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INTRINSIC EQUATIONS FOR A GENERALIZED RELAXED ELASTIC LINE ON AN ORIENTED SURFACE

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Abstract

H. K. Nickerson and Gerald S. Manning (*Intrinsic equations for a relaxed elastic line on an oriented surface*, Geometriae Dedicate **27**, 127–136, 1988) derived the intrinsic equations for a relaxed elastic line on an oriented surface in Euclidean 3-dimensional space E^3. In this paper, we define a generalized relaxed elastic line and derive the intrinsic equations for a generalized relaxed elastic line on an oriented surface in Euclidean 3-dimensional space E^3, and give some applications of the result.

Keywords: Elastic line, Intrinsic equation, Variational problem.

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1. Introduction

A brief mathematical background for curvature, including fundamental definitions and theorems, may be found in [1, 2, 4, 7, 9] and [10]. In this section, we will recall some fundamental definitions and theorems. Let κ denote the curvature of a curve α and let $P(\kappa)$ be a smooth function. The geometric importance of minimizing a curvature energy functional of the type $\Theta(\alpha) = \int_\alpha P(\kappa)\, ds$ is discussed for a certain space of curves in the Euclidean 3-dimensional space E^3 in [1].

The natural variational integrals in geometry are the common integrals on space curves $\alpha(s)$. These include the length $L(\alpha) = \int ds$, total torsion $T(\alpha) = \int \tau\, ds$, total squared curvature $K(\alpha) = \int \kappa^2\, ds$, used in [6, 8], and the integral $H(\alpha) = \int \kappa^2 \tau\, ds$.

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Let $\alpha(s)$ denote an arc on a connected oriented surface S in E^3, parameterized by arc length, $0 \leq s \leq l$, with curvature $\kappa(s)$ and torsion $\tau(s)$. Let the energy density be given as some function $f(\kappa, \tau)$ of the curvature and torsion. Then

$$H = \int f(\kappa, \tau)\, ds \tag{1.1}$$

define Hamiltonians for the curve [3]. Thus the following integral can be taken as a special Hamiltonian for the curve α:

$$H = \int_0^l \kappa^2 \tau\, ds. \tag{1.2}$$

Also, the filament model (FM) is often known as localized induction. The Hamiltonians are given simply by

$$\mathcal{F}_n = \frac{1}{n-2} \int_\alpha f_{n-1}\, ds, \;\; n = 1, 3, 4, 5, \ldots,$$

where f_n is obtained in terms of $X_1, X_2, \ldots, X_{n-1}$ from $\partial f_n = \langle X_1, JX_n \rangle$ since $J^2 = -Id$ on a normal vector field, and the X_n depends on the n derivatives of $T(s) = \alpha'(s)$. It is known that $\mathcal{F}_n$ is a FM constant of motion in involution [5].

1.1. Definition. The arc α is called a *generalized relaxed elastic line* if it is extremal for the variational problem of minimizing the value of H within the family of all arcs of length l on S having the same initial point and initial direction as α.

We shall require that the coordinate functions of S are sufficiently smooth and that the equations of α as functions of s, are sufficiently smooth in these coordinates.

In this study, we would like to calculate the intrinsic equations for a curve α that is extremal for (1.2).

At a point $\alpha(s)$ of α, let $T(s) = \alpha'(s)$ denote the unit tangent vector to α, $N(s)$ the unit normal to S, and let $Q(s) = N \times T$. Then $\{T, Q, N\}$ gives an orthonormal basis for all vectors at $\alpha(s)$ and $\{T, Q\}$ gives a basis for the vectors tangent to S at $\alpha(s)$. Let Π denote the second fundamental form of S. The surface analogue of the Frenet-Serret formulas is

$$\begin{bmatrix} T' \\ Q' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ -k_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ Q \\ N \end{bmatrix} \tag{1.3}$$

