma,t,\beta,n) > \theta$. For all $a \neq e_0, b, c \in M$ and all $\gamma,\beta,\beta' \in \Gamma$,

$$\begin{aligned}((a*\gamma*c)\circ(a*\gamma*b^{-1}))(e_0) &\\ &\geq S(a,\gamma,c,\beta,v^{-1}) \wedge S(a,\gamma,b^{-1},\beta',v^{-1}) \wedge R(v,v^{-1},e_0)\\ &> \theta\end{aligned}$$

and

$$(a*\gamma*(c\circ b^{-1}))(n) \geq R(c,b^{-1},t) \wedge S(a,\gamma,t,\beta,n) > \theta.$$

Thus we get $n = e_0$ by $(M,\Gamma)_3\,(i)$, and so $S(a,\gamma,t,\beta,e_0) > \theta$. Since $a \neq e_0$ and $(M,\Gamma,R,S)$ has no zero divisor, we get $t = e_0$ and so

$$R(c,b^{-1},e_0) > \theta. \tag{3.12}$$

From (3.11) and (3.12), we have $b = c$ by Definition 2.5. Similarly, it may be shown that $(b * \gamma * a)(v) > \theta$ and $(c * \gamma * a)(v) > \theta$ implies $b = c$.

$\Longleftarrow$ Suppose one of the cancellation laws holds, say, the left one, i.e., if $a, b \in M$ with $a \neq e_0$ and $\gamma \in \Gamma$, then $(a*\gamma*b)(e_0) > \theta$ and $(a*\gamma*e_0)(e_0) > \theta$ implies $b = e_0$. Similarly, the right cancellation law implies $b = e_0$. Thus, $(M, \Gamma, R, S)$ has no zero divisors. □

Now, we introduce the idea of a fuzzy gamma subring of a fuzzy gamma ring.

Let $(M, \Gamma, R, S)$ be a fuzzy gamma ring and $N$ a nonempty subset of $M$. Let $R_N(a, b, c) = R(a, b, c)$ and $S_N(a, \gamma, b, \beta, c) = S(a, \gamma, b, \beta, c)$ for all $a, b, c \in N$ and all $\gamma, \beta \in \Gamma$. Then we have

$$(3.13) \qquad (a \triangle b)(c) = R_N(a, b, c) = R(a, b, c), \text{ for all } a, b, c \in N$$

$$(3.14) \qquad (a \diamond \gamma \diamond b)(c) = \bigvee_{\beta \in \Gamma} S_N(a, \gamma, b, \beta, c) = \bigvee_{\beta \in \Gamma} S(a, \gamma, b, \beta, c) \text{ for all } a, b, c \in N,\ \gamma \in \Gamma.$$

**3.9. Definition.** Let $(M, \Gamma, R, S)$ be a fuzzy gamma ring and $N$ be nonempty subset of $M$ for which:

(i) $(a \circ b)(c) > \theta$ implies $c \in N$ and $(a * \gamma * b)(c) > \theta$ implies $c \in N$ for all $a, b \in N$, all $c \in M$ and all $\gamma \in \Gamma$, and

(ii) $(N, \Gamma, R_N, S_N)$ is fuzzy gamma ring.

Then, $(N, \Gamma, R_N, S_N)$ is called a *fuzzy gamma subring* of $(M, \Gamma, R, S)$.

**3.10. Proposition.** *Let $(M, \Gamma, R, S)$ be a fuzzy gamma ring and $N$ a nonempty subset of $M$. Then $(N, \Gamma, R_N, S_N)$ is a fuzzy gamma subring of $M$ if and only if*

(i) *$(a \circ b)(c) > \theta$ implies $c \in N$ and $(a * \gamma * b)(c) > \theta$ implies $c \in N$, for all $a, b \in N$, all $c \in M$ and all $\gamma \in \Gamma$*

(ii) *$a^{-1} \in N$ for all $a \in N$.*

*Proof.* Straightforward. □

**3.11. Theorem.** *Let $(M, \Gamma, R, S)$ be a fuzzy gamma ring and $x$ an element of $M$. If*

$$C(x) = \{a \in M \mid (x * \gamma * a)(c) > \theta \iff (a * \gamma * x)(c) > \theta,\ \forall\, c \in M\ \forall\, \gamma \in \Gamma\}$$

*then $C(x)$ is a fuzzy gamma subring of $M$.*

*Proof.* Clearly $e_0 \in C(x)$ and so $C(x) \neq \emptyset$.

