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SHARPENING AND GENERALIZATIONS OF CARLSON'S INEQUALITY FOR THE ARC COSINE FUNCTION[§]

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Abstract

In this paper, we sharpen and generalize Carlson's double inequality for the arc cosine function.

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1. Introduction and main results

In [1, p. 700, (1.14)] and [3, p. 246, 3.4.30], it was listed that

$$\frac{6(1-x)^{1/2}}{2\sqrt{2}+(1+x)^{1/2}} < \arccos x < \frac{\sqrt[3]{4}\,(1-x)^{1/2}}{(1+x)^{1/6}},\ 0 \le x < 1. \tag{1.1}$$

In [2], the right-hand side inequality in (1.1) was sharpened and generalized.

On the other hand, the left-hand side inequality in (1.1) was also generalized slightly in [2] as follows: For $x \in (0,1)$, the function

$$F_{1/2,1/2,2\sqrt{2}}(x) = \frac{2\sqrt{2}+(1+x)^{1/2}}{(1-x)^{1/2}} \arccos x \tag{1.2}$$

is strictly decreasing. Consequently, the double inequality

$$\frac{6(1-x)^{1/2}}{2\sqrt{2}+(1+x)^{1/2}} < \arccos x < \frac{\left(1/2+\sqrt{2}\,\right)\pi(1-x)^{1/2}}{2\sqrt{2}+(1+x)^{1/2}} \tag{1.3}$$

holds on $(0,1)$ and the constants 6 and $\left(\frac{1}{2}+\sqrt{2}\,\right)\pi$ are the best possible.

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The aim of this paper is to further generalize the left-hand side inequality in (1.1). Our main results may be stated as follows.

1.1. Theorem. *Let a be a real number and*

$$F_a(x)=\frac{a+(1+x)^{1/2}}{(1-x)^{1/2}}\arccos x,\ x\in(0,1). \tag{1.4}$$

(1) *If $a\le\frac{2(\pi-2)}{4-\pi}$, the function $F_a(x)$ is strictly increasing;*
(2) *If $a\ge2\sqrt2$, then the function $F_a(x)$ is strictly decreasing;*
(3) *If $\frac{2(\pi-2)}{4-\pi}<a<2\sqrt2$, the function $F_a(x)$ has a unique minimum.*

1.2. Theorem. *For $a\le\frac{2(\pi-2)}{4-\pi}$,*

$$\frac{[\pi(1+a)/2](1-x)^{1/2}}{a+(1+x)^{1/2}}<\arccos x<\frac{(2+\sqrt2\,a)(1-x)^{1/2}}{a+(1+x)^{1/2}},\ x\in(0,1). \tag{1.5}$$

For $\frac{2(\pi-2)}{4-\pi}<a<2\sqrt2$,

$$\begin{aligned}\frac{8(1-2/a^2)(1-x)^{1/2}}{a+(1+x)^{1/2}}&<\arccos x\\&<\frac{\max\{2+\sqrt2\,a,\pi(1+a)/2\}(1-x)^{1/2}}{a+(1+x)^{1/2}},\ x\in(0,1).\end{aligned} \tag{1.6}$$

For $a\ge2\sqrt2$, the inequality (1.5) reverses on $(0,1)$.

Moreover, the constants $2+\sqrt2\,a$ and $\frac{\pi}{2}(1+a)$ in (1.5) and (1.6) are the best possible.

2. Remarks

Before proving our theorems, we give several remarks on them as follows.

2.1. Remark. The left-hand side inequality in (1.1) and the double inequality (1.3) are the special case $a=2\sqrt2$ of the double inequality (1.6). This shows that Theorem 1.1 and Theorem 1.2 sharpen and generalize the left-hand side inequality in (1.1).

2.2. Remark. It is easy to verify that the function $a\mapsto\frac{1+a}{a+(1+x)^{1/2}}$ is increasing and the function $a\mapsto\frac{2+\sqrt2\,a}{a+(1+x)^{1/2}}$ is decreasing. Therefore, the sharp inequalities deduced from (1.5) are

$$\begin{aligned}\frac{\pi^2(1-x)^{1/2}}{2\left[2(\pi-2)+(4-\pi)(1+x)^{1/2}\right]}&<\arccos x\\&<\frac{2\left[2\left(2-\sqrt2\right)+\left(\sqrt2-1\right)\pi\right](1-x)^{1/2}}{2(\pi-2)+(4-\pi)(1+x)^{1/2}}\end{aligned} \tag{2.1}$$

and

$$\frac{\pi\left(1+2\sqrt2\right)(1-x)^{1/2}}{2\left[2\sqrt2+(1+x)^{1/2}\right]}>\arccos x>\frac{6(1-x)^{1/2}}{2\sqrt2+(1+x)^{1/2}} \tag{2.2}$$

on $(0,1)$.