Here, k_g is the geodesic curvature of α, $k_n = \Pi(T, T)$ the normal curvature, and $\tau_g = \Pi(T, Q)$ the geodesic torsion. The square curvature κ^2 of α is given by

$$\kappa^2 = \langle T', T' \rangle = k_g^2 + k_n^2 \text{ [2, 9].} \tag{1.4}$$

1.2. Theorem. *For any regular curve α the following formulas hold:*

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3} \text{ and } \tau = \frac{\langle \alpha' \times \alpha'',\ \alpha''' \rangle}{\|\alpha' \times \alpha''\|^2} \text{ [4].}$$

2. Obtaining the equations

Now, assume that α lies in a coordinate patch $(u, v) \to x(u, v)$ of S, and let $x_u = \dfrac{\partial x}{\partial u}$, $x_v = \dfrac{\partial x}{\partial v}$. Then α is expressed as $\alpha(s) = x(u(s), v(s)),\ 0 \leq s \leq l$, with

$$T(s) = \alpha'(s) = \frac{du}{ds} x_u + \frac{dv}{ds} x_v$$

and

$$Q(s) = p(s)x_u + q(s)x_v$$

for suitable scalar functions $p(s)$ and $q(s)$.

Next, we must define variational fields for our problem. In order to obtain variational arcs of length l, it is generally necessary to extend α to an arc $\alpha^*(s)$ defined for $0 \le s \le l^*$, with $l^* > l$ but sufficiently close to l so that α^* lies in the coordinate patch. Let $\mu(s)$, $0 \le s \le l^*$, be a scalar function which is sufficiently smooth and does not vanish identically. Define

$$\eta(s) = \mu(s)p^*(s), \ \xi(s) = \mu(s)q^*(s).$$

Then

$$\eta(s)x_u + \xi(s)x_v = \mu(s)Q(s) \tag{2.1}$$

along α. Also, assume that

$$\mu(0) = 0, \ \mu'(0) = 0 \text{ and } \mu''(0) = 0. \tag{2.2}$$

No further restrictions need to be placed on μ. Now define

$$\beta(\sigma; t) = x\left(u(\sigma) + t\eta(\sigma), v(\sigma) + t\xi(\sigma)\right), \tag{2.3}$$

for $0 \le \sigma \le l^*$. For $|t| < \varepsilon$ (where $\varepsilon > 0$ depends upon the choice of α^* and of μ), the point $\beta(\sigma; t)$ lies in the coordinate patch. For fixed t, $\beta(\sigma; t)$ gives an arc with the same initial point and initial direction as α, because of (2.2).

For $t = 0$, $\beta(\sigma; 0)$ is the same as α^* and σ is the arc length. For $t \ne 0$, the parameter σ is not the arc length in general.

For fixed t, $|t| < \varepsilon$, let $L^*(t)$ denote the length of the arc $\beta(\sigma; t)$, $0 \le \sigma \le l^*$. Then

$$L^*(t) = \int_0^{l^*} \sqrt{\left\langle \frac{\partial \beta}{\partial \sigma}(\sigma; t), \frac{\partial \beta}{\partial \sigma}(\sigma; t) \right\rangle}\, d\sigma \tag{2.4}$$

with

$$L^*(0) = l^* > l. \tag{2.5}$$

It is clear from (2.3) and (2.4) that $L^*(t)$ is continuous in t. In particular, it follows from (2.5) that

$$L^*(t) > \frac{l + l^*}{2} > l, \ |t| < \varepsilon_* \tag{2.6}$$

for a suitable ε_* satisfying $0 < \varepsilon_* \le \varepsilon$. Because of (2.6), we can restrict $\beta(\sigma; t)$, $0 \le |t| < \varepsilon_*$, to an arc of length l by restricting the parameter σ to an interval $0 \le \sigma \le \lambda(t) \le l^*$ by requiring

$$\int_0^{\lambda(t)} \sqrt{\left\langle \frac{\partial \beta}{\partial \sigma}, \frac{\partial \beta}{\partial \sigma} \right\rangle}\, d\sigma = l. \tag{2.7}$$

Note that $\lambda(0) = l$. The function $\lambda(t)$ need not be determined explicitly, but we shall need $\left.\frac{d\lambda}{dt}\right|_{t=0}$.