(i) $a_1, a_2 \in C(x)$ and $(a_1 \circ a_2)(b) = R(a_1, a_2, b) > \theta$ implies $b \in C(x)$. Let $x, b, c, b_1, b_2, d_1, d_2 \in M$ be such that $S(b, \gamma, x, \beta, c) > \theta$, $S(x, \gamma, b, \beta, d_1) > \theta$, $S(x, \gamma, a_1, \beta, b_1) > \theta$, $S(x, \gamma, a_2, \beta, b_2) > \theta$, and $R(b_1, b_2, d_2) > \theta$.

From $R(a_1, a_2, b) > \theta$ and $R(a_2, a_1, b) > \theta$, we have

$$(x * \gamma * (a_1 \circ a_2))(d_1) \geq R(a_1, a_2, b) \wedge S(x, \gamma, b, \beta, d_1) > \theta,$$

and

$$((x*\gamma*a_1) \circ (x*\gamma*a_2))(d_2) \geq S(x, \gamma, a_1, \beta, b_1) \wedge S(x, \gamma, a_2, \beta, b_2) \wedge R(b_1, b_2, d_2) > \theta.$$

Thus, $d_1 = d_2$ by $(M, \Gamma)_3\,(i)$ and $R(b_1, b_2, d_1) > \theta$. For $a_1, a_2 \in C(x)$, $S(a_1, \gamma, x, \beta, b_1) > \theta$, $S(a_2, \gamma, x, \beta, b_2) > \theta$ and $R(b_2, b_1, d_1) > \theta$, and so

$$((a_2 \circ a_1) * \gamma * x)(c) \geq R(a_2, a_1, b) \wedge S(b, \gamma, x, \beta, c) > \theta,$$

and

$$((a_2*\gamma*x) \circ (a_1*\gamma*x))(d_1) \geq S(a_2, \gamma, x, \beta, b_2) \wedge S(a_1, \gamma, x, \beta, b_1) \wedge R(b_2, b_1, d_1) > \theta.$$

Thus $c = d_1$ by $(M,\Gamma)_3\,(iii)$ and $S(x,\gamma,b,\beta,c) > \theta$. Therefore, we get that $S(b,\gamma,x,\beta,c) > \theta$ implies $S(x,\gamma,b,\beta,c) > \theta$. Similarly, it may be shown that $S(x,\gamma,b,\beta,c) > \theta$ implies $S(b,\gamma,x,\beta,c) > \theta$, and so $b \in C(x)$.

Let $x, b, c, b_2, d, d_1 \in M$ and $\alpha, \beta, \gamma_2, \tau, \beta_1 \in \Gamma$ be such that $S(x,\alpha,b,\tau,c) > \theta$, $S(x,\alpha,a_2,\gamma_2,b_2) > \theta$, $S(b,\alpha,x,\tau,d) > \theta$ and $S(a_1,\alpha,a_2,\beta,b) > \theta$ and $S(a_1,\alpha,b_2,\beta_1,d_1) > \theta$. From $S(a_2,\alpha,x,\gamma_2,b_2) > \theta$, we have

$$((a_1 * \alpha * a_2) * \alpha * x))(d) \geq S(a_1,\alpha,a_2,\beta,b) \wedge S(b,\alpha,x,\tau,d) > \theta,$$

and

$$(a_1 * \alpha * (a_2) * \alpha * x)(d_1) \geq S(a_2,\alpha,x,\gamma_2,b_2) \wedge S(a_1,\alpha,b_2,\beta_1,d_1) > \theta.$$

Thus $d_1 = d$ by $(M,\Gamma)_2$, and so $S(a_1,\alpha,b_2,\beta_1,d) > \theta$. Let $b_1, d_2 \in M$ and $\gamma_1, \beta_2 \in \Gamma$ be such that $S(a_1,\alpha,x,\gamma_1,b_1) > \theta$ and $S(b_1,\alpha,a_2,\beta_2,d_2) > \theta$. From $S(x,\alpha,a_2,\gamma_2,b_2) > \theta$, we have

$$((a_1 * \alpha * a_2) * \alpha * x)(d_2) \geq S((a_1,\alpha,a_2,\beta,b) \wedge S(b,\alpha,x,\tau,d) > \theta$$

and

$$(a_1 * \alpha * (x * \alpha * a_2))(d) \geq S(x,\alpha,a_2,\gamma_2,b_2) \wedge S(a_1,\alpha,b_2,\beta_1,d) > \theta.$$

Thus, $d_2 = d$ by $(M,\Gamma)_2$, and so $S(b_1,\alpha,a_2,\beta_2,d) > \theta$. From $S(a_1,\alpha,x,\gamma_1,b_1) > \theta$ we have

$$((x * \alpha * a_1) * \alpha * a_2)(d) \geq S(a_1,\alpha,x,\gamma_1,b_1) \wedge S(b_1,\alpha,a_2,\beta_2,d) > \theta$$

and

$$(x * \alpha * (a_1 * \alpha * a_2))(c) \geq S(a_1,\alpha,a_2,\beta,b) \wedge S(x,\alpha,b,\tau,c) > \theta.$$

Thus $c = d$ by $(M,\Gamma)_2$, and so $S(b,\alpha,x,\tau,c) > \theta$. Similarly, $S(b,\alpha,x,\tau,c) > \theta$ implies $S(x,\alpha,b,\tau,c) > \theta$, and then $b \in C(x)$.

