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NUMERICAL SOLUTION OF BRATU-TYPE EQUATIONS BY THE VARIATIONAL ITERATION METHOD

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Abstract

In this paper, the variational iteration method (VIM) is applied to obtain approximate analytical solution of Bratu-type equations without any discretization. Comparisons with the exact solutions reveal that VIM is very effective and convenient.

Keywords: Variational iteration method, Bratu's problem, Boundary value problems, Initial value problems.

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1. Introduction

Nonlinear phenomena are of fundamental importance in various fields of science and engineering. The nonlinear models of real-life problems are still difficult to solve either numerically or analytically. There has recently been much attention devoted to the search for better and more efficient solution methods for determining a solution, approximate or exact, analytical or numerical, to nonlinear models, [27, 21, 26].

In this paper, we consider the Bratu's boundary value problem in one-dimensional planar coordinates in the form

$$\begin{aligned} & u'' + \alpha e^u = 0,\ 0 < x < 1,\ \alpha > 0, \\ & \quad u(0) = u(1) = 0, \end{aligned} \tag{1.1}$$

which is used to model a combustion problem in a numerical slab.

Several numerical techniques, like the Adomain decomposition method (ADM) [28], the one-dimensional differential transform method [10], the finite difference method, finite element approximation, weighted residual method [3], the shooting method [4], the Laplace Adomain decomposition method [25] and the Laplace transform decomposition

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numerical algorithm [22] have been implemented independently to handle the Bratu model numerically.

The variational iteration method (VIM) is a simple and yet powerful method for solving a wide class of nonlinear problems, first envisioned by He [13] (see also [15, 18, 19, 14, 17]). The VIM has successfully been applied to many situations. For example, He [15] solved the classical Blasius' equation using VIM. He [18] used VIM to give approximate solutions for some well-known nonlinear problems. He [19] used VIM to solve autonomous ordinary differential systems. He [14] coupled the iteration method with the perturbation method to solve the well-known Blasius equation. He [17] solved strongly non-linear equations using VIM. Soliman [24] applied the VIM to solve the KdV-Burger's and Lax's seventh-order KdV equations. The VIM has recently been applied for solving nonlinear coagulation problem with mass loss by Abulwafa *et al.* [2]. The VIM has been applied for solving nonlinear differential equations of fractional order by Odibat *et al.* [23]. Bildik *et al.* [7] used VIM for solving different types of nonlinear partial differential equations. Dehghan and Tatari [11] employed VIM to solve a Fokker-Planck equation. Wazwaz [29] presented a comparative study between the variational iteration method and Adomian decomposition method. Tamer *et al.* [12] introduced a modification of VIM. Batiha *et al.* [5] used VIM to solve the generalized Burgers-Huxley equation. Batiha *et al.* [6] applied VIM to the generalized Huxley equation. Abbasbandy [1] solved one example of the quadratic Riccati differential equation (with constant coefficient) by He's variational iteration method by using Adomian's polynomials.

The exact solution for Eq. (1.1) is given by [4, 8, 9] as

$$u(x) = -2\ln\left[\frac{\cosh((x-\frac{1}{2})\frac{\theta}{2})}{\cosh(\frac{\theta}{4})}\right], \tag{1.2}$$

where θ satisfies

$$\theta = \sqrt{2\alpha}\cosh\left(\frac{\theta}{4}\right). \tag{1.3}$$

The problem has zero, one or two solutions when $\alpha > \alpha_c$, $\alpha = \alpha_c$, and $\alpha < \alpha_c$, respectively, where α_c satisfies

$$1 = \frac{1}{4}\sqrt{2\alpha_c}\sinh\left(\frac{\theta_c}{4}\right). \tag{1.4}$$

The critical value α_c was evaluated in [4, 8, 9], and is given by

$$\alpha_c = 3.513830719. \tag{1.5}$$

In this paper, we shall apply VIM to find the approximate analytical solution of the boundary and initial value problem of the Bratu-type model. Comparisons with the exact solution shall be performed.

