Hacettepe Journal of Mathematics and Statistics
Volume 39 (3) (2010), 417 – 428

SOLUTION OF A MIXED PROBLEM WITH PERIODIC BOUNDARY CONDITION FOR A QUASI-LINEAR EULER-BERNOULLI EQUATION

H. Halilov*†, K. Kutlu*, B. Ö. Güler*

Received 04 : 02 : 2010 : Accepted 22 : 02 : 2010

Abstract

In this paper, the existence and uniqueness of the weak generalized solution of a mixed problem with periodic boundary condition for a quasi-linear Euler-Bernoulli equation are examined, and an estimation of the differences between the exact and approximate solution is obtained. In order to solve the problem, first the test functions are given, then the weak generalized solution of the problem is defined in terms of these functions. The weak solution is expressed as a Fourier series with undetermined variable coefficients, and a system of non-linear infinite integral equations for the coefficients mentioned above is obtained. The existence and uniqueness of the solution of the system are proved by the successive approximation method on the Banach space B_T. Finally, in view of the practical importance of the problem, the norm of the difference between the exact solution and successive approximations of the infinite system is estimated on the space B_T.

Keywords: Partial derivative, Periodic boundary condition, Quasi-linear, Mixed problem, Euler-Bernoulli equation, Fourier method, Non-linear infinite integral equations.

2000 AMS Classification: 35 K 55, 35 K 70.

*Rize University, Faculty of Science and Letters, Department of Mathematics, 53050 Rize, Turkey. E-mail: (H. Halilov) `huseyin.halilov@rize.edu.tr` (K. Kutlu) `kkutlu@ttmail.com` (B. Ö. Güler) `bahadir.guler@rize.edu.tr`

†Corresponding Author.

1. Introduction

The investigation of various problems concerning 4 th order homogeneous, linear and quasi-linear equations has been one of the most attractive areas for mathematicians and engineers due to their importance in the solution of several engineering problems. The reader is refereed to [1, 2, 3, 8, 9, 11, 14] for some relevant previous work on linear and quasi-linear equations, and to [5, 6, 7, 10, 16] for applications. The textbooks [4, 12, 13, 15] also contain important results.

In this study, the existence and uniqueness of a weak generalized solution of a mixed problem with periodic boundary condition for the quasi-linear Euler-Bernoulli equation are examined by the non-linear Fourier method for the first time. We hope that, in addition to being of interest to mathematicians, the examination, results and method applied in the study will be useful to engineers who are dealing with the solution of problems involving dynamic stability, free and forced vibration of bars consisting of composite materials and carbon nanotubes.

2. Establishing the problem

We consider the following mixed problem with periodic boundary condition:

$$\frac{\partial^2 u}{\partial t^2} - \varepsilon b^2 \frac{\partial^4 u}{\partial t^2 \partial x^2} + a^2 \frac{\partial^4 u}{\partial^4 x} = f(t,x,u), \quad (t,x) \in D\{0 < t \le T,\ 0 < x < \pi\}, \tag{2.1}$$

$$u(0,x,\varepsilon) = \varphi(x), \quad u_t(0,x,\varepsilon) = \psi(x),\ (0 \le x \le T), \tag{2.2}$$

$$\begin{aligned} &u(t,0,\varepsilon) = u(t,\pi,\varepsilon), \quad u_x(t,0,\varepsilon) = u_x(t,\pi,\varepsilon), \\ &u_{x^2}(t,0,\varepsilon) = u_{x^2}(t,\pi,\varepsilon), \quad u_{x^3}(t,0,\varepsilon) = u_{x^3}(t,\pi,\varepsilon),\ (0 \le x \le T), \end{aligned} \tag{2.3}$$

where a, b are constants, $\varepsilon \in [0, \varepsilon_0]$ is a parameter, $\varphi(x), \psi(x)$ and $f(t,x,u)$ are functions defined on $[0,\pi]$ and $\bar{D}\{0 \le t \le T,\ 0 \le x \le \pi\} \times (-\infty,\ \infty)$ respectively, and $u(t,x,\varepsilon)$ is a solution of the problem considered.

2.1. Definition. The function $v(t,x) \in C(\bar{D})$ is called a *test function* if it has continuous partial derivatives of orders involved in Equation (2.1), and satisfies both the following conditions

$$v(T,x) = v_t(T,x) = v_{x^2}(T,x) = v_{x^2 t}(T,x) = 0$$

and the boundary condition (2.3).

We give the definition below from H. I. Chandirov [1] who, for the first time, introduced the applicability of the Fourier method to non-linear mixed problems.

2.2. Definition. The function $u(t,x,\varepsilon) \in C(\bar{D}) \times [0, \varepsilon_0]$ satisfying the integral identity

$$\begin{aligned} &\int_0^T \int_0^\pi \left\{ u \left[\frac{\partial^2 v}{\partial t^2} - \varepsilon b^2 \frac{\partial^4 v}{\partial x^2 \partial t^2} + a^2 \frac{\partial^4 v}{\partial x^4} \right] - f(t,x,u) v \right\} dx\, dt + \\ &\int_0^\pi \varphi(x) \left[v_t(0,x) - \varepsilon b^2 v_{x^2 t}(0,x) \right] dx - \int_0^\pi \psi(x) \left[v(0,x) - \varepsilon b^2 v_{x^2}(0,x) \right] dx = 0 \end{aligned} \tag{2.4}$$

for an arbitrary test function $v(t,x)$ is called a *weak generalized solution* of problem (2.1)-(2.3).