(ii) Let $a \in C(x)$. If $b, c, d, b, b_2, d_1 \in M$ are such that $S(x,\gamma,a^{-1},\beta,d) > \theta$, $S(a^{-1},\gamma,x,\beta,c) > \theta$, $S(x,\gamma,a,\gamma',b_1) > \theta$, $R(b_1,d,d_1) > \theta$ and $R(b_1,d,d_1) > \theta$, then we get

$$((a^{-1} \circ a) * \gamma * x)(e_0) \geq R(a^{-1},a,e_0) \wedge S(e_0,\gamma,x,\alpha,e_0) > \theta,$$

and

$$((a^{-1}*\gamma*x)\circ(a*\gamma*x))(b_2) \geq S(a^{-1},\gamma,x,\beta,c)\wedge S(a,\gamma,x,\gamma',b_1)\wedge R(c,b_1,b_2) > \theta.$$

Thus $e_0 = b_2$ and so $R(c,b_1,e_0) > \theta$. Also,

$$(x * \gamma * (a^{-1} \circ a))(e_0) \geq R(a^{-1},a,e_0) \wedge S(x,\gamma,e_0,\alpha,e_0) > \theta,$$

and

$$((x*\gamma*a^{-1})\circ(x*\gamma*a)(d_1) \geq S(x,\gamma,a^{-1},\beta,d)\wedge(x,\gamma,a,\gamma',b_1)\wedge R(d,b_1,d_1) > \theta,$$

from which we get $e_0 == d_1$ and $R(d,b_1,e_0) > \theta$. Since $R(c,b_1,e_0) > \theta$, $R(d,b_1,e_0) > \theta$ and $(M,R)$ is an abelian fuzzy group, e have $d = c$. Similarly, $a \in C(x)$ and $S(a^{-1},\gamma,x,\beta,d) > \theta$ implies $S(x,\gamma,a^{-1},\beta,d) > \theta$. Hence, $a^{-1} \in C(x)$. Then, $C(x)$ is a fuzzy gamma subring of $M$ by Proposition 3.10. □

**3.12. Definition.** Let $(M,\Gamma,R,S)$ be a fuzzy gamma ring. A nonempty subset $I$ of $M$ is called a *left* (*right*) *fuzzy ideal of* $M$ if for all $a, b \in I$, all $n, m \in M$, and all $\gamma \in \Gamma$, $(a \circ b)(m) > \theta$ implies $m \in I$, $a^{-1} \in I$, $(n*\gamma*a)(m) > \theta$ implies $m \in I$ $((a*\gamma*n)(m) > \theta$ implies $m \in I)$.

A nonempty subset $I$ of a fuzzy gamma ring $(M,\Gamma,R,S)$ is called a *fuzzy* (*two-sided*) *ideal of* $(M,\Gamma,R,S)$ if $I$ is both a left and a right ideal of $(M,\Gamma,R,S)$.

**3.13. Remark.** From the definition of a fuzzy left (right) ideal $I$ of $(M, \Gamma, R, S)$, then $I$ is a fuzzy gamma subring of $(M, \Gamma, R, S)$. Also, if $M$ is a commutative fuzzy gamma ring, then every left fuzzy ideal is a right fuzzy ideal and every right fuzzy ideal is a left fuzzy ideal.

**3.14. Proposition.** *Let $I_i$, $i \in \Lambda$, be a fuzzy ideal of fuzzy gamma ring $(M, \Gamma, R, S)$, where $\Lambda$ is a index set. Then $\bigcap_{i\in\Lambda} I_i$ is a fuzzy ideal of $M$.*

*Proof.* Straightforward. □

Let $I$ be a fuzzy ideal of fuzzy gamma ring $(M, \Gamma, R, S)$ and $\Delta = \{a \circ I \mid a \in M\}$, where $(a \circ I)(u) = \bigvee_{x\in I} R(a, x, u)$ for all $a \in M$. We define a relation over $\Delta$ by

$$a_1 \circ I \sim a_2 \circ I \iff \exists\, u \in I \text{ such that } R\big(a_1^{-1}, a_2, u\big) > \theta.$$

