CONTRA SEMI-I-CONTINUOUS FUNCTIONS

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Abstract

In this paper, we apply the notion of semi-I-open sets in ideal topological spaces to present and study a new class of functions called contra semi-I-continuous functions. Relationships between this new class and other classes of functions are established.

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1. Introduction

In 1996, Dontchev [1] introduced a new class of functions called contra-continuous functions. He defined a function \( f : X \rightarrow Y \) to be contra-continuous if the preimage of every open set of \( Y \) is closed in \( X \). A new weaker form of this class of functions, called contra-semi-continuous functions, is introduced and investigated by Dontchev and Noiri [2]. In this direction, we will introduce the concept of contra semi-I-continuous functions via the notion of semi-I-open sets. We also obtain some properties of such functions.

The subject of ideals in topological spaces has been studied by Kuratowski [6] and Vaidyanathaswamy [9]. An ideal on a topological space \((X, \tau)\) is defined as a non-empty collection \( I \) of subsets of \( X \) satisfying the following two conditions:

1. If \( A \in I \) and \( B \subseteq A \), then \( B \in I \);
2. If \( A \in I \) and \( B \in I \), then \( A \cup B \in I \).

An ideal topological space is a topological space \((X, \tau)\) with an ideal \( I \) on \( X \), and is denoted by \((X, \tau, I)\). For a subset \( A \subseteq X \), the set

\[ A^*(I) = \{ x \in X : U \cap A \notin I \} \]

for every \( U \in \tau \) with \( x \in U \), is called the local function of \( A \) with respect to \( I \) and \( \tau \) [5]. We simply write \( A^* \) instead of \( A^*(I) \) in case there is no chance of confusion. It is well known that \( \text{Cl}^*(A) = A \cup A^* \) defines a Kuratowski closure operator for \( \tau^*(I) \).

Throughout this paper, for a subset \( A \) of a topological space \((X, \tau)\), \( \text{Cl}(A) \) and \( \text{Int}(A) \) denote the closure and the interior of \( A \), respectively.

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A subset \( A \) of an ideal topological space \((X, \tau, I)\) is said to be semi-\(I\)-open \([3]\) (resp., semi-open \([7]\), \(I\)-open \([5]\)) if \( A \subseteq \text{Cl}^*\left(\text{Int}(A)\right) \) (resp., \( A \subseteq \text{Cl}(\text{Int}(A)), A \subseteq \text{Int}(A^*) \)). The complement of a semi-\(I\)-open (resp., semi-open, \(I\)-open) set is called semi-\(I\)-closed \([3]\) (resp., semi-closed \([7]\), \(I\)-closed \([5]\)). If the set \( A \) is semi-\(I\)-open and semi-\(I\)-closed, then it is called semi-\(I\)-clopen.

2. Contra semi-\(I\)-continuous functions

2.1. Definition. A function \( f : (X, \tau, I) \rightarrow (Y, \rho) \) is called contra semi-\(I\)-continuous if \( f^{-1}(V) \) is semi-\(I\)-closed in \( X \) for each open set \( V \) in \( Y \).

2.2. Theorem. For a function \( f : (X, \tau, I) \rightarrow (Y, \rho) \), the following are equivalent:
   
   a) \( f \) is contra semi-\(I\)-continuous.
   
   b) For every closed subset \( F \) of \( Y \), \( f^{-1}(F) \) is semi-\(I\)-open in \( X \).
   
   c) For each \( x \in X \) and each closed subset \( F \) of \( Y \) with \( f(x) \in F \), there exists a semi-\(I\)-open subset \( U \) of \( X \) with \( x \in U \) such that \( f(U) \subseteq F \).

Proof. The implications (a) \( \implies \) (b) and (b) \( \implies \) (c) are obvious.

   (c) \( \implies \) (b) Let \( F \) be any closed subset of \( Y \). If \( x \in f^{-1}(F) \) then \( f(x) \in F \), and there exists a semi-\(I\)-open subset \( U_x \) of \( X \) with \( x \in U_x \) such that \( f(U_x) \subseteq F \). Therefore, we obtain \( f^{-1}(F) = \cup\{U_x : x \in f^{-1}(F)\} \). Now, by [4, Theorem 3.4] we have that \( f^{-1}(F) \) is semi-\(I\)-open. \( \square \)

Recall that a function \( f : (X, \tau, I) \rightarrow (Y, \rho) \) is called contra semi-continuous \([2]\) if the preimage of every open subset of \( Y \) is semi-closed in \( X \).