The set

$$\{\bar{u}(t,\varepsilon)\} = \left\{ \frac{1}{2} u_0(t,\varepsilon), u_{c1}(t,\varepsilon), u_{s1}(t,\varepsilon), \dots, u_{ck}(t,\varepsilon), u_{sk}(t,\varepsilon), \dots \right\}$$

of functions continuous on $[0,T]\times[0,\varepsilon_0]$ satisfying the condition

$$\frac{1}{2}\max_{t\in[0,T]}|u_0(t,\varepsilon)|+\sum_{k=1}^{\infty}\Big[\max_{t\in[0,T]}|u_{ck}(t,\varepsilon)|+\max_{t\in[0,T]}|u_{sk}(t,\varepsilon)|\Big]<\infty$$

will be denoted by B_T. Let

$$\left\|\overline{u}(t,\varepsilon)\right\|_{B_T}=\frac{1}{2}\max_{t\in[0,T]}|u_0(t,\varepsilon)|+\sum_{k=1}^{\infty}\Big[\max_{t\in[0,T]}|u_{ck}(t,\varepsilon)|+\max_{t\in[0,T]}|u_{sk}(t,\varepsilon)|\Big]$$

be the norm in B_T. It can be shown that B_T is Banach space.

3. The solution

We search formerly for the weak generalized solution of problem (2.1)-(2.3) as

$$u(t,x,\varepsilon)=\frac{1}{2}u_0(t,\varepsilon)+\sum_{k=1}^{\infty}\left[u_{ck}(t,\varepsilon)\cos 2kx+u_{sk}(t,\varepsilon)\sin 2kx\right] \tag{3.1}$$

where $u_0(t,\varepsilon)$, $u_{ck}(t,\varepsilon)$, $u_{sk}(t,\varepsilon)$, $(k=\overline{1,\infty})$, are unknown functions. Employing the equality (2.4) we get the following infinite non-linear system of integral equations:

$$\begin{aligned}
u_0(t,\varepsilon)&=\varphi_0+\psi_0 t+\frac{2}{\pi}\int_0^t\int_0^\pi (t-\tau)f\Big\{\tau,\xi,\frac{1}{2}u_0(\tau,\varepsilon)\\
&\qquad+\sum_{n=1}^{\infty}\left[u_{cn}(\tau,\varepsilon)\cos 2n\xi+u_{sn}(\tau,\varepsilon)\sin 2n\xi\right]\Big\}\,d\xi\,d\tau,\\
u_{ck}(t,\varepsilon)&=\varphi_{ck}\cos\alpha_k t+\frac{\psi_{ck}}{\alpha_k}\sin\alpha_k t+\frac{2}{\pi\alpha_k}\int_0^t\int_0^\pi f\Big\{\tau,\xi,\frac{1}{2}u_0(\tau,\varepsilon)\\
&\qquad+\sum_{n=1}^{\infty}\left[u_{cn}(\tau,\varepsilon)\cos 2n\xi+u_{sn}(\tau,\varepsilon)\sin 2n\xi\right]\Big\}\\
&\qquad\times\cos 2k\xi\sin\alpha_k(t-\tau)\,d\xi\,d\tau,\\
u_{sk}(t,\varepsilon)&=\varphi_{sk}\cos\alpha_k t+\frac{\psi_{sk}}{\alpha_k}\sin\alpha_k t+\frac{2}{\pi\alpha_k}\int_0^t\int_0^\pi f\Big\{\tau,\xi,\frac{1}{2}u_0(\tau,\varepsilon)\\
&\qquad+\sum_{n=1}^{\infty}\left[u_{cn}(\tau,\varepsilon)\cos 2n\xi+u_{sn}(\tau,\varepsilon)\sin 2n\xi\right]\Big\}\\
&\qquad\times\sin 2k\xi\sin\alpha_k(t-\tau)\,d\xi\,d\tau,
\end{aligned} \tag{3.2}$$

$$\alpha_k=\frac{a(2k)^2}{\sqrt{1+\varepsilon b^2(2k)^2}},\quad k=\overline{1,\infty}.$$

3.1. Theorem. *Suppose the following conditions are satisfied:*

a) $f(t,x,u)$ *is continuous respect to all arguments on* $\overline{D}\times(-\infty,\infty)$,
b) $|f(t,x,u)-f(t,x,v)|\le b(t,x)|u-v|$, *where* $b(t,x)\in L_2(D)$, $b(t,x)>0$,
c) $f(t,x,0)\in L_2(D)$,
d) $\varphi(0)=\varphi(\pi)$, $\varphi'(0)=\varphi'(\pi)$, $\psi(0)=\psi(\pi)$, *where* $\varphi(x)\in C^1[0,\pi],\psi(x)\in C[0,\pi]$.

Then the system (3.2) has a unique solution in B_T.

Proof. We will prove the theorem by the successive approximation method. The successive approximations for system (3.2) are as follows:

$$(3.3)\quad \begin{aligned} u_0^{(N+1)}(t,\varepsilon) &= u_0^{(0)}(t,\varepsilon) + \frac{2}{\pi}\int_0^t\int_0^\pi (t-\tau) f\Big\{\tau,\xi,\frac{1}{2}u_0^{(N)}(\tau,\varepsilon) \\ &\qquad + \sum_{n=1}^{\infty}\Big[u_{cn}^{(N)}(\tau,\varepsilon)\cos 2n\xi + u_{sn}^{(N)}(\tau,\varepsilon)\sin 2n\xi\Big]\Big\}\, d\xi\, d\tau, \\ u_{ck}^{(N+1)}(t,\varepsilon) &= u_{ck}^{(0)}(t,\varepsilon) + \frac{2}{\pi\alpha_k}\int_0^t\int_0^\pi f\Big\{\tau,\xi,\frac{1}{2}\ u_0^{(N)}(\tau,\varepsilon) \\ &\qquad + \sum_{n=1}^{\infty}\Big[u_{cn}^{(N)}(\tau,\varepsilon)\cos 2n\xi + u_{sn}^{(N)}(\tau,\varepsilon)\sin 2n\xi\Big]\Big\} \\ &\qquad \times \cos 2k\xi \sin\alpha_k(t-\tau)\, d\xi\, d\tau, \\ u_{sk}^{(N+1)}(t,\varepsilon) &= u_{sk}^{(0)}(t,\varepsilon) + \frac{2}{\pi\alpha_k}\int_0^t\int_0^\pi f\Big\{\tau,\xi,\frac{1}{2}u_0^{(N)}(\tau,\varepsilon) \\ &\qquad + \sum_{n=1}^{\infty}\Big[u_{cn}^{(N)}(\tau,\varepsilon)\cos 2n\xi + u_{sn}^{(N)}(\tau,\varepsilon)\sin 2n\xi\Big]\Big\} \\ &\qquad \times \sin 2k\xi \sin\alpha_k(t-\tau)\, d\xi\, d\tau, \\ N &= \overline{0,\infty}, \end{aligned}$$