The fuzzy relation $\sim$ on the set $\Delta$ is a fuzzy equivalence relation by [17, Theorem 4.1]. Let $[a \circ I] = \{a' \circ I \mid a' \circ I \sim a \circ I\}$, $\bar{a} = \{a' \mid a' \in M,\ a' \circ I \sim a \circ I\}$ and $M/I = \{[a \circ I] \mid a \in M\}$. Also, $(I, R)$ is a fuzzy subgroup of $(M, R)$, and since $(M, R)$ is abelian, $(I, R)$ is a normal fuzzy group of $(M, R)$ by [17, Theorem 3.1]. Hence $(M/I, \overline{R})$ is a commutative fuzzy group by [17, Theorem 4.2], where

$$\big([a \circ I] \oplus [b \circ I]\big)\big([c \circ I]\big) = \overline{R}\big([a \circ I], [b \circ I], [c \circ I]\big) = \bigvee_{(a',b',c')\in\bar{a}\times\bar{b}\times\bar{c}} R(a', b', c') \tag{3.15}$$

Given the fuzzy groups $(M/I, \overline{R})$ and $(\Gamma, R)$, let $\overline{S}$ be a fuzzy binary operation on $(M/I, \Gamma)$, that is a fuzzy subset of $M/I \times \Gamma \times M/I \times \Gamma \times M/I$ with the same value of $\theta$ as for $R$ and $\overline{R}$. Then we may associate with $\overline{S}$ the mapping $\overline{S} : F(M/I) \times F(\Gamma) \times F(M/I) \to F(M/I)$ given by

$$\overline{S}(\overline{A}, G, \overline{B})(c') = \bigvee_{\substack{a',b'\in M/I\\ \gamma,\beta\in\Gamma}} \big(\overline{A}(a') \wedge G(\gamma) \wedge \overline{B}(b') \wedge \overline{S}(a', \gamma, b', \beta, c')\big)$$

where

$$F(M/I) = \{\overline{A} \mid \overline{A} : M/I \to [0, 1] \text{ is a mapping}\}.$$

With $\overline{R}$ and $\overline{S}$ as above, we have

$$\begin{aligned}
&\big([a \circ I] \otimes \gamma \otimes [b \circ I]\big)([c \circ I])\\
&\qquad = \overline{S}\big([a \circ I], [b \circ I], [c \circ I]\big)\\
&\qquad = \bigvee_{(a',\gamma,b',\beta,c')\in\bar{a}\times\gamma\times\bar{b}\times\beta\times\bar{c}} S(a', \gamma, b', \beta, c'),
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
&\big(([a \circ I] \oplus [b \circ I]) \oplus [c \circ I]\big)([u \circ I])\\
&\qquad = \bigvee_{d\in M} \big(\overline{R}\big([a \circ I], [b \circ I], [d \circ I]\big) \wedge \overline{R}\big([d \circ I], [c \circ I], [u \circ I]\big),\\
&\big([a \circ I] \oplus ([b \circ I] \oplus [c \circ I])\big)([w \circ I])\\
&\qquad = \bigvee_{d\in M} \big(\overline{R}\big([b \circ I], [c \circ I], [d \circ I]\big) \wedge \overline{R}\big([a \circ I], [d \circ I], [w \circ I]\big)\big),\\
&\big([a \circ I] \otimes \gamma \otimes ([b \circ I] \otimes [c \circ I])\big)([z \circ I])\\
&\qquad = \bigvee_{d\in M,\beta\in\Gamma} \big(\overline{R}\big([b \circ I], [c \circ I], [d \circ I\big) \wedge \overline{S}\big([a \circ I], \gamma, [d \circ I], \beta, [z \circ I]\big)\big)
\end{aligned}$$