2. Variational iteration method

VIM is based on the general Lagrange multiplier method [20]. The main feature of the method is that the solution of a mathematical problem with a linearization assumption is used as initial approximation or trial function. Then a more highly precise approximation at some special point can be obtained. This approximation converges rapidly to an accurate solution [16].

To illustrate the basic concepts of VIM, we consider the following nonlinear differential equation:

$$Lu + Nu = g(x), \tag{2.1}$$

where L is a linear operator, N a nonlinear operator, and $g(x)$ is an inhomogeneous term. According to VIM [18, 19, 17, 16], we can construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\{Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)\}\mathrm{d}\tau, \quad n \geq 0, \tag{2.2}$$

where λ is a general Lagrangian multiplier [20], which can be identified optimally via the variational theory, the subscript n denotes the nth-order approximation, and $\tilde{u}_n$ is considered as a restricted variation [18, 19], i.e. $\delta\tilde{u}_n = 0$.

3. Analysis of the Bratu-type model

In this section, we present the solution of Eq. (1.1) by means of VIM. To do so, we first construct a correction functional,

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\left[\frac{\mathrm{d}u_n^2(s)}{\mathrm{d}s^2} + \alpha \mathrm{e}^{\tilde{u}_n(s)}\right]\mathrm{d}s, \; n \geq 0, \tag{3.1}$$

where $\tilde{u}_n$ are considered as restricted variations, which means $\tilde{u}_n = 0$. To find the optimal $\lambda(s)$, we proceed as follows:

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta \int_0^x \lambda(s)\left[\frac{\mathrm{d}u_n^2(s)}{\mathrm{d}s^2} + \alpha \mathrm{e}^{\tilde{u}_n(s)}\right]\mathrm{d}s, \tag{3.2}$$

and consequently

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta \int_0^x \lambda(s)\left[\frac{\mathrm{d}u_n^2(s)}{\mathrm{d}s^2}\right]\mathrm{d}s, \tag{3.3}$$

which results in

$$\delta u_{n+1}(x) = \delta u_n(x)(1-\lambda'(s)) + \delta(u_n)_s\lambda(s) + \int_0^t \delta u_n(x)\lambda''(s)\,\mathrm{d}s = 0.$$

The stationary conditions can be obtained as follows:

$$1-\lambda'(s) = 0|_{s=x}, \; \lambda(s) = 0|_{s=x}, \; \lambda''(s) = 0|_{s=x}. \tag{3.4}$$

The Lagrange multipliers, therefore, can be identified as

$$\lambda(s) = s - x, \tag{3.5}$$

and the iteration formula is given as

$$u_{n+1}(x) = u_n(x) + \int_0^x (s-x)\left[\frac{\mathrm{d}u_n^2(s)}{\mathrm{d}s^2} + \alpha \mathrm{e}^{u_n(s)}\right]\mathrm{d}s, \; n \geq 0. \tag{3.6}$$

4. Applications

4.1. Example. We first consider the Bratu-type model

$$\begin{aligned} u'' - \pi^2\mathrm{e}^u = 0, \; 0 < x < 1, \\ u(0) = u(1) = 0. \end{aligned} \tag{4.1}$$

Here $\alpha = -\pi^2$.

The exact solution for Eq. (4.1) was found to be [28]:

$$u(x) = -\ln\left(1 + \cos\left[\left(\frac{1}{2} + x\right)\pi\right]\right). \tag{4.2}$$

Using VIM, the iteration formula for Eq. (4.1) is (see (3.6)),

$$u_{n+1}(x) = u_n(x) + \int_0^x (s-x)\left[\frac{\mathrm{d}u_n^2(s)}{\mathrm{d}s^2} - \pi^2 \mathrm{e}^{u_n(s)}\right]\mathrm{d}s, \; n \geq 0. \tag{4.3}$$

We can take an initial approximation $u_0(x) = \pi x$.