2.1. Lemma.

$$\left.\frac{d\lambda}{dt}\right|_{t=0} = \int_0^l \mu k_g \, ds. \tag{2.8}$$

Proof. The proof of (2.8) and other results below will depend on results obtained from (2.3); such as

$$\left.\frac{\partial \beta}{\partial \sigma}\right|_{t=0} = T,\ 0 \leq \sigma \leq l. \tag{2.9}$$

Further differentiation of (2.9) gives

$$\left.\frac{\partial^2 \beta}{\partial \sigma^2}\right|_{t=0} = T' = k_g Q + k_n N, \tag{2.10}$$

and

$$\left.\frac{\partial^3 \beta}{\partial \sigma^3}\right|_{t=0} = \left(-k_g^2 - k_n^2\right) T + \left(k_g' - k_n \tau_g\right) Q + \left(k_n' + k_g \tau_g\right) N. \tag{2.11}$$

Also,

$$\left.\frac{\partial \beta}{\partial t}\right|_{t=0} = \mu Q, \tag{2.12}$$

because of (2.1). Further differentiation of (2.12) gives

$$\left.\frac{\partial^2 \beta}{\partial t \partial \sigma}\right|_{t=0} = \left.\frac{\partial^2 \beta}{\partial \sigma \partial t}\right|_{t=0} = -\mu k_g T + \mu' Q + \mu \tau_g N \tag{2.13}$$

using (1.3), and

$$\begin{aligned}\left.\frac{\partial^3 \beta}{\partial t \partial \sigma^2}\right|_{t=0} = {} & \left(-2\mu' k_g - \mu k_g' - \mu \tau_g k_n\right) T + \left(\mu'' - \mu k_g^2 - \mu \tau_g^2\right) Q \\ & + \left(2\mu' \tau_g - \mu k_g k_n + \mu \tau_g'\right) N\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}\left.\frac{\partial^4 \beta}{\partial t \partial \sigma^3}\right|_{t=0} = {} & \big(\mu k_g^3 - 3\mu' k_g' - 3\mu'' k_g - 3\mu' \tau_g k_n - 2\mu \tau_g' k_n - \mu \tau_g k_n' \\ & + \mu \tau_g^2 k_g + \mu k_g k_n^2 - \mu k_g''\big) T \\ & + \big(- 3\mu' k_g^2 - 3\mu' \tau_g^2 - 3\mu k_g k_g' - 3\mu \tau_g \tau_g' + \mu'''\big) Q \\ & + \big(- 2\mu k_g' k_n - \mu k_g k_n' - \mu \tau_g^3 + 3\mu' \tau_g' + 3\mu'' \tau_g \\ & - \mu k_g^2 \tau_g + \mu \tau_g'' - 3\mu' k_g k_n - \mu \tau_g k_n^2\big) N.\end{aligned} \tag{2.15}$$

To prove (2.8), differentiate (2.7) with respect to t, remembering that l is constant, and evaluate at $t = 0$ using (2.9) and (2.13), with $\lambda(0) = l$. Since

$$\left.\tfrac{d\lambda}{dt}\right|_{t=0} \sqrt{\left\langle \left.\tfrac{\partial \beta}{\partial \sigma}\right|_{t=0},\ \left.\tfrac{\partial \beta}{\partial \sigma}\right|_{t=0} \right\rangle} + \int_0^l \left\langle \left.\tfrac{\partial \beta}{\partial \sigma}\right|_{t=0},\ \left.\tfrac{\partial^2 \beta}{\partial \sigma \partial t}\right|_{t=0} \right\rangle \left\langle \left.\tfrac{\partial \beta}{\partial \sigma}\right|_{t=0},\ \left.\tfrac{\partial \beta}{\partial \sigma}\right|_{t=0} \right\rangle^{-1/2} ds$$
$$= 0,$$

we obtain that

$$\left.\frac{d\lambda}{dt}\right|_{t=0} = \int_0^l \mu k_g \, ds.$$

□

2.2. Theorem. *The intrinsic equations for a generalized relaxed elastic line of length l on a connected oriented surface in E^3 are given by the equalities*