$$\big(([a\circ I]\otimes\gamma\otimes[b\circ I])\oplus([a\circ I]\otimes\gamma\otimes[c\circ I])\big)([z\circ I])$$
$$= \bigvee_{\substack{d_1,d_2\in M\\ \beta,\beta'\in\Gamma}} \big(\overline{S}\big([a\circ I],\gamma,[b\circ I],\beta,[d\circ I]\big)\wedge\overline{S}\big([a\circ I],\gamma,[c\circ I],\beta',[d_2\circ I]\big)$$
$$\wedge\,\overline{R}\big([d_1\circ I],[d_2\circ I],[z\circ I]\big)\big),$$
$$\big([a\circ I]\otimes(\gamma\circ\gamma')\otimes[b\circ I]\big)([z\circ I])$$
$$= \bigvee_{\beta,\beta'\in\Gamma} \big(R(\gamma,\gamma',\beta)\wedge\overline{S}\big([a\circ I],\beta,[b\circ I],\beta',[z\circ I]\big)\big),$$
$$\big(([a\circ I]\otimes\gamma\otimes[b\circ I])\oplus([a\circ I]\otimes\gamma'\otimes[b\circ I])\big)([z\circ I])$$
$$= \bigvee_{\substack{c,d\in M\\ \alpha,\alpha'\in\Gamma}} \big(\overline{S}\big([a\circ I],\gamma,[b\circ I],\alpha,[c\circ I]\big)\wedge\overline{S}\big([a\circ I],\gamma',[b\circ I],\alpha',[d\circ I]\big)$$
$$\wedge\,\overline{R}\big([c\circ I],[d\circ I],[z\circ I]\big)\big),$$
$$\big(([a\circ I]\oplus[b\circ I])\otimes\beta\otimes(c\circ I\big)([z\circ I])$$
$$= \bigvee_{\substack{d\in M\\ \beta'\in\Gamma}} \big(\overline{R}\big([a\circ I],[b\circ I],[d\circ I]\big)\wedge\overline{S}\big([d\circ I],\beta,[c\circ I],\beta',[z\circ I]\big)\big),$$
$$\big(([a\circ I]\otimes\beta\otimes[c\circ I])\oplus([b\circ I]\otimes\beta\otimes[c\circ I])\big)([z\circ I])$$
$$= \bigvee_{\substack{d_1,d_2\in M\\ \beta',\beta''\in\Gamma}} \big(\overline{S}\big([a\circ I],\beta,[c\circ I],\beta',[d_1\circ I]\big)\wedge\overline{S}\big([b\circ I],\beta,[c\circ I],\beta'',[d_2\circ I]\big)$$
$$\wedge\,\overline{R}\big([d_1\circ I],[d_2\circ I],[z\circ I]\big)\big).$$

**3.15. Theorem.** *Let $(M,\Gamma,R,S)$ be a fuzzy gamma ring and $I$ a fuzzy ideal of $M$. Then the quotient fuzzy group $(M/I,\overline{R})$ is a fuzzy gamma ring with*

$$\big([a\circ I]\otimes\gamma\otimes[b\circ I]\big)\big([c\circ I]\big) = \overline{S}\big([a\circ I],\gamma,[b\circ I],\beta,[c\circ I]\big)$$
$$= \bigvee_{(a',\gamma,b',\beta,c')\in\bar a\times\gamma\times\bar b\times\beta\times\bar c} S\big(a',\gamma,b',\beta,c'\big)$$

*Proof.* The proof of $(M,\Gamma)_2$ is similar to the proof of [17, Theorem 4.3], and is omitted. It only remains to check that $(M,\Gamma)_3$ is satisfied.

(i) Let

$$\big([a\circ I]\otimes\gamma\otimes([b\circ I]\oplus[c\circ I])\big)\big([d\circ I]\big) > \theta$$

and

$$\big(([a\circ I]\otimes\gamma\otimes[b\circ I])\oplus([a\circ I]\otimes\gamma\otimes[c\circ I])\big)\big([w\circ I]\big) > \theta.$$

Thus, we have $a_1,a_1',b_1,b_1',c_1,c_1',d_1,w_1\in M$ such that $a_1\circ I\sim a_1'\circ I\sim a\circ I$, $b_1\circ I\sim b_1'\circ I\sim b\circ I$, $c_1\circ I\sim c_1'\circ I\sim c\circ I$, $d_1\circ I\sim d\circ I$, $w_1\circ I\sim w\circ I$, and there exist elements $u_1,u_2,u_3\in I$, $x_1',x_2',x_3'\in M$ and $\alpha,\beta,\alpha',\beta'\in\Gamma$ such that

$$R(b_1,c_1,x_1')\wedge S(a_1,\gamma,x_1',\alpha,d_1) > \theta,$$
$$S(a_1',\gamma,b_1',\beta,x_2')\wedge S(a_1',\gamma,c_1',\alpha',x_3')\wedge R(x_2',x_3',w_1) > \theta,$$
$$R(a_1',u_1,a_1) > \theta,\ R(b_1',u_2,b_1) > \theta \text{ and } R(c_1',u_3,c_1) > \theta$$

by (3.15) and (3.16).

Let $z_1 \in M$ be such that $R(b_1', c_1', z_1) > \theta$. Then by $R(b_1, c_1, x_1') > \theta$, $R(b_1', u_2, b_1) > \theta$, $R(b_1', c_1', z_1) > \theta$, $R(c_1', u_3, c_1) > \theta$, and the proof of [17, Theorem 4.2], we have $R(z_1, u, x_1') > \theta$ for any $u \in I$.