The first iterate is easily obtained from (3.6) and is given by:

$$u_1(x) = \mathrm{e}^{\pi x} - 1. \tag{4.4}$$

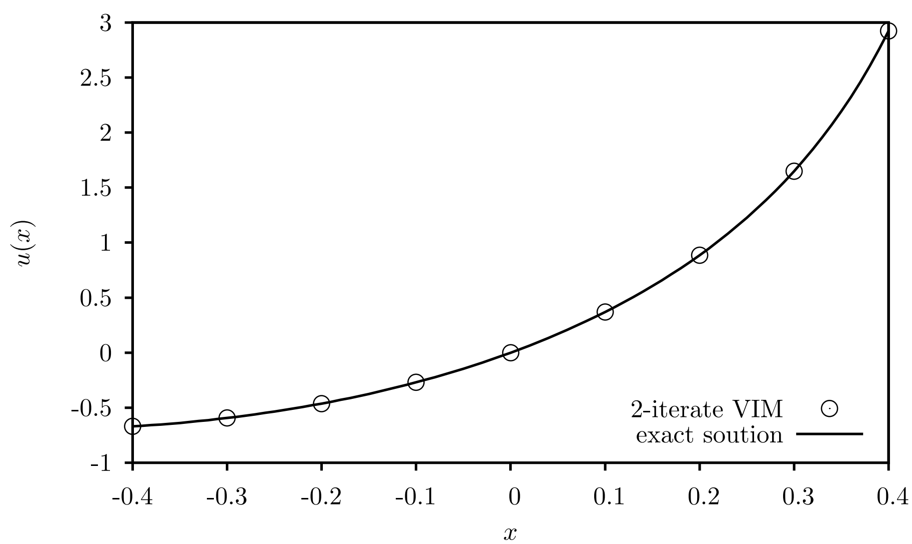
In Table 1, we compare the two-iteration VIM solution with the exact solution. Figure 1 shows graphically the comparison between the 2-iterate of VIM and the exact solution (4.2).

Based on these results we can conclude that the VIM is very accurate and convenient.

Table 1. Comparison between the exact solution and the numerical results for $u(x)$ by means of a 2-iterate VIM solution with absolute errors

t	Exact	VIM	Absolute error
-0.4	-0.6683710291	-0.6700998320	1.728803E-3
-0.3	-0.5927836007	-0.5931666781	3.830774E-4
-0.2	-0.4623401221	-0.4623836732	4.355110E-5
-0.1	-0.2692764696	-0.2692773941	9.246000E-7
0.0	0.0000000000	0.0000000000	0.000000000
0.1	0.3696400494	0.3696379582	2.091200E-6
0.2	0.8862108331	0.8859757749	2.350583E-4
0.3	1.6555708307	1.6499543970	5.616434E-3
0.4	3.0170890403	2.9248018688	9.228717E-2

Figure 1. Comparison between the exact solution and the numerical results for $u(x)$ by means of a 2-iterate VIM solution



4.2. Example. Now, we will find the approximate analytical solution of the the Bratu-type model

$$\begin{aligned} u'' + \pi^2 \mathrm{e}^u = 0,\ 0 < x < 1,\\ u(0) = u(1) = 0. \end{aligned} \tag{4.5}$$

Here $\alpha = \pi^2$.

The exact solution for Eq. (4.5) was found to be [28]:

$$u(x) = \ln\left(1 + \sin[1 + \pi x]\right). \tag{4.6}$$

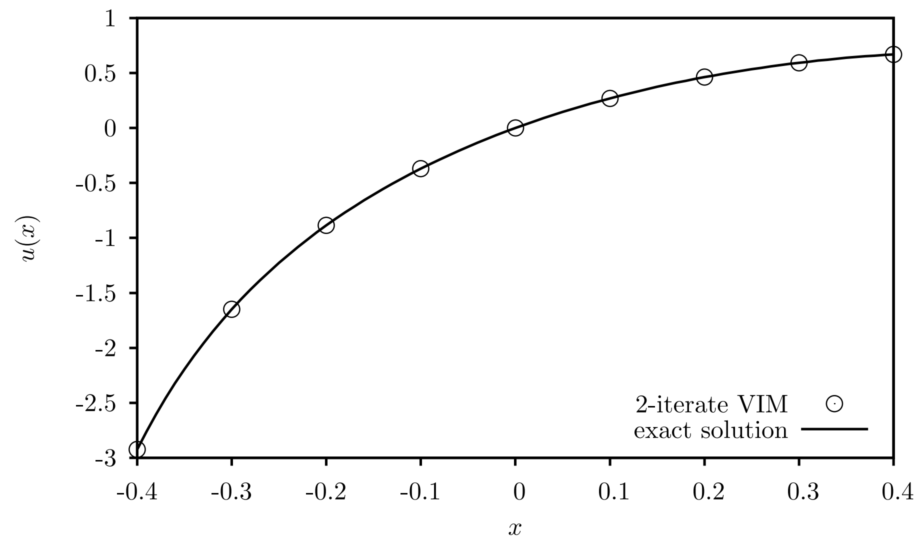
We can take an initial approximation $u_0(x) = \pi x$.