(BC I) $k_n(l) = 0$,

(BC II) $k_n'(l) = -2k_g(l)\tau_g(l)$,

(BC III) $2k_n''(l) = -k_n(l)[k_g^2(l) - 5\tau_g^2(l) + k_n^2(l)] - 6\tau_g(l)k_g'(l) - k_g(l)\tau_g'(l),$

and the differential equation

$$\text{(DE)} \quad \begin{aligned} &2k_n''' + 6\tau_g k_g'' + 3k_n^2 k_n' + 3k_g^2 k_n' - 2k_g\tau_g^3 + 2k_g\tau_g'' - 6k_n\tau_g\tau_g' \\ &\quad - 6\tau_g^2 k_n' + 3k_g\tau_g k_n^2 + 3k_g^3\tau_g + 6\tau_g' k_g' + k_g[-k_n(l)k_g'(l) \\ &\quad + k_n^2(l)\tau_g(l) + k_g^2(l)\tau_g(l) + k_g(l)k_n'(l)] = 0. \end{aligned}$$

Here k_g, k_n and τ_g are the functions giving the geodesic curvature, the normal curvature and the geodesic torsion as functions of arc length along the line, respectively.

Proof. We begin by calculating $H(t)$ for the arc $\beta(\sigma;t)$, $0 \le \sigma \le \lambda(t)$, $|t| < \varepsilon_*$. Since σ is not generally the arc length for $t \neq 0$, $H(t)$ is given by

$$H(t) = \int_0^{\lambda(t)} \left\langle \frac{\partial\beta}{\partial\sigma} \times \frac{\partial^2\beta}{\partial\sigma^2},\ \frac{\partial^3\beta}{\partial\sigma^3} \right\rangle \left\langle \frac{\partial\beta}{\partial\sigma},\ \frac{\partial\beta}{\partial\sigma} \right\rangle^{-5/2} d\,\sigma.$$

A necessary condition for α to be extremal is that $H'(0) = 0$ for arbitrary μ satisfying (2.2). We now calculate $H'(t)$:

$$\begin{aligned} H'(t) = {} & \frac{d\lambda}{dt}\left\{ \left\langle \frac{\partial\beta}{\partial\sigma} \times \frac{\partial^2\beta}{\partial\sigma^2},\ \frac{\partial^3\beta}{\partial\sigma^3} \right\rangle \left\langle \frac{\partial\beta}{\partial\sigma},\ \frac{\partial\beta}{\partial\sigma} \right\rangle^{-5/2} \right\}_{\sigma=\lambda(t)} \\ & + \int_0^{\lambda(t)} \left\{ \left\langle \frac{\partial^2\beta}{\partial t\partial\sigma} \times \frac{\partial^2\beta}{\partial\sigma^2}, \frac{\partial^3\beta}{\partial\sigma^3} \right\rangle + \left\langle \frac{\partial\beta}{\partial\sigma} \times \frac{\partial^3\beta}{\partial t\partial\sigma^2},\ \frac{\partial^3\beta}{\partial\sigma^3} \right\rangle \right. \\ & \qquad \left. + \left\langle \frac{\partial\beta}{\partial\sigma} \times \frac{\partial^2\beta}{\partial\sigma^2},\ \frac{\partial^4\beta}{\partial t\partial\sigma^3} \right\rangle \right\} \left\langle \frac{\partial\beta}{\partial\sigma},\ \frac{\partial\beta}{\partial\sigma} \right\rangle^{-5/2} d\sigma \\ & \qquad - 5\int_0^{\lambda(t)} \left\langle \frac{\partial\beta}{\partial\sigma} \times \frac{\partial^2\beta}{\partial\sigma^2},\ \frac{\partial^3\beta}{\partial\sigma^3} \right\rangle \left\langle \frac{\partial\beta}{\partial\sigma},\ \frac{\partial\beta}{\partial\sigma} \right\rangle^{-7/2} \left\langle \frac{\partial^2\beta}{\partial t\partial\sigma},\ \frac{\partial\beta}{\partial\sigma} \right\rangle d\sigma. \end{aligned}$$