Since $I$ is a fuzzy ideal, there exist elements $u', u_3', u_4, u_5, u_6 \in I$ such that $S(u_3, \gamma, z_1, \beta, u_3') > \theta$, $S(u, \gamma, b_1', \beta', u') > \theta$, $S(u, \gamma, u_3, \beta, u_4) > \theta$, $R(u_3', u', u_5) > \theta$ and $R(u_5, u_4, u_6) > \theta$.

Let $z_1 \in M$ be such that $S(a_1', \gamma, z_1, \beta, z_2) > \theta$. By $S(a_1, \gamma, x_1', \alpha, d_1) > \theta$, $R(z_1, u, x_1') > \theta$, $R(c_1', u_3, c_1) > \theta$, $S(a_1', \gamma, z_1, \beta, z_2) > \theta$, and similarly to the proof of [17, Theorem 4.2], we have $R(z_2, u_6, d_1) > \theta$. Hence,

$$(a_1' * \gamma * (b_1' \circ c_1'))(z_2) \geq R(b_1', c_1', z_1) \wedge S(a_1', \gamma, z_1, \beta, z_2) > \theta$$

and

$$\begin{aligned}((a_1' * \gamma * b_1') \circ (a_1' * \gamma * c_1'))(w_2) &\geq S(a_1', \gamma, b_1', \beta, x_2') \wedge S(a_1', \gamma, c_1', \beta', x_3') \\ &\quad \wedge R(x_2', x_3', w_2) \\ &> \theta.\end{aligned}$$

Therefore, $z_2 = w_2$ and $R(w_2, u_6, d_1) > \theta$. In this case, $w_1 \circ I \sim d_1 \circ I$, and so $[w_1 \circ I] = [d_1 \circ I]$.

Similarly, it may be shown that (ii) and (iii) of $(M, \Gamma)_3$ also hold. □

**3.16. Definition.** Let $(M, \Gamma, R, S)$ be a fuzzy gamma ring and $I$ a fuzzy ideal of $M$. Then the fuzzy gamma ring $(M/I, \Gamma, \overline{R}, \overline{S})$ is called the *fuzzy quotient gamma ring of $M$ by $I$*.

Finally, we introduce the notion of a fuzzy gamma homomorphism of fuzzy gamma rings. This concept is the analog of homomorphism for rings.

**3.17. Definition.** Let $(M_1, \Gamma, R_1, S_1)$ and $(M_2, \Gamma, R_2, S_2)$ be fuzzy gamma rings and $f$ a function from $M_1$ into $M_2$. Then $f$ is called a *fuzzy gamma homomorphism* of $M_1$ into $M_2$ if

(i) $R_1(a, b, c) > \theta$ implies $R_2(f(a), f(b), f(c)) > \theta$,
(ii) $S(a, \gamma, b, \beta, c) > \theta$ implies $S(f(a), \gamma, f(b), \beta, f(c)) > \theta$,

for all $a, b, c \in M_1$, and all $\gamma, \beta \in \Gamma$.

A homomorphism $f$ of the fuzzy gamma ring $M_1$ into the fuzzy gamma ring $M_2$ is called

(1) A *monomorphism* if $f$ is one-one,
(2) An *epimorphism* if $f$ is onto $M_2$, and
(3) An *isomorphism* if $f$ is a one-one and map of $M_1$ onto $M_2$.

If $f$ is an isomorphism of $M_1$ onto $M_2$, then the fuzzy gamma rings $M_1$ and $M_2$ are called *isomorphic*, denoted by $M_1 \cong M_2$.

**3.18. Theorem.** *Let $(M_1, \Gamma, R_1, S_1)$ and $(M_2, \Gamma, R_2, S_2)$ be fuzzy gamma rings, and let $f$ be a fuzzy gamma homomorphism of $M_1$ into $M_2$. Then*

(i) *$f(e_0) = e_0'$, where $e_0'$ is the zero of $M_2$,*
(ii) *$f(a^{-1}) = f(a)^{-1}$ for all $a \in M_1$,*
(iii) *$\mathrm{Im} f = \{f(a) \mid a \in M_1\}$ is a fuzzy gamma subring of $M_2$.*

*Proof.* (i) Since $f$ is a fuzzy gamma ring homomorphism, for all $a \in M_1$,

$$R_1(a, e_0, a) > \theta \text{ implies } R_2(f(a), f(e_0), f(a)) > \theta,$$

and for $f(a) \in M_2$ we get $R_2(f(a), e'_0, f(a)) > \theta$. Now

$$(f(a)^{-1} \circ (f(a) \circ f(e_0)))(f(e_0)) \geq R_2(f(a), f(e_0), f(a)) \wedge R_2(f(a)^{-1}, f(a), f(e_0)) > \theta.$$

Therefore, we get that $f(e_0) = e'_0$ by $G1$.