Furthermore, it is not difficult to see that the double inequalities (2.1) and (2.2) do not include each other.

2.3. Remark. Let

$$h_x(a)=\frac{1-2/a^2}{a+(1+x)^{1/2}}$$

for $\frac{2(\pi-2)}{4-\pi}<a<2\sqrt{2}$ and $x\in(0,1)$. Direct calculation yields

$$h_x'(a)=\frac{4\sqrt{1+x}+6a-a^3}{a^3\big(a+\sqrt{1+x}\,\big)^2}$$

which satisfies

$$\begin{aligned}(2+a)\big(\sqrt{3}-1+a\big)\big(1+\sqrt{3}-a\big)&=4+6a-a^3\\&<a^3\big(a+\sqrt{1+x}\,\big)^2h_x'(a)\\&=4\sqrt{1+x}+6a-a^3\\&<4\sqrt{2}+6a-a^3\\&=\big(a+\sqrt{2}\,\big)^2\big(2\sqrt{2}-a\big).\end{aligned}$$

Accordingly,

(1) When $\frac{2(\pi-2)}{4-\pi}<a\le 1+\sqrt{3}$, the function $a\mapsto h_x(a)$ is increasing;
(2) When $1+\sqrt{3}<a<2\sqrt{2}$, the function $a\mapsto h_x(a)$ attains its maximum

$$\frac{4\cos^2\Big(\frac13\arctan\frac{\sqrt{1-x}}{\sqrt{1+x}}\Big)-1}{4\Big[2\sqrt{2}\cos\Big(\frac13\arctan\frac{\sqrt{1-x}}{\sqrt{1+x}}\Big)+\sqrt{1+x}\Big]\cos^2\Big(\frac13\arctan\frac{\sqrt{1-x}}{\sqrt{1+x}}\Big)}$$

at the point

$$2\sqrt{2}\cos\bigg(\frac13\arctan\frac{\sqrt{1-x}}{\sqrt{1+x}}\bigg).$$

As a result, the sharp inequalities deduced from (1.6) are

$$\frac{8\big[1-2/\big(1+\sqrt{3}\,\big)^2\big](1-x)^{1/2}}{1+\sqrt{3}+(1+x)^{1/2}}<\arccos x<\frac{\pi\big(2-\sqrt{2}\,\big)(1-x)^{1/2}}{4-\pi+\big(\pi-2\sqrt{2}\,\big)(1+x)^{1/2}}\tag{2.3}$$

and

$$\frac{2\Big[4\cos^2\Big(\frac13\arctan\frac{\sqrt{1-x}}{\sqrt{1+x}}\Big)-1\Big](1-x)^{1/2}}{\Big[2\sqrt{2}\cos\Big(\frac13\arctan\frac{\sqrt{1-x}}{\sqrt{1+x}}\Big)+\sqrt{1+x}\Big]\cos^2\Big(\frac13\arctan\frac{\sqrt{1-x}}{\sqrt{1+x}}\Big)}<\arccos x\tag{2.4}$$

on $(0,1)$.

2.4. Remark. By the famous software MATHEMATICA 7.0 and standard computation, we show that

(1) The inequality (2.4) includes the right-hand side inequality in (2.2) and the left-hand side inequality in (2.3);
(2) The left-hand side inequality (2.1) and the inequality (2.4) are not included in each other;
(3) The upper bound in (2.3) is better than those in (2.1) and (2.2).