The first iterate is easily obtained from (3.6) and is given by:

$$u_1(x) = \pi x - \mathrm{e}^{(-\pi x)}\left(\pi x \mathrm{e}^{(\pi x)} - \mathrm{e}^{(\pi x)} + 1\right). \tag{4.7}$$

Figure 2 shows graphically the comparison between the 2-iterate of VIM and the exact solution (4.6).

Figure 2. Comparison between the exact solution and the numerical results for $u(x)$ by means of a 2-iterate VIM solution



4.3. Example. We finally consider the initial value problem of Bratu-type model

$$\begin{aligned} &u'' - 2\mathrm{e}^u = 0,\ 0 < x < 1, \\ &\quad u(0) = u'(0) = 0. \end{aligned} \tag{4.8}$$

Here $\alpha = -2$.

The exact solution for Eq. (4.8) was found to be [28]:

$$u(x) = -2\ln(\cos x). \tag{4.9}$$

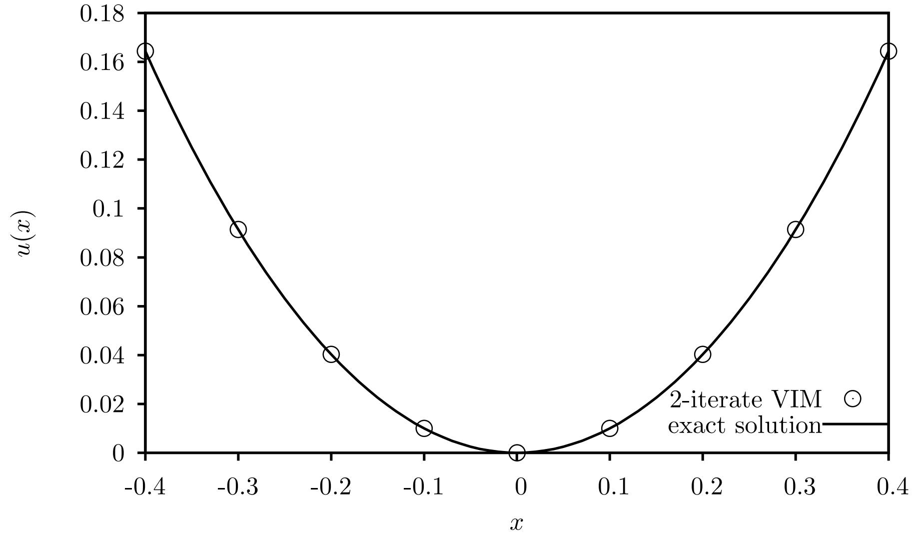
For simplicity, we can take an initial approximation $u_0(x) = 0$.

The first iterate is easily obtained from (3.6) and is given by:

$$u_1(x) = x^2. \tag{4.10}$$

Figure 3 shows the comparison between the 2-iterate of VIM and the exact solution (4.9).

Figure 3. Comparison between the exact solution and the numerical results for $u(x)$ by means of a 2-iterate VIM solution



5. Conclusions

In this paper, the variation iteration method (VIM) has been successfully employed to obtain the approximate analytical solutions of the Bratu-type equations. The method has been applied directly without using linearization, or any restrictive assumptions. Comparisons with the exact solution reveal that VIM is very effective and convenient. It may be concluded that VIM is very powerful and efficient in finding analytical as well as numerical solutions for a wide class of linear and nonlinear differential equations. VIM provides more realistic series solutions that converge very rapidly in real physical problems.

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