where

$$u_0^{(0)}(t,\varepsilon) = \varphi_0 + \psi_0 t,\ u_{ck}^{(0)}(t,\varepsilon) = \varphi_{ck}\cos\alpha_k t + \frac{\psi_{ck}}{\alpha_k}\sin\alpha_k t,$$
$$u_{sk}^{(0)}(t,\varepsilon) = \varphi_{sk}\cos\alpha_k t + \frac{\psi_{sk}}{\alpha_k}\sin\alpha_k t,\ (k=\overline{1,\infty}).$$

For simplicity, letting

$$Au^{(N)}(t,\xi,\varepsilon) = \frac{1}{2}\ u_0^{(N)}(t,\varepsilon) + \sum_{n=1}^{\infty}\Big[u_{cn}^{(N)}(t,\varepsilon)\cos 2n\xi + u_{sn}^{(N)}(t,\varepsilon)\sin 2n\xi\Big]$$

and

$$\big\{\overline{u}^{(N)}(t,\xi)\big\} = \Big\{\frac{1}{2}u_0^{(N)}(t,\varepsilon), u_{c1}^{(N)}(t,\varepsilon), u_{s1}^{(N)}(t,\varepsilon),\dots, u_{cn}^{(N)}(t,\varepsilon), u_{sn}^{(N)}(t,\varepsilon),\dots\Big\},$$

the successive approximations of (3.2) become

$$(3.4)\quad \begin{aligned} u_0^{(N+1)}(t,\varepsilon) &= u_0^{(0)}(t,\varepsilon) + \frac{2}{\pi}\int_0^t\int_0^\pi (t-\tau) f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\, d\xi\, d\tau, \\ u_{ck}^{(N+1)}(t,\varepsilon) &= u_{ck}^{(0)}(t,\varepsilon) + \frac{2}{\pi\alpha_k}\int_0^t\int_0^\pi f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)] \\ &\qquad \times \cos 2k\xi \sin\alpha_k(t-\tau)\, d\xi\, d\tau, \\ u_{sk}^{(N+1)}(t,\varepsilon) &= u_{sk}^{(0)}(t,\varepsilon) + \frac{2}{\pi\alpha_k}\int_0^t\int_0^\pi f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)] \\ &\qquad \times \sin 2k\xi \sin\alpha_k(t-\tau)\, d\xi\, d\tau, \\ (k &= \overline{1,\infty}). \end{aligned}$$

It is clear that

$$\max_{0\le t\le T}|Au^{(N)}(\tau,\xi,\varepsilon)| \le \frac{1}{2}\max_{0\le t\le T}|u_0^{(N)}(t,\varepsilon)| + \sum_{n=1}^{\infty}\left[\max_{0\le t\le T}|u_{cn}^{(N)}(t,\varepsilon)| + \max_{0\le t\le T}|u_{sn}^{(N)}(t,\varepsilon)|\right] = \|\overline{u}^{(N)}(t,\varepsilon)\|_{B_T}. \tag{3.5}$$

First, let us prove $\overline{u}^{(N)}(t,\varepsilon)\in B_T$. From the conditions of the theorem it is easily seen that

$$\begin{aligned}\|\overline{u}^{(0)}(t,\varepsilon)\|_{B_T} =& \frac{1}{2}\max_{0\le t\le T}|u_0^{(0)}(t,\varepsilon)| + \sum_{n=1}^{\infty}\left[\max_{0\le t\le T}|u_{cn}^{(0)}(t,\varepsilon)| + \max_{0\le t\le T}|u_{sn}^{(0)}(t,\varepsilon)\right]\\ &\le \frac{1}{2}(|\varphi_0|+|\psi_0|T) + \sum_{k=1}^{\infty}\Big[(|\varphi_{ck}| + \frac{1}{\alpha_k}|\psi_{ck}|)\\ &\qquad + (|\varphi_{sk}| + \frac{1}{\alpha_k}|\psi_{sk}|)\Big]\\ &< \infty.\end{aligned}$$

Taking $N=0$ in the equalities (3.4), the first equality obtained may be written as

$$\begin{aligned}u_0^{(1)}(t,\varepsilon) = u_0^{(0)}(t,\varepsilon) &+ \frac{2}{\pi}\int_0^t\int_0^{\pi}(t-\tau)\big\{f[\tau,\xi,Au^{(0)}(t,\xi,\varepsilon)] - f(\tau,\xi,0)\big\}\,d\xi\,d\tau\\ &+ \frac{2}{\pi}\int_0^t\int_0^{\pi}(t-\tau)f(\tau,\xi,0)\,d\xi\,d\tau,\end{aligned}$$

then applying the Cauchy inequality with respect to t to both integrals on the right hand side we get

$$\begin{aligned}|u_0^{(1)}(t,\varepsilon)| \le |u_0^{(0)}(t,\varepsilon)| &+ \frac{2}{\pi}\left[\int_0^t\int_0^{\pi}(t-\tau)^2\,d\tau\right]^{1/2}\\ &\times\left(\int_0^t\left[\int_0^{\pi}\big\{f[\tau,\xi,Au^{(0)}(t,\xi,\varepsilon)] - f(\tau,\xi,0)\big\}\,d\xi\right]^2 d\tau\right)^{1/2}\\ &+ \frac{2}{\pi}\left[\int_0^t(t-\tau)^2\,d\tau\right]^{1/2}\left(\int_0^t\left[\int_0^{\pi} f(\tau,\xi,0)\,d\xi\right]^2 d\tau\right)^{1/2}.\end{aligned}$$