Using (2.7), (2.8), (2.9), (2.10), (2.11), (2.13), (2.14) and (2.15), we find

$$\begin{aligned} H'(0) = {} & \left(\int_0^l \mu k_g\, ds \right) \big(- k_n(l)k_g'(l) + k_n^2(l)\tau_g(l) + k_g^2(l)\tau_g(l) + k_g(l)k_n'(l) \big) \\ & + \int_0^l \big\{ \big(k_g\tau_g'' + 2k_g^2 k_n' - 2k_g\tau_g^3 + 3k_g^3\tau_g - \tau_g' k_g' + 4k_n\tau_g\tau_g' \\ & \quad + 3k_g\tau_g k_n^2 - 2k_n k_g k_g' - \tau_g^2 k_n' \big)\mu + (-k_n k_g^2 + 5k_n\tau_g^2 - k_n^3 \\ & \quad - 2\tau_g k_g' + 3k_g\tau_g')\mu' + \big(4k_g\tau_g + k_n'\big)\mu'' - k_n\mu''' \big\}\, ds. \end{aligned}$$

However, using integration by parts and (2.2),

$$-\int_0^l \mu''' k_n ds = -\mu''(l)k_n(l) + \mu'(l)k_n'(l) - \mu(l)k_n''(l) + \int_0^l \mu k_n''' \, ds,$$

$$\begin{aligned}\int_0^l \mu''(4k_g\tau_g + k_n')\, ds = \mu'(l)\big[4k_g(l)\tau_g(l) + k_n'(l)\big] \\ - \mu(l)\big[4\tau_g(l)k_g'(l) + 4k_g(l)\tau_g'(l) + k_n''(l)\big] \\ + \int_0^l \mu(8\tau_g' k_g' + 4\tau_g k_g'' + 4k_g\tau_g'' + k_n''')\, ds\end{aligned}$$

and

$$\begin{aligned}&\int_0^l \mu'\big[-k_n k_g^2 + 5k_n\tau_g^2 - k_n^3 - 2\tau_g k_g' + 3k_g\tau_g'\big]\, ds \\ &\quad = \mu(l)\big[-k_n(l)k_g^2(l) + 5k_n(l)\tau_g^2(l) - k_n^3(l) - 2\tau_g(l)k_g'(l) \\ &\qquad + 3k_g(l)\tau_g'(l)\big] - \int_0^l \mu\big[-k_g^2 k_n' - 2k_n k_g k_g' + 5\tau_g^2 k_n' \\ &\qquad + 10k_n\tau_g\tau_g' - 3k_n^2 k_n' + \tau_g' k_g' - 2\tau_g k_g'' + 3k_g\tau_g''\big]\, ds.\end{aligned}$$

Thus $H'(0)$ can be written as

$$\begin{aligned}H'(0) = \int_0^l \mu\big\{&2k_n''' + 6\tau_g k_g'' + 3k_n^2 k_n' + 3k_g^2 k_n' - 2k_g\tau_g^3 + 2k_g\tau_g'' \\ &- 6k_n\tau_g\tau_g' - 6\tau_g^2 k_n' + 3k_g\tau_g k_n^2 + 3k_g^3\tau_g + 6\tau_g' k_g' \\ &+ k_g[-k_n(l)k_g'(l) + k_n^2(l)\tau_g(l) + k_g^2(l)\tau_g(l) + k_g(l)k_n'(l)]\big\}\, ds \\ &+ \mu(l)\big[-k_n(l)k_g^2(l) + 5k_n(l)\tau_g^2(l) - 6\tau_g(l)k_g'(l) - k_g(l)\tau_g'(l) \\ &- 2k_n''(l) - k_n^3(l)\big] + \mu'(l)\big[4k_g(l)\tau_g(l) + 2k_n'(l)\big] - \mu''(l)k_n(l).\end{aligned} \tag{2.16}$$