Since every semi-\(I\)-open set is semi-open, then every contra semi-\(I\)-continuous function is contra semi-continuous, but the converse need not be true as shown by the following example:

2.3. Example. Let \( X = \{1, 2, 3, 4\}, \tau = \{\emptyset, X, \{2\}, \{3\}, \{2, 3\}, \{1, 2\}, \{1, 2, 3\}, \{2, 3, 4\}\} \) and \( I = \{\emptyset, \{3\}\} \). Let \( f : (X, \tau, I) \rightarrow (X, \tau) \) be defined by \( f(1) = 2, f(2) = 1, f(3) = 4 \) and \( f(4) = 3 \). Observe that \( f \) is contra semi-continuous. But \( f \) is not contra semi-\(I\)-continuous, since \( \{1, 2\} \) is open and \( f^{-1}(\{1, 2\}) = \{1, 2\} \) is not semi-\(I\)-closed.

A function \( f : (X, \tau, I) \rightarrow (Y, \rho) \) is called contra \(I\)-continuous if the preimage of every open subset of \( Y \) is an \( I \)-closed subset of \( X \).

The following two examples show that the concepts of contra-\(I\)-continuity and contra semi-\(I\)-continuity are independent of each other.

2.4. Example. Let \( X = \{1, 2\}, \tau = \{\emptyset, X, \{1\}\} \) and \( I = \{\emptyset, \{1\}\} \). Let \( f : (X, \tau, I) \rightarrow (X, \tau) \) be defined by \( f(1) = 2 \) and \( f(2) = 1 \). Observe that \( f \) is contra semi-\(I\)-continuous. But \( f \) is not contra-\(I\)-continuous since \( \{1\} \) is open and \( f^{-1}(\{1\}) = \{2\} \) is not \( I \)-closed.

2.5. Example. Let \( X = Y = \{1, 2\}, \tau = \{\emptyset, X\}, I = \{\emptyset, \{1\}\} \) and \( \rho = \{\emptyset, Y, \{1\}\} \). Let \( f : (X, \tau, I) \rightarrow (Y, \rho) \) be the identity function. Observe that \( f \) is contra \(I\)-continuous. But \( f \) is not contra semi-\(I\)-continuous, since \( \{1\} \) is open in \( Y \) and \( f^{-1}(\{1\}) = \{1\} \) is not semi-\(I\)-closed in \( X \).

Recall that a function \( f : (X, \tau, I) \rightarrow (Y, \rho) \) is called semi-\(I\)-continuous \([3]\) if the preimage of every open subset of \( Y \) is semi-\(I\)-open in \( X \).

The following two examples show that the concepts of semi-\(I\)-continuity and contra semi-\(I\)-continuity are independent of each other.
2.6. Example. Let \( X = \{1, 2\} \), \( \tau = \{\emptyset, X, \{1\}\} \) and \( I = \{\emptyset, \{2\}\} \). Let \( f : (X, \tau, I) \rightarrow (X, \pi) \) be defined by \( f(1) = 2 \) and \( f(2) = 1 \). Observe that \( f \) is contra semi-I-continuous. But \( f \) is not semi-I-continuous, since \( \{1\} \) is open and \( f^{-1}(\{1\}) = \{2\} \) is not semi-I-open.

2.7. Example. Let \( X = \mathbb{R}, \tau \) the usual topology and \( I = \{\emptyset, \{1\}\} \). The identity function \( f : (\mathbb{R}, \tau, I) \rightarrow (\mathbb{R}, \pi) \) is semi-I-continuous. But \( f \) is not contra semi-I-continuous, since the inverse image of \((0, 1)\) is not semi-I-closed.

2.8. Theorem. If a function \( f : (X, \tau, I) \rightarrow (Y, \rho) \) is contra semi-I-continuous and \( Y \) is regular, then \( f \) is semi-I-continuous.

Proof. Let \( x \in X \) and let \( V \) be an open subset of \( Y \) with \( f(x) \in V \). Since \( Y \) is regular, there exists an open set \( W \) in \( Y \) such that \( f(x) \in W \subseteq \text{Cl}(W) \subseteq V \). Since \( f \) is contra semi-I-continuous, by Theorem 2.2, there exists a semi-I-open set \( U \) in \( X \) with \( x \in U \) such that \( f(U) \subseteq \text{Cl}(W) \subseteq V \). Hence, by [4, Theorem 4.1], \( f \) is semi-I-continuous.