Calculating the fist integral in both summands on the right hand side containing integrals, taking the first factor as 1 in the second integrals and applying Cauchy inequality with respect to ξ, we have

$$\begin{aligned}|u_0^{(1)}(t,\varepsilon)| \le& |u_0^{(0)}(t,\varepsilon)|\\ &+ \frac{2}{\pi}\sqrt{\frac{\pi T^3}{3}}\left(\int_0^t\int_0^{\pi}\big\{f[\tau,\xi,Au^{(0)}(t,\xi,\varepsilon)] - f(\tau,\xi,0)\big\}^2\,d\xi\,d\tau\right)^{1/2}\\ &+ \frac{2}{\pi}\sqrt{\frac{\pi T^3}{3}}\left[\int_0^t\int_0^{\pi} f^2(\tau,\xi,0)\,d\xi\,d\tau\right]^{1/2}.\end{aligned}$$

Applying Lipschitz condition to the first integral on the right hand side and making some calculations we get

$$\begin{aligned}|u_0^{(1)}(t,\varepsilon)| \le |u_0^{(0)}(t,\varepsilon)| + \frac{2}{\pi}\sqrt{\frac{T^3\pi}{3}}\Bigg[&\left(\int_0^t\int_0^{\pi} b^2(\tau,\xi)\left[Au^{(0)}(t,\xi,\varepsilon)\right]^2 d\xi\,d\tau\right)^{1/2}\\ &+ \|f(\tau,x,0)\|_{L_2(D)}\Bigg],\end{aligned}$$

hence we have

$$|u_0^{(1)}(t,\varepsilon)| \le |u_0^{(0)}(t,\varepsilon)| + \frac{2}{\pi}\sqrt{\frac{\pi T^3}{3}}\Big[\|b(t,x)\|_{L_2(D)}\|\overline{u}^{(0)}(t,\varepsilon)\|_{B_T} + \|f(t,x,0)\|_{L_2(D)}\Big]. \tag{3.6}$$

The second equality obtained from (3.4) for $N = 0$ can be written as

$$u_{ck}^{(1)}(t,\varepsilon) = u_{ck}^{(0)}(t,\varepsilon) + \frac{2}{\pi\alpha_k}\int_0^t\int_0^\pi \big\{f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)] - f(\tau,\xi,0)\big\} \times \cos 2k\xi \sin\alpha_k(t-\tau)\,d\xi\,d\tau + \frac{2}{\pi\alpha_k}\int_0^t\int_0^\pi f(\tau,\xi,0)\cos 2k\xi\sin\alpha_k(t-\tau)\,d\xi\,d\tau.$$

Then applying the Cauchy inequality with respect to t to both integrals we get

$$|u_{ck}^{(1)}(t,\varepsilon)| \le |u_{ck}^{(0)}(t,\varepsilon)| + \frac{\sqrt{T}}{\alpha_k}\bigg(\int_0^t\bigg[\frac{2}{\pi}\int_0^\pi \big\{f[\tau,\xi,Au^{(0)}(t,\xi,\varepsilon)] - f(\tau,\xi,0)\big\}\cos 2k\xi\,d\xi\bigg]^2 d\tau\bigg)^{1/2} + \frac{\sqrt{T}}{\alpha_k}\bigg(\int_0^t\bigg[\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)\cos 2k\xi\,d\xi\bigg]^2 d\tau\bigg)^{1/2}.$$

Summing both sides with respect to $k = \overline{1,\infty}$, we have

$$\sum_{k=1}^\infty |u_{ck}^{(1)}(t,\varepsilon)| \le \sum_{k=1}^\infty |u_{ck}^{(0)}(t,\varepsilon)| + \sqrt{T}\sum_{k=1}^\infty\frac{1}{\alpha_k}\bigg(\int_0^t\bigg[\frac{2}{\pi}\int_0^\pi\big\{f[\tau,\xi,Au^{(0)}(t,\xi,\varepsilon)] - f(\tau,\xi,0)\big\}\cos 2k\xi\,d\xi\bigg]^2 d\tau\bigg)^{1/2} + \sqrt{T}\sum_{k=1}^\infty\frac{1}{\alpha_k}\bigg(\int_0^t\bigg[\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)\cos 2k\xi\,d\xi\bigg]^2 d\tau\bigg)^{1/2}.$$

Applying Hölder's inequality to the second and third sums after the required processes, using Bessel's inequality related to the Fourier coefficients, and taking the maximum of the integrals on the right hand side respect to t, we have

$$\sum_{k=1}^\infty |u_{ck}^{(1)}(t,\varepsilon)| \le \sum_{k=1}^\infty |u_{ck}^{(0)}(t,\varepsilon)| + M\sqrt{T}\Big[\|b(t,x)\|_{L_2(D)}\|\overline{u}^{(0)}(t,\varepsilon)\|_{B_T} + \|f(t,x,0)\|_{L_2(D)}\Big], \tag{3.7}$$

where $M = \Big(\sum_{k=1}^\infty \frac{1}{\alpha_k^2}\Big)^{1/2}$.