In conclusion, we obtain the following best and sharp double inequality

$$\begin{aligned}&\frac{\pi\big(2-\sqrt{2}\,\big)(1-x)^{1/2}}{4-\pi+\big(\pi-2\sqrt{2}\,\big)(1+x)^{1/2}}\\ &\qquad>\arccos x\\ &\qquad>\max\biggl\{\frac{2\big[4\lambda^2(x)-1\big](1-x)^{1/2}}{\big[2\sqrt{2}\,\lambda(x)+(1+x)^{1/2}\big]\lambda^2(x)},\frac{\pi^2(1-x)^{1/2}}{2\big[2(\pi-2)+(4-\pi)(1+x)^{1/2}\big]}\biggr\}\end{aligned}\tag{2.5}$$

for $x\in(0,1)$, where

$$\lambda(x)=\cos\biggl(\frac{1}{3}\arctan\frac{\sqrt{1-x}}{\sqrt{1+x}}\biggr),\ x\in(0,1).\tag{2.6}$$

2.5. Remark. Letting $\arccos x=t$ in (2.5) leads to

$$\begin{aligned}&\max\biggl\{\frac{2\big[4\cos^2(t/6)-1\big]\sin(t/2)}{\big[2\cos(t/6)+\cos(t/2)\big]\cos^2(t/6)},\frac{\pi^2\sin(t/2)}{2\big[\sqrt{2}\,(\pi-2)+(4-\pi)\cos(t/2)\big]}\biggr\}\\ &\qquad<t\\ &\qquad<\frac{2\pi\big(\sqrt{2}\,-1\big)\sin(t/2)}{4-\pi+\sqrt{2}\,\big(\pi-2\sqrt{2}\,\big)\cos(t/2)},\ 0<t<\frac{\pi}{2}.\end{aligned}\tag{2.7}$$

This may be rearranged as

$$\begin{aligned}&\max\biggl\{\frac{\big[2\cos(t/6)+\cos(t/2)\big]\cos^2(t/6)}{4\cos^2(t/6)-1},\frac{4\big[\sqrt{2}\,(\pi-2)+(4-\pi)\cos(t/2)\big]}{\pi^2}\biggr\}\\ &\qquad>\frac{\sin(t/2)}{t/2}\\ &\qquad>\frac{4-\pi+\sqrt{2}\,\big(\pi-2\sqrt{2}\,\big)\cos(t/2)}{\pi\big(\sqrt{2}\,-1\big)},\ 0<t<\frac{\pi}{2}.\end{aligned}\tag{2.8}$$

Therefore, we have

$$\begin{aligned}&\max\biggl\{\frac{\big[2\cos(t/3)+\cos t\big]\cos^2(t/3)}{4\cos^2(t/3)-1},\frac{4\big[\sqrt{2}\,(\pi-2)+(4-\pi)\cos t\big]}{\pi^2}\biggr\}\\ &\qquad>\frac{\sin t}{t}\\ &\qquad>\frac{4-\pi+\sqrt{2}\,\big(\pi-2\sqrt{2}\,\big)\cos t}{\pi\big(\sqrt{2}\,-1\big)},\ 0<t<\frac{\pi}{4}.\end{aligned}\tag{2.9}$$

It is noted that the double inequality (2.9) improves related inequalities surveyed in [4, Section 3] and [8, Section 1.7].

2.6. Remark. The approach used in this paper to prove Theorem 1.1 and Theorem 1.2 has been utilized in [2, 5, 6, 7, 9, 10] to establish similar monotonicity and inequalities related to the arc sine, arc cosine and arc tangent functions. For more information on this topic, please see the expository and survey article [8].

3. Proofs of Theorem 1.1 and Theorem 1.2

Now we are in a position to verify our theorems.

Proof of Theorem 1.1. Straightforward differentiation yields

$$\begin{aligned} F_a'(x) &= \frac{\sqrt{1-x^2}\big(a\sqrt{x+1}\,+2\big)}{2(x-1)^2(x+1)}\left[\frac{2(x-1)\big(a\sqrt{x+1}\,+x+1\big)}{\sqrt{1-x^2}\,\big(a\sqrt{x+1}\,+2\big)}+\arccos x\right] \\ &\triangleq \frac{\sqrt{1-x^2}\big(a\sqrt{x+1}\,+2\big)}{2(x-1)^2(x+1)}G_a(x), \end{aligned}$$

and

$$\begin{aligned} G_a'(x) &= \frac{\big(a^2\sqrt{x+1}\,-ax-a-4\sqrt{x+1}\big)\sqrt{1-x}}{(1+x)\big(a\sqrt{x+1}\,+2\big)^2} \\ &\triangleq \frac{H_a(x)\sqrt{1-x}}{(1+x)\big(a\sqrt{x+1}\,+2\big)^2} \end{aligned}$$