(ii) Since $f$ is a fuzzy gamma homomorphism, for all $a \in M_1$,

$$R_1(a, a^{-1}, e_0) > \theta \text{ implies } R_2(f(a), f(a^{-1}), f(e_0)) > \theta,$$

and so $R_2(f(a), f(a^{-1}), e'_0) > \theta$ by (i). Hence we get that $f(a^{-1}) = f(a)^{-1}$.

(iii) Since $f$ is a fuzzy gamma homomorphism, we have $f(e_0) = e'_0 \in \mathrm{Im}f$, $e_0 \in M_1$, by (i). Hence $\mathrm{Im}f \neq \emptyset$.

(1) If $a_1, a_2, a \in M_1$ are such that $R_1(a_1, a_2, a) > \theta$, then $R_2(f(a_1), f(a_2), f(a)) > \theta$, and so $f(a) \in \mathrm{Im}f$. Let $a_1, a_2, a \in M_1$ and $\gamma, \beta \in \Gamma$ be such that $S_1(a_1, \gamma, a_2, \beta, a) > \theta$. Then $S_2(f(a_1), \gamma, f(a_2), \beta, f(a)) > \theta$, and so $f(a) \in \mathrm{Im}f$.

(2) Let $b \in \mathrm{Im}f$ be such that $b = f(a)$, $a \in M_1$. Since $f$ is a fuzzy $\Gamma$ homomorphism and $a^{-1} \in M_1$, we get $b^{-1} = f(a)^{-1} = f(a^{-1}) \in \mathrm{Im}(f)$. □

**3.19. Theorem.** *Let $(M_1, \Gamma, R_1, S_1)$ and $(M_2, \Gamma, R_2, S_2)$ be fuzzy gamma rings, and let $f$ be a fuzzy gamma homomorphism of $M_1$ into $M_2$. Then*

(i) $\mathrm{Ker}f = \{a \in M_1 \mid f(a) = e'_0\}$ *is a fuzzy ideal of $M_1$*
(ii) *If $B$ is a fuzzy ideal of $M_2$, then $f^{-1}(B)$ is a fuzzy ideal of $M_1$,*
(iii) *If $f$ is surjective and $A$ is a fuzzy ideal of $M_1$, then $f(A)$ is a fuzzy ideal of $M_2$.*

*Proof.* (i) Since $f(e_0) = e'_0$, $e_0 \in \mathrm{Ker}f$, and so $\mathrm{Ker}f \neq \emptyset$.
If $a, b \in \mathrm{Ker}f$ are such that $R_1(a, b, m_1) > \theta$, $m_1 \in M_1$, then

$$R_2(f(a), f(b), f(m_1)) = R_2(e'_0, e'_0, f(m_1)) > \theta$$

since $f$ is a fuzzy gamma homomorphism. Therefore, $f(m_1) = e'_0$ and so $m_1 \in \mathrm{Ker}f$.

If $a \in \mathrm{Ker}f$ is such that $R_1(a, a^{-1}, e_0) > \theta$, then

$$R_2(f(a), f(a^{-1}), f(e_0)) = R_2(f(a), f(a^{-1}), e'_0) > \theta,$$

and so $f(a^{-1}) = e'_0$, i.e. $a^{-1} \in \mathrm{Ker}f$.

Finally, if $S_1(a, \gamma, m_1, \beta, w) > \theta$ for all $m_1, w \in M_1$ and all $\gamma, \beta \in \Gamma$, then

$$S_2(f(a), \gamma, f(m_1), \beta, f(w)) > \theta.$$

Since $f(a) = e'_0$, $S_2(e'_0, \gamma, f(m_1), \beta, f(w)) > \theta$. In this case, we have $(e'_0 * \gamma * f(m_1))(f(w)) > \theta$ and $(e'_0 * \gamma * f(m_1))(e'_0) > \theta$, and so $f(w) = e'_0$ by Theorem 3.4.

Similarly, if $S_1(m_1, \gamma, a, \beta, u) > \theta$, then $S_2(f(m_1), \gamma, f(a), \beta, f(u)) > \theta$. Since $f(a) = e'_0$,

$$S_2(f(m_1), \gamma, f(a), \beta, f(u)) = (f(m_1) * \gamma * e'_0)(f(u)) > \theta.$$

Also, since $(f(m_1) * \gamma * e'_0)(e'_0) > \theta$, we have $f(u) = e'_0$ by Theorem 3.4. Therefore, we get that $w, u \in \mathrm{Ker}f$, and so $\mathrm{Ker}f$ is a fuzzy ideal of $M_1$.