Analogously, for $u_{sk}^{(1)}(t,\varepsilon)$ we get

$$\sum_{k=1}^\infty |u_{sk}^{(1)}(t,\varepsilon)| \le \sum_{k=1}^\infty |u_{sk}^{(0)}(t,\varepsilon)| + M\sqrt{T}\Big[\|b(t,x)\|_{L_2(D)}\|\overline{u}^{(0)}(t,\varepsilon)\|_{B_T} + \|f(t,x,0)\|_{L_2(D)}\Big]. \tag{3.8}$$

Using the inequalities (3.6), (3.7) and (3.8) as follows

$$\frac{|u_0^{(1)}(t,\varepsilon)|}{2}+\sum_{k=1}^{\infty}\Big[|u_{ck}^{(1)}(t,\varepsilon)|+|u_{sk}^{(1)}(t,\varepsilon)|\Big]$$
$$\leq \frac{|u_0^{(0)}(t,\varepsilon)|}{2}+\sum_{k=1}^{\infty}\Big[|u_{ck}^{(0)}(t,\varepsilon)|+|u_{sk}^{(0)}(t,\varepsilon)|\Big]$$
$$+\left(\sqrt{\frac{T^3}{3\pi}}+2M\sqrt{T}\right)\Big[\|b(t,x)\|_{L_2(D)}\|\overline{u}^{(0)}(t,\varepsilon)\|_{B_T}+\|f(t,x,0)\|_{L_2(D)}\Big],$$

then taking the maximum over t we obtain

$$\|\overline{u}^{(1)}(t,\varepsilon)\|_{B_T}$$
$$\leq \|\overline{u}^{(0)}(t,\varepsilon)\|_{B_T}$$
$$+\left(\sqrt{\frac{T^3}{3\pi}}+2M\sqrt{T}\right)\Big[\|b(t,x)\|_{L_2(D)}\|\overline{u}^{(0)}(t,\varepsilon)\|_{B_T}+\|f(t,x,0)\|_{L_2(D)}\Big].$$

Hence, according to the hypothesis of the theorem we obtain

$$\|\overline{u}^{(1)}(t,\varepsilon)\|_{B_T}<\infty.$$

By the principle of mathematical induction, we obtain that

$$\|\overline{u}^{(N)}(t,\varepsilon)\|_{B_T}$$
$$\leq \|\overline{u}^{(0)}(t,\varepsilon)\|_{B_T}+\sqrt{\frac{T}{3\pi}}(T+2\sqrt{6\pi}M)\|b(t,x)\|_{L_2(D)}\|u^{(N-1)}(t,\varepsilon)\|_{B_T}$$
$$+\sqrt{\frac{T}{3\pi}}(T+2\sqrt{6\pi}M)\|f(t,x,0)\|_{L_2(D)}.$$

Proceeding in the same way, it can be shown analogously that if $\|\overline{u}^{(N)}(t,\varepsilon)\|_{B_T}<\infty$ then

$$\|\overline{u}^{(N+1)}(t,\varepsilon)\|_{B_T}<\infty.$$

Therefore, we have proven that

$$\overline{u}^{(N+1)}(t,\varepsilon)=\big\{\tfrac{1}{2}u_0^{(N+1)}(t,\varepsilon),u_{c1}^{(N+1)}(t,\varepsilon),u_{s1}^{(N+1)}(t,\varepsilon),\ldots$$
$$\ldots,u_{ck}^{(N+1)}(t,\varepsilon),u_{sk}^{(N+1)}(t,\varepsilon),\ldots\big\}\in B_T.$$

Now, let us make an estimation of the differences

$$|u_0^{(N+1)}(t,\varepsilon)-u_0^{(N)}(t,\varepsilon)|,|u_{ck}^{(N+1)}(t,\varepsilon)-u_{ck}^{(N)}(t,\varepsilon)|,|u_{sk}^{(N+1)}(t,\varepsilon)-u_{sk}^{(N)}(t,\varepsilon)|,$$

where ($N=\overline{0,\infty}$, $k=\overline{1,\infty}$), respectively, in order to prove the convergence of the successive approximation sequence $\{\overline{u}^{(N)}(t,\varepsilon)\}$ in B_T. Take

$$|u_0^{(1)}(t,\varepsilon)-u_0^{(0)}(t,\varepsilon)|=\left|\frac{2}{\pi}\int_0^t\int_0^\pi(t-\tau)f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]\,d\xi\,d\tau\right|$$
$$\leq\left|\frac{2}{\pi}\int_0^t\int_0^\pi(t-\tau)\{f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0)\}\,d\xi\,d\tau\right|$$
$$+\left|\frac{2}{\pi}\int_0^t(t-\tau)\int_0^\pi f(\tau,\xi,0)\,d\xi\,d\tau\right|,$$

then apply Cauchy's inequality to the integrals on the right hand side with respect to t to give

$$\begin{aligned}&|u_0^{(1)}(t,\varepsilon)-u_0^{(0)}(t,\varepsilon)|\\&\quad\le\Big[\int_0^t(t-\tau)^2\,d\tau\Big]^{1/2}\\&\qquad\times\Big(\int_0^t\Big[\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au^{(0)}(\tau,\xi,\varepsilon)]-f(\tau,\xi,0)\}\,d\xi\Big]^2d\tau\Big)^{1/2}\\&\qquad+\Big[\int_0^t(t-\tau)^2\,d\tau\Big]^{1/2}\Big(\int_0^t\Big[\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)\,d\xi\Big]^2d\tau\Big)^{1/2}.\end{aligned}$$

We calculate the first integral in the summands on the right hand side, and apply Cauchy's inequality to the second integrals with respect to ξ. This gives

$$\begin{aligned}&|u_0^{(1)}(t,\varepsilon)-u_0^{(0)}(t,\varepsilon)|\\&\quad\le 2T\sqrt{\frac{T}{3\pi}}\Big[\int_0^t\int_0^\pi\{f[\tau,\xi,Au^{(0)}(t,\xi,\varepsilon)]-f(\tau,\xi,0)\}^2\,d\xi\,d\tau\Big]^{1/2}\\&\qquad+2T\sqrt{\frac{T}{3\pi}}\Big[\int_0^t\int_0^\pi f^2(\tau,\xi,0)\,d\xi\,d\tau\Big]^{1/2}.\end{aligned}$$

Applying the Lipschitz inequality to the first term on the right hand side and performing some calculations, we get

$$|u_0^{(1)}(t,\varepsilon)-u_0^{(0)}(t,\varepsilon)|\le 2T\sqrt{\frac{T}{3\pi}}\big(\|b(t,x)\|_{L_2(D)}\|\overline{u}^{(0)}(t,\varepsilon)\|_{B_T}+\|f(t,x,0)\|_{L_2(D)}\big).\tag{3.9}$$