It is clear that only if $a \notin \big(-2,-\sqrt{2}\big)$ the denominators of $G_a'(x)$ and $G_a(x)$ do not equal zero on $(0,1)$ and that the function $H_a(x)$ has two zeros

$$a_1(x) = \frac{x+1-\sqrt{x^2+18x+17}}{2\sqrt{x+1}} \quad \text{and} \quad a_2(x) = \frac{x+1+\sqrt{x^2+18x+17}}{2\sqrt{x+1}}$$

whose derivatives are

$$a_1'(x) = \frac{\sqrt{x^2+18x+17}\,-x-1}{4\sqrt{(1+x)(x^2+18x+17)}} > 0$$

and

$$a_2'(x) = \frac{1+x+\sqrt{x^2+18x+17}}{4\sqrt{(1+x)(x^2+18x+17)}} > 0$$

with

$$\begin{aligned} &\lim_{x\to 0^+} a_1(x) = \frac{1-\sqrt{17}}{2}, \quad \lim_{x\to 1^-} a_1(x) = -\sqrt{2}\,, \\ &\lim_{x\to 0^+} a_2(x) = \frac{1+\sqrt{17}}{2}, \quad \lim_{x\to 1^-} a_2(x) = 2\sqrt{2}\,. \end{aligned}$$

Since the functions $a_1(x)$ and $a_2(x)$ are strictly increasing on $(0,1)$, the following conclusions can be derived:

(1) When $a \le -2 < \frac{1-\sqrt{17}}{2} < -\sqrt{2}$ or $a \ge 2\sqrt{2}$, the function $H_a(x)$ and the derivative $G_a'(x)$ are always positive on $(0,1)$, and so the function $G_a(x)$ is strictly increasing on $(0,1)$. From

$$\text{(3.1)} \qquad \lim_{x\to 0^+} G_a(x) = \frac{(\pi-4)a+2(\pi-2)}{2(a+2)} \quad \text{and} \quad \lim_{x\to 1^-} G_a(x) = 0,$$

it follows that the functions $G_a(x)$ and $F_a'(x)$ are negative, and so the function $F_a(x)$ is strictly decreasing on $(0,1)$.

(2) When $-\sqrt{2} \le a \le \frac{1+\sqrt{17}}{2}$, the function $H_a(x)$ and the derivative $G_a'(x)$ are negative on $(0,1)$, and so the function $G_a(x)$ is strictly decreasing on $(0,1)$. From (3.1), it is obtained that the function $G_a(x)$ and the derivative $F_a'(x)$ are positive. So the function $F_a(x)$ is strictly increasing on $(0,1)$.

(3) When $\frac{1+\sqrt{17}}{2} < a < 2\sqrt{2}$, the functions $H_a(x)$ and $G_a'(x)$ have a unique zero which is the unique maximum point of $G_a(x)$. From (3.1), it is deduced that
   (a) If $\frac{1+\sqrt{17}}{2} < a \le \frac{2(\pi-2)}{4-\pi}$, the functions $G_a(x)$ and $F_a'(x)$ are positive, and so the function $F_a(x)$ is strictly increasing on $(0,1)$.
   (b) If $\frac{2(\pi-2)}{4-\pi} < a < 2\sqrt{2}$, the functions $G_a(x)$ and $F_a'(x)$ have a unique zero which is the unique minimum point of the function $F_a(x)$ on $(0,1)$.

On the other hand, the derivative $F_a'(x)$ can be rearranged as

$$\begin{aligned} F_a'(x) &= \frac{\sqrt{1-x^2}}{2(x-1)^2(x+1)}\left[\frac{2(x-1)\big(a\sqrt{x+1}\,+x+1\big)}{\sqrt{1-x^2}} + \big(a\sqrt{x+1}\,+2\big)\arccos x\right] \\ &\triangleq \frac{\sqrt{1-x^2}}{2(x-1)^2(x+1)}Q_a(x), \end{aligned}$$