(ii) and (iii) may be proved similarly. □

**3.20. Theorem.** *Let $(M, \Gamma, R, S)$ be a fuzzy gamma ring and $I$ a fuzzy ideal of $M$. Then, the mapping $\Pi : M \to M/I$ defined by $\Pi(a) = a \circ I$ for all $a \in M$ is a fuzzy gamma homomorphism, called the* fuzzy canonical gamma homomorphism.

*Proof.* Let $a, b, c \in M$ be such that $R(a, b, c) > \theta$. Then

$$\begin{aligned}
\overline{R}\big(\Pi(a), \Pi(b), \Pi(c)\big) &= \overline{R}(a \circ I, b \circ I, c \circ I) = \big([a \circ I] \oplus [b \circ I]\big)\big[c \circ I\big] \\
&= \bigvee_{(a', b', c') \in \bar{a} \times \bar{b} \times \bar{c}} R\big(a', b', c'\big) \\
&\geq R(a, b, c) \\
&> \theta
\end{aligned}$$

by (3.15). If $a, b, c \in M$ and $\gamma, \beta \in \Gamma$ are such that $S(a, \gamma, b, \beta, c) > \theta$, then

$$\begin{aligned}
\overline{S}\big(\Pi(a), \gamma, \Pi(b), \beta, \Pi(c)\big) &= \overline{S}\big(a \circ I, \gamma, b \circ I, \beta, c \circ I\big) \\
&= \big([a \circ I] \otimes \gamma \otimes [b \circ I]\big)[c \circ I] \\
&= \bigvee_{(a', \gamma, b', \beta, c') \in \bar{a} \times \gamma \times \bar{b} \times \beta \times \bar{c}} S\big(a', \gamma, b', \beta, c'\big) \\
&\geq S(a', \gamma, b', \beta, c') \\
&> \theta
\end{aligned}$$

by (3.16). □

**3.21. Theorem.** *Let $f : (M_1, \Gamma, R_1, S_1) \to (M_2, \Gamma, R_2, S_2)$ be a fuzzy gamma epimorphism. Then $M_1/N \cong M_2$, where $N = \mathrm{Ker} f$*

*Proof.* Define the mapping $\varphi : M_1/N \to M_2$ by $\varphi([a \circ N]) = f(a)$ for all $a \in M_1$. In this case, $\varphi$ is a well defined one-to-one fuzzy group homomorphism by [17, Theorem 5.3]. Therefore, it is only remains to show that if $\overline{S}_1([a \circ N], \gamma, [b \circ N], \beta, [c \circ N]) > \theta$, then $S_2(\varphi([a \circ N]), \gamma, \varphi([b \circ N]), \beta, \varphi([c \circ N])) > \theta$. In this case, there exist $a_1, b_1, c_1 \in M_1$, $\gamma, \beta \in \Gamma$ and $n_1, n_2, n_3 \in N$ such that $R_1(a, n_1, a_1) > \theta$, $R_1(b, n_2, b_1) > \theta$, $R_1(c, n_3, c_1) > \theta$, and $S_1(a_1, \gamma, b_1, \beta, c_1) > \theta$.

Let $u \in M_1$ be such that $S_1(a, \gamma, b, \beta, u) > \theta$. Then, as in the proof of [17, Theorem 4.2] we have $R_1(u, n', c) > \theta$ for any $n' \in N$. Thus, $w \circ N \sim c \circ N$ and so $f(c) = f(w)$. Since $S_1(a, \gamma, b, \beta, u) > \theta$, we have $S_2(f(a), \gamma, f(b), \beta, f(u)) > \theta$. Then, $S_2(f(a), \gamma, f(b), \beta, f(c)) > \theta$. □

**3.22. Theorem.** *Let $f : (M_1, \Gamma, R_1, S_1) \to (M_2, \Gamma, R_2, S_2)$ be a fuzzy gamma homomorphism, and let $A$ and $B$ be fuzzy ideals of $M_1$ and $M_2$, respectively such that $A \subseteq f^{-1}(B)$. Then there exists a fuzzy gamma homomorphism $f^* : M_1/A \to M_2/B$ such that the following diagram commutes:*

$$\begin{array}{ccc}
M_1 & \xrightarrow{\;f\;} & M_2 \\
\Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi'} \\
M_1/A & \xrightarrow{\;f^*\;} & M_2/B
\end{array}$$

*Proof.* Left to the reader. □

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