We obtain the following estimation ia a similar way

$$\begin{aligned}&|u_{ck}^{(1)}(t,\varepsilon)-u_{ck}^{(0)}(t,\varepsilon)|\\&\quad\le\frac{\sqrt{T}}{\alpha_k}\Big(\int_0^t\Big[\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au^{(0)}(t,\xi,\varepsilon)]-f(\tau,\xi,0)\}\cos 2k\xi\,d\xi\Big]^2\,d\tau\Big)^{1/2}\\&\qquad+\frac{\sqrt{T}}{\alpha_k}\Big(\int_0^t\Big[\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)\cos 2k\xi\,d\xi\Big]^2\,d\tau\Big)^{1/2}.\end{aligned}$$

Taking the sum of both side with respect to k, and applying Hölder's inequality to the integrals, we have the following ($k=\overline{1,\infty}$),

$$\begin{aligned}&\sum_{k=1}^\infty|u_{ck}^{(1)}(t,\varepsilon)-u_{ck}^{(0)}(t,\varepsilon)|\\&\quad\le\sqrt{T}\Big(\sum_{k=1}^\infty\frac{1}{\alpha_k^2}\Big)^{1/2}\Big(\sum_{k=1}^\infty\int_0^t\Big[\frac{2}{\pi}\int_0^\pi\{f[\tau,\xi,Au^{(0)}(t,\xi,\varepsilon)]\\&\qquad-f(\tau,\xi,0)\}\cos 2k\xi\,d\xi\Big]^2d\tau\Big)^{1/2}\\&\qquad+\sqrt{T}\Big(\sum_{k=1}^\infty\frac{1}{\alpha_k^2}\Big)^{1/2}\Big(\sum_{k=1}^\infty\int_0^t\Big[\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)\cos 2k\xi\,d\xi\Big]^2d\tau\Big)^{1/2}.\end{aligned}$$

From the conditions of the theorem, the series on the right hand side are integrable term by term. Hence employing Bessel's inequality we obtain:

$$\sum_{k=1}^{\infty} \left|u_{ck}^{(1)}(t,\varepsilon) - u_{ck}^{(0)}(t,\varepsilon)\right| \le M\sqrt{\frac{2T}{\pi}}\Bigg[\bigg(\int_0^t\int_0^\pi \{f[\tau,\xi,Au^{(0)}(t,\xi,\varepsilon)] - f(\tau,\xi,0)\}^2\,d\xi\,d\tau\bigg)^{1/2} + \bigg(\int_0^t\int_0^\pi f^2(\tau,\xi,0)\,d\xi\,d\tau\bigg)^{1/2}\Bigg].$$

Applying the Lipschitz inequality to the first integral, and maximizing with respect to t, we get

$$\sum_{k=1}^{\infty} |u_{ck}^{(1)}(t,\varepsilon) - u_{ck}^{(0)}(t,\varepsilon)| \le M\sqrt{\frac{2T}{\pi}}\Big[\|b(t,x)\|_{L_2(D)}\|\overline{u}^{(0)}(t,\varepsilon)\|_{B_T} + \|f(t,x,0)\|_{L_2(D)}\Big]. \tag{3.10}$$

In a similar manner we obtain

$$\sum_{k=1}^{\infty} |u_{sk}^{(1)}(t,\varepsilon) - u_{sk}^{(0)}(t,\varepsilon)| \le M\sqrt{\frac{2T}{\pi}}\Big[\|b(t,x)\|_{L_2(D)}\|\overline{u}^{(0)}(t,\varepsilon)\|_{B_T} + \|f(t,x,0)_{L_2(D)}\|\Big]. \tag{3.11}$$

From the inequalities (3.9), (3.10) and (3.11) we have

$$\begin{aligned}&\frac{1}{2}|u_0^{(1)}(t,\varepsilon) - u_0^{(0)}(t,\varepsilon)| + \sum_{k=1}^{\infty}\Big[|u_{ck}^{(1)}(t,\varepsilon) - u_{ck}^{(0)}(t,\varepsilon)| + |u_{sk}^{(1)}(t,\varepsilon) - u_{sk}^{(0)}(t,\varepsilon)|\Big]\\ &\quad\le (T+2\sqrt{6}M)\sqrt{\frac{T}{3\pi}}\Big[\|b(t,x)\|_{L_2(D)}\|\overline{u}^{(0)}(t,\varepsilon)\|_{B_T} + \|f(t,x,0)\|_{L_2(D)}\Big]\\ &\quad := A_T,\end{aligned} \tag{3.12}$$

where it is clear that A_T is a positive number. Taking the maximum with respect to t on the left hand side of the last inequality we obtain

$$\|\overline{u}_0^{(1)}(t,\varepsilon) - \overline{u}_0^{(0)}(t,\varepsilon)\|_{B_T} \le A_T.$$

Following the process above and using the principle of mathematical induction the inequality

$$\|\overline{u}_0^{(N+1)}(t,\varepsilon) - \overline{u}_0^{(N)}(t,\varepsilon)\|_{B_T} \le A_T\left[(T+2\sqrt{6}M)\sqrt{\frac{T}{3\pi}}\right]^N \frac{\|b(t,x)\|_{L_2(D)}^N}{\sqrt{N!}} \tag{3.13}$$

can be proved ($N = \overline{1,\infty}$). It is understood from (3.13) that the sequence

$$\sum_{n=0}^{\infty} |\overline{u}^{(N+1)}(t,\varepsilon) - \overline{u}^{(N)}(t,\varepsilon)|$$

is uniformly convergent in B_T. Therefore the successive approximation sequence $\{\overline{u}^{(N+1)}(t,\varepsilon)\}$, whose general term is

$$\overline{u}^{(N+1)}(t,\varepsilon) = \overline{u}^{(0)}(t,\varepsilon) + \sum_{n=1}^{N}\Big[\overline{u}^{(n+1)}(t,\varepsilon) - \overline{u}^{(n)}(t,\varepsilon)\Big]$$

is uniformly convergent in B_T. Let

$$\lim_{n\to\infty} \overline{u}^{(N+1)}(t,\varepsilon) = \overline{u}(t,\varepsilon) = \Big\{ \frac{1}{2}u_0(t,\varepsilon), u_{c1}(t,\varepsilon), u_{s1}(t,\varepsilon), \dots, u_{ck}(t,\varepsilon), u_{sk}(t,\varepsilon), \dots \Big\}.$$