Note that if \( f : (X, \tau, I) \rightarrow (Y, \rho) \) is semi-I-continuous and \( Y \) is regular, then \( f \) need not be contra semi-I-continuous, as shown in Example 2.7.

2.9. Definition. A topological space \((X, \tau, I)\) is said to be semi-I-connected if \( X \) is not the union of two disjoint non-empty semi-I-open subsets of \( X \).

2.10. Theorem. If \( f : (X, \tau, I) \rightarrow (Y, \rho) \) is a contra semi-I-continuous function from a semi-I-connected space \( X \) onto any space \( Y \), then \( Y \) is not a discrete space.

Proof. Suppose that \( Y \) is discrete. Let \( A \) be a proper non-empty clopen set in \( Y \). Then \( f^{-1}(A) \) is a proper non-empty semi-I-clopen subset of \( X \), which contradicts the fact that \( X \) is semi-I-connected.

2.11. Theorem. A contra semi-I-continuous image of a semi-I-connected space is connected.

Proof. Let \( f : (X, \tau, I) \rightarrow (Y, \rho) \) be a contra semi-I-continuous function from a semi-I-connected space \( X \) onto a space \( Y \). Assume that \( Y \) is disconnected. Then \( Y = A \cup B \), where \( A \) and \( B \) are non-empty clopen sets in \( Y \) with \( A \cap B = \emptyset \). Since \( f \) is contra semi-I-continuous, we have that \( f^{-1}(A) \) and \( f^{-1}(B) \) are semi-I-open non-empty sets in \( X \) with \( f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(Y) = X \) and \( f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\emptyset) = \emptyset \). This means that \( X \) is not semi-I-connected, which is a contradiction. Then \( Y \) is connected.

2.12. Theorem. Let \((X, \tau, I)\) be a semi-I-connected space and \((Y, \rho)\) a \( T_1 \)-space. If \( f : (X, \tau, I) \rightarrow (Y, \rho) \) is a contra semi-I-continuous function, then \( f \) is constant.

Proof. Let \( \Delta = \{f^{-1}(\{y\}) : y \in Y\} \). Since \((Y, \rho)\) is \( T_1 \), \( \Delta \) is a disjoint semi-I-open partition of \( X \). If \( |\Delta| \geq 2 \), then \( X \) is the union of two non-empty semi-I-open sets. Since \((X, \tau, I)\) is semi-I-connected, \( |\Delta| = 1 \). Therefore, \( f \) is constant.

2.13. Definition. A space \((X, \tau, I)\) is said to be semi-I-\( T_2 \) if for each pair of distinct points \( x \) and \( y \) in \( X \), there exists two semi-I-open sets \( U \) and \( V \) in \( X \) such that \( x \in U \), \( y \in V \) and \( U \cap V = \emptyset \).

2.14. Theorem. Let \( f : (X, \tau, I) \rightarrow (Y, \rho) \) be a contra semi-I-continuous injection. If \( Y \) is a Urysohn space, then \( X \) is semi-I-\( T_2 \).
2.2. Corollary. If $K$ exists a finite subset $U \subseteq X$ is semi-$\alpha$.

2.23. Definition. A function $f : (X, \tau, I) \to (Y, \rho, J)$ is called contra $I$-irresolute if $f^{-1}(V)$ is semi-$I$-closed in $X$ for each semi-$I$-open set $V$ of $Y$.

Recall that a function $f : (X, \tau, I) \to (Y, \rho, J)$ is said to be $I$-irresolute [4] if $f^{-1}(V)$ is semi-$I$-open in $X$ for each semi-$J$-open set $V$ of $Y$. In fact, contra $I$-irresoluteness and $I$-irresoluteness are independent, as shown by the following two examples.

2.24. Example. The function $f$ in Example 2.6 is contra $I$-irresolute but not $I$-irresolute.

2.25. Example. Let $(X, \tau, I)$ be the ideal topological space in Example 2.6. Then the identity function on the space $(X, \tau, I)$ is an example of an $I$-irresolute function which is not contra $I$-irresolute.
Also, every contra I-irresolute function is contra semi-I-continuous, but the converse is not true as shown by the following example.