with

$$\begin{aligned} Q_a'(x) &= \frac{\arccos x}{2\sqrt{x+1}}\left(a-\frac{4\sqrt{1-x}}{\arccos x}\right) \\ &\triangleq \frac{\arccos x}{2\sqrt{x+1}}[a-P(x)], \\ P'(x) &= \frac{2(x+1)}{\sqrt{x+1}\,\sqrt{1-x^2}\,(\arccos x)^2}\left[\frac{2\sqrt{1-x^2}}{x+1}-\arccos x\right] \\ &\triangleq \frac{2(x+1)}{\sqrt{x+1}\,\sqrt{1-x^2}\,(\arccos x)^2}R(x) \end{aligned}$$

and

$$R'(x) = \frac{x-1}{(x+1)\sqrt{1-x^2}} < 0.$$

From $\lim_{x\to1^-} R(x) = 0$ and the decreasingly monotonic property of $R(x)$, we obtain that $R(x) > 0$, and so the function $P(x)$ is strictly increasing. Since

$$\lim_{x\to0^+} P(x) = \frac{8}{\pi} \quad \text{and} \quad \lim_{x\to1^-} P(x) = 2\sqrt{2}\,,$$

the function $Q_a(x)$ is strictly decreasing (or increasing, respectively) with respect to $x \in (0,1)$ for $a \le \frac{8}{\pi}$ (or $a \ge 2\sqrt{2}$, respectively). By virtue of $\lim_{x\to1^-} Q_a(x) = 0$, it follows that

(1) If $a \le \frac{8}{\pi}$, the function $Q_a(x)$ is positive on $(0,1)$;
(2) If $a \ge 2\sqrt{2}$, the function $Q_a(x)$ is negative on $(0,1)$.

These imply that the function $F_a(x)$ is strictly increasing for $a \le \frac{8}{\pi} < \frac{2(\pi-2)}{4-\pi}$ and strictly decreasing for $a \ge 2\sqrt{2}$. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. Easy calculation gives

$$\lim_{x\to0^+} F_a(x) = \frac{\pi}{2}(1+a) \quad \text{and} \quad \lim_{x\to1^-} F_a(x) = 2+\sqrt{2}\,a.$$

By the monotonicity of $F_a(x)$ procured in Theorem 1.1, it follows that

(1) If $a \le \frac{2(\pi-2)}{4-\pi}$, then

$$\frac{\pi}{2}(1+a) < F_a(x) < 2+\sqrt{2}\,a$$

on $(0,1)$, which can be rearranged as the inequality (1.5);

(2) If $a \ge 2\sqrt{2}$, the inequality (1.5) is reversed;
(3) If $\frac{2(\pi-2)}{4-\pi} < a < 2\sqrt{2}$, the function $F_a(x)$ has a unique minimum, so

$$F_a(x) < \max\left\{\frac{\pi}{2}(1+a), 2+\sqrt{2}\,a\right\}$$

on $(0,1)$, which is equivalent to the right-hand side inequality (1.6).

Furthermore, the minimum point $x_0 \in (0,1)$ of the function $F_a(x)$ satisfies

$$\arccos x_0 = \frac{2(1-x_0)\big(a\sqrt{x_0+1}\,+x_0+1\big)}{\sqrt{1-x_0^2}\,\big(a\sqrt{x_0+1}\,+2\big)},$$

and so

$$F_a(x_0) = \frac{2\bigl(a+\sqrt{x_0+1}\,\bigr)\bigl(a\sqrt{x_0+1}\,+x_0+1\bigr)}{\sqrt{1+x_0}\,\bigl(a\sqrt{x_0+1}\,+2\bigr)} \triangleq \frac{2(a+u)^2}{au+2} \ge 8\left(1-\frac{2}{a^2}\right),$$

where $u=\sqrt{1+x_0}\in\bigl(1,\sqrt{2}\,\bigr)$. The left-hand side inequality in (1.6) follows. The proof of Theorem 1.2 is complete. □

4. An open problem

Finally, we propose the following open problem.

4.1. Open Problem. For real numbers α, β and γ, let

$$F_{\alpha,\beta,\gamma}(x) = \frac{\gamma+(1+x)^\beta}{(1-x)^\alpha}\arccos x,\ \ x\in(0,1). \tag{4.1}$$

Find the ranges of the constants α, β and γ such that the function $F_{\alpha,\beta,\gamma}(x)$ is monotonic on $(0,1)$.

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