In order to prove that $\overline{u}(t,\varepsilon)$ satisfies the system (3.2), substitute $\overline{u}(t,\varepsilon)$ in the system (3.2), and let σ denote the absolute value of the difference of the systems (3.2) and (3.3). By the previous scheme applied above we have

$$\begin{aligned}
\sigma \le{}& \frac{2}{\pi}\Big| \int_0^t \int_0^\pi (t-\tau)\{f[\tau,\xi,Au(\tau,\xi,\varepsilon)] - f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\}\, d\xi\, d\tau \Big| \\
&+ \sum_{k=1}^{\infty} \frac{2}{\pi}\frac{1}{\alpha_k}\Big| \int_0^t\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi,\varepsilon)] \\
&\qquad - f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\} \cos 2k\xi \sin\alpha_k(t-\tau)\, d\xi\, d\tau \Big| \\
&+ \sum_{k=1}^{\infty} \frac{2}{\pi}\frac{1}{\alpha_k}\Big| \int_0^t\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi,\varepsilon)] \\
&\qquad - f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\} \sin 2k\xi \sin\alpha_k(t-\tau)\, d\xi\, d\tau \Big| \\
\le{}& \sqrt{\frac{T^3}{3}}\Big(\int_0^t \frac{2}{\pi}\Big[\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi,\varepsilon)] \\
&\qquad - f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\} \cos 2k\xi\, d\xi \Big]^2 d\tau \Big)^{1/2} \\
&+ \sqrt{T}\Big(\sum_{k=1}^{\infty} \frac{1}{\alpha_k^2} \Big)^{1/2} \Big[\sum_{k=1}^{\infty} \int_0^t \Big(\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi,\varepsilon)] \\
&\qquad - f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\} \cos 2k\xi\, d\xi \Big)^2 \Big]^{1/2} \\
&+ \sqrt{T}\Big(\sum_{k=1}^{\infty} \frac{1}{\alpha_k^2} \Big)^{1/2} \Big[\sum_{k=1}^{\infty} \int_0^t \Big(\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi,\varepsilon)] \\
&\qquad - f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\} \sin 2k\xi\, d\xi \Big)^2 \Big]^{1/2}.
\end{aligned}$$

By means of the inequality $(a+b+c)^2 \le 3(a^2+b^2+c^2)$ we get

$$\begin{aligned}
\sigma^2 \le{}& T^3 \int_0^t \Big(\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi,\varepsilon)] - f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\}\, d\xi \Big)^2 d\tau \\
&+ 3M^2T \sum_{k=1}^{\infty} \int_0^t \Big(\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi,\varepsilon)] \\
&\qquad - f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\} \cos 2k\xi\, d\xi \Big)^2 d\tau \\
&+ 3M^2T \sum_{k=1}^{\infty} \int_0^t \Big(\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi,\varepsilon)] \\
&\qquad - f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\} \sin 2k\xi\, d\xi \Big)^2 d\tau.
\end{aligned}$$

Let $\max(2T^3, 3M^2T) = M_T$. Hence

$$\sigma^2 \le M_T \int_0^t \frac{1}{2}\left(\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi,\varepsilon)] - f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\}\,d\xi\right)^2 d\tau$$
$$+ M_T \sum_{k=1}^\infty \int_0^t \left(\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi,\varepsilon)] - f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\}\cos 2k\xi\,d\xi\right)^2 d\tau$$
$$+ M_T \sum_{k=1}^\infty \int_0^t \left(\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi,\varepsilon)] - f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\}\sin 2k\xi\,d\xi\right)^2 d\tau.$$

Applying Bessel's inequality to the right hand side of the inequality, and then the Lipschitz condition, we get

$$\sigma^2 \le \frac{2}{\pi} M_T \int_0^t \int_0^\pi \{f[\tau,\xi,Au(\tau,\xi,\varepsilon)] - f[\tau,\xi,Au^{(N)}(\tau,\xi,\varepsilon)]\}^2\,d\xi\Big)^2 d\tau$$
$$\le \frac{2}{\pi} M_T \int_0^t \int_0^\pi b^2(\tau,\xi)\left[Au(\tau,\xi,\varepsilon) - Au^{(N)}(\tau,\xi,\varepsilon)\right]^2 d\xi\,d\tau$$
$$\le \frac{2}{\pi} M_T \|b^2(t,x)\|_{L_2(D)} \|\overline{u}(t,\varepsilon) - \overline{u}^{(N)}(t,\varepsilon)\|_{B_T}.$$

Considering $\lim_{n\to\infty} \|\overline{u}(t,\varepsilon) - \overline{u}^{(N)}(t,\varepsilon)\| = 0$, the norm $\|\overline{u}(t,\varepsilon) - \overline{u}^{(N+1)}(t,\varepsilon)\|$, which is formed by the difference of (3.2) and (3.3), tends to zero as $N \to \infty$, i.e. the limit function $\overline{u}(t,\varepsilon)$ is a solution of the system (3.2).