2.26. Example. Let $X = \{1, 2, 3\}$, $\tau = \{\emptyset, X, \{1\}\}$ and $I = \{\emptyset, \{2\}\}$. Define a function $f : (X, \tau, I) \to (Y, \rho, J)$ by $f(1) = 2$, $f(2) = 1$ and $f(3) = 3$. Then $f$ is contra semi-I-continuous but $f$ is not contra I-irresolute.

The following results can be easily verified and the proofs are omitted.

2.27. Theorem. A function $f : (X, \tau, I) \to (Y, \rho, J)$ is contra I-irresolute if and only if the inverse image of each semi-I-closed set in $Y$ is semi-I-open in $X$. \hfill \Box$

2.28. Theorem. Let $f : (X, \tau, I) \to (Y, \rho, J)$ and $g : (Y, \rho, J) \to (Z, \sigma, K)$. Then

1) $gof$ is contra I-irresolute if $g$ is I-irresolute and $f$ is contra I-irresolute.

2) $gof$ is contra I-irresolute if $g$ is contra I-irresolute and $f$ is I-irresolute. \hfill \Box

2.29. Theorem. Let $f : (X, \tau, I) \to (Y, \rho, J)$ and $g : (Y, \rho, J) \to (Z, \sigma)$. Then

1) $gof$ is contra semi-I-continuous if $g$ is continuous and $f$ is contra semi-I-continuous.

2) $gof$ is contra semi-I-continuous if $g$ is semi-I-continuous and $f$ is contra I-irresolute. \hfill \Box

Recall that a function $f : (X, \tau) \to (Y, \rho, J)$ is called semi-I-open [4] if for each $U \in \tau$, $f(U)$ is semi-I-open in $Y$.

2.30. Theorem. Let $f : (X, \tau, I) \to (Y, \rho, J)$ be surjective, I-irresolute and semi-I-open, and let $g : (Y, \rho, J) \to (Z, \sigma)$ be any function. Then $gof$ is contra semi-I-continuous if and only if $g$ is contra semi-I-continuous.

Proof. $\implies$ Let $gof$ be contra semi-I-continuous and $C$ a closed subset of $Z$. Then $(gof)^{-1}(C)$ is a semi-I-open subset of $X$. That is $f^{-1}(g^{-1}(C))$ is semi-I-open. Since $f$ is semi-I-open, $f(f^{-1}(g^{-1}(C)))$ is a semi-I-open subset of $Y$. So $g^{-1}(C)$ is semi-I-open in $Y$. Therefore, $g$ is contra semi-I-continuous.

$\impliedby$ Straightforward. \hfill \Box

2.31. Definition. A function $f : (X, \tau, I) \to (Y, \rho, J)$ is called perfectly contra I-irresolute if the inverse image of every semi-I-open set in $Y$ is semi-I-clopen in $X$.

2.32. Example. Let $X = Y = \{1, 2, 3, 4\}$, $\tau = \{\emptyset, X, \{2\}, \{4\}, \{2, 4\}\}$, $I = \{\emptyset, \{4\}\}$, $\rho = \{\emptyset, Y, \{2\}\}$ and $J = \{\emptyset, \{2\}\}$. Let $f : (X, \tau, I) \to (Y, \rho, J)$ be defined by $f(1) = 3$, $f(2) = 4$, $f(3) = 1$ and $f(4) = 2$. Then $f$ is perfectly contra I-irresolute.

Every perfectly contra I-irresolute function is contra I-irresolute and I-irresolute. The following two examples show that a contra I-irresolute function may not be perfectly contra I-irresolute, and an I-irresolute function may not be perfectly contra I-irresolute.

2.33. Example. The function in Example 2.6 is contra I-irresolute, but not perfectly contra I-irresolute.

2.34. Example. Let $X = \{1, 2, 3\}$, $\tau = \{\emptyset, X, \{1\}\}$ and $I = \{\emptyset, \{2\}\}$. Let $f : (X, \tau, I) \to (X, \tau, I)$ be the identity function. Then $f$ is I-irresolute, but not perfectly contra I-irresolute.

2.35. Theorem. For a function $f : (X, \tau, I) \to (Y, \rho, J)$ the following conditions are equivalent:

1) $f$ is perfectly contra I-irresolute.

2) $f$ is contra I-irresolute and I-irresolute. \hfill \Box
References