In order that $H'(0) = 0$ for all choices of the function $\mu(s)$ satisfying (2.2), with arbitrary values of $\mu(l), \mu'(l)$ and $\mu''(l)$, the given arc α must satisfy the three boundary conditions

(BC I) $k_n(l) = 0$,
(BC II) $k_n'(l) = -2k_g(l)\tau_g(l)$,
(BC III) $2k_n''(l) = -k_n(l)[k_g^2(l) - 5\tau_g^2(l) + k_n^2(l)] - 6\tau_g(l)k_g'(l) - k_g(l)\tau_g'(l)$,

and the differential equation

$$\begin{aligned}&2k_n''' + 6\tau_g k_g'' + 3k_n^2 k_n' + 3k_g^2 k_n' - 2k_g\tau_g^3 + 2k_g\tau_g'' - 6k_n\tau_g\tau_g' \\ &\quad - 6\tau_g^2 k_n' + 3k_g\tau_g k_n^2 + 3k_g^3\tau_g + 6\tau_g' k_g' + k_g[-k_n(l)k_g'(l) \\ &\quad + k_n^2(l)\tau_g(l) + k_g^2(l)\tau_g(l) + k_g(l)k_n'(l)] = 0.\end{aligned} \tag{DE}$$

Although the derivation of the equations uses a particular local coordinate system, the final equations, namely the differential equation (DE) together with the boundary conditions (BC I), (BC II) and (BC III) at the free end are coordinate free, so they must hold in general. □

3. Applications

3.1. Theorem. *If α is any ruling of the ruled surface, then α is a generalized relaxed elastic line.*

Proof. Since any ruling of the ruled surface is both asymptotic and geodesic, it follows that $k_g = k_n = 0$. Hence the proof is clear. □

3.2. Theorem. *In the plane, any arc is a generalized relaxed elastic line.*

Proof. Since $k_n = \tau_g = 0$, the proof is clear. □

3.3. Theorem. *On a sphere of radius R, there is no generalized relaxed elastic line.*

Proof. For any arc on a sphere, $k_n = \frac{1}{R}$ and $\tau_g = 0$. Therefore (BC I), (BC II) and (DE) cannot be satisfied. □

3.4. Theorem. *On a right circular cylinder, an arc of an oblique geodesic (helix) cannot be a generalized elastic line.*

Proof. Let the cylinder be parameterized by

$$X(u,v) = (R\cos\tfrac{u}{R}, R\sin\tfrac{u}{R}, v),$$

where R is the radius of the cylinder. For an arbitrary arc $\alpha(s)$ we find $k_g = \frac{d\theta}{ds} = 0$, $k_n = -\frac{1}{R}\cos^2\theta =$ constant and $\tau_g = \frac{1}{R}\cos\theta\sin\theta$ [8]. The geodesics on the cylinder are characterized by $\theta =$ constant and the boundary condition (BC I) can be satisfied only if $\theta = \pm\pi/2$. □

4. Conclusion

In [8], "Intrinsic equations for a relaxed elastic line on an oriented surface", such equations were studied by H. K. Nickerson and G. S. Manning. In their study, the authors calculated the internal equations of elastic lines with the aid of k_g by using just the curvature of the elastic curve. In this study, since the energy density is given as some function $f(\kappa,\tau)$ of the curvature and torsion, the equations are given in E^3 with the aid of k_n by using both the curvature and the torsion of the elastic curve.

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