In order to prove the uniqueness of the solution of the system (3.2), by contradiction suppose that $\overline{v}(t,\varepsilon)$ is another solution. Evaluating the difference $|\overline{u}(t,\varepsilon) - \overline{v}(t,\varepsilon)|$ in accordance with the scheme above we get

$$[\overline{u}(t,\varepsilon) - \overline{v}(t,\varepsilon)]^2 \le \frac{2}{\pi} M_T \int_0^t \left(\int_0^\pi b^2(\tau,\xi)\,d\xi\right)\left[\overline{u}(t,\varepsilon) - \overline{v}(t,\varepsilon)\right]^2 d\tau.$$

However, $|\overline{u}(t,\varepsilon) - \overline{v}(t,\varepsilon)| \le 0$ in view of the Cronwall inequality gives $\overline{u}(t,\varepsilon) = \overline{v}(t,\varepsilon)$. Thus the theorem is proven. □

By Theorem 3.1, the following theorem related to the weak generalized solution of problem (2.1) - (2.3) is also true.

3.2. Theorem. *Suppose that the conditions of Theorem 3.1 are satisfied. Then there is a unique weak generalized solution of problem* (2.1) - (2.3)*, and this solution can be found as a uniformly convergent series* (3.1) *in* $C(D)$.

Due to the practical significance of the problem handled, it is useful to obtain an estimation of the difference between the exact solution

$$\bar{u}(t,\varepsilon) = \left\{\frac{1}{2}u_0(t,\varepsilon), u_{c1}(t,\varepsilon), u_{s1}(t,\varepsilon), \ldots, u_{ck}(t,\varepsilon), u_{sk}(t,\varepsilon), \ldots\right\}$$

and the $(N+1)$-th successive approximation

$$\bar{u}^{(N+1)}(t,\varepsilon) = \left\{\frac{1}{2}u_0^{(N+1)}(t,\varepsilon), u_{c1}^{(N+1)}(t,\varepsilon), u_{s1}^{(N+1)}(t,\varepsilon), \ldots \ldots, u_{ck}^{(N+1)}(t,\varepsilon), u_{sk}^{(N+1)}(t,\varepsilon), \ldots\right\}$$

of system (3.2). The following theorem may be proved by the method applied above.

3.3. Theorem. *Suppose that the conditions of Theorem 3.1 are satisfied. Then the following inequality is true for the difference between the exact solution* $\bar{u}(t,\varepsilon)$ *and the approximate solution* $\bar{u}(t,\varepsilon)$ *of problem (3.2)*

$$\|\overline{u}(t,\varepsilon) - \overline{u}^{(N+1)}(t,\varepsilon)\|_{B_T} \leq \sqrt{\frac{2}{\pi}\frac{M_T}{N!}}\left[T + 2\sqrt{6}M\right]^N \times \|b(t,x)\|^{N+1}_{L_2(D)} exp\frac{M_T}{\pi}\|b(t,x)\|_{L_2(D)}.$$

4. Conclusion

In this work, the existence and uniqueness of the weak generalized solution of a mixed problem with periodic boundary condition for a quasi-linear Euler-Bernoulli equation are examined, and an estimation of the difference between the exact and approximate solution is given.

References

[1] Chandrov, H. I. *On Mixed Problem for A Class of Quasilinear Hyperbolic Equation* (Tbilisi, 1970).

[2] Ciftci, I and Halilov, H. *Fourier method for a quasilinear parabolic equation with periodic boundary condition*, Hacettepe J. Math. Stat. **37** (2), 69–79, 2008.

[3] Ciftci, I and Halilov, H. *Dependency of the solution of quasilinear pseudo-parabolic equation with periodic boundary condition*, Int. J. Math. Anal. **2**, 881–888, 2008.

[4] Conzalez-Valesco, E. A. *Fourier Analysis and Boundary Value Problems* (Academic Press, New York, 1995).

[5] Demir C., Akgöz B. and Civelek O. *Free vibration and bending analysis of carbon nanotubes using Euler beam theory*, Proceeding International Symp. on Engineering and Architectural Sciences of Balkan and Turkic Republics, Vol. III, 50–55, 2009.

[6] Elishakoff, I. and Candan, S. *Apparently first closed-form solution for vibrating inhomogeneous beams*, Internat. J. Solids Structures **38** (19), 3411–3441, 2001.

[7] Gibson, R. F., Ayorinde, E. O. and Weng, Y. *Vibration of carbon nano-tubes and their composites: A review*, Compos Sci. Tech. **67**, 1–28, 2007.

[8] Halilov, H. M. *Solution of the mixed non-linear problem for a class of quasi-linear equation 4th order*, J. Mathematical Physics and Functional Analysis, Alma Ata, 27–32, 1966.

[9] Halilov, H., *On the Mixed Problem for a class of quasilinear pseudo-parabolic equations*, Applicable Analysis **75** (1-2), 61–71, 2000.

[10] Halilov, H., Kutlu, K. and Güler, B. O. *Investigation of the non-linear vibration problem*, Proceedings of the Symposium on Engineering and Architectural Sciences of Balkan, Caucasus and Turkic Republics, Isparta, 79–84, 2009.

[11] Il'in, V. A. *Solvability of mixed problem for hyperbolic and parabolic equations*, Uspekhi Math. Nauk. **15-2** (92), 97–154, 1960 (in Russian).

[12] Ladyzhenskaya, D. A. *Boundary Value Problem of Mathematical Physics* (Springer, New York, 1985).

[13] Lattes R. and Lions, J.-L. *Methode de Quasi-Reversibilitè et Applications* (Dunod, Paris, 1967).

[14] Shabadikov, K. H. *Issledovanie Rashennie Smashannikh Zadach dlya Kvazilineaynikh Differentsialnikh Urevneniy Malim Parametrom pri Starshey SmessonoiPreizvodnoi* (PhD Thesis, Fargana, 1984).

[15] Strutt, J. W. and Rayleigh B. *The Theory of Sound* (Dover Publication, New York, 1945).

[16] Wang, C. M., Tan, V. B. C. and Zhang, Y. Y. *Timoshenko beam model for vibration analysis of multi-walled carbon nanotubes*, J. Sound and Vibration **294**, 1060–1072, 2006.