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A Study on Controllability and Periodicity Solutions for Nonlinear Neutral Integrodifferential System

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Abstract

The objective of this paper is to present sufficient conditions for controllability and periodicity solutions of an integrodifferential system in Banach space. The main results are obtained by using resolvent operators and a fixed point approach. Further, the mild solution of the integrodifferential system has been shown to be compact asymptotically almost automorphic $(\mathscr{A}\mathscr{A}\mathscr{A}_c)$. Then uniqueness of the $\mathscr{A}\mathscr{A}\mathscr{A}_c$ solution has been shown by the Banach contraction principle. Finally, an example is provided to show the effectiveness of the obtained theoretical result.

Keywords: *Controllability; neutral integrodifferential equation; delay differential equation; asymptotically almost automorphic*
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1. Introduction

In recent years, in the fields of science and engineering, many problems are connected with mathematical modeling. It is applied to define physical, biological or chemical phenomena, one of the usual popular results is either a differential equation or a system of differential equations, commonly with the suitable boundary and initial conditions. These differential equations may be ordinary or partial, and finding and interpreting their solution is at the heart of applied mathematics. A delay differential equations are similar to ordinary differential equations, but their evolution involves the past of their state variable. It is a special type of functional differential equations. Neutral differential equations have numerous utilization. They can create a bundle of problems emerging from engineering, such as transmission line, immune response, population dynamics, distribution of albumin in the blood. For instance, in the theory of heat conduction in fading memory material [18]. Evolution type of equations appears in several applications such as in thermodynamics, shear in second-order fluids and flow of fluid through fissured rocks.

The problem of controllability is to find an objective can be reached by some suitable control function. It occurs when a system described by a state $x(t)$ is controlled by a given differential equation. Balachandran [1] appeared on the problem of controllability nonlinear and integrodifferential systems in Banach spaces using fixed point principles. Most of those products are in associate with finite delays. Many authors expressed the controllability of nonlinear systems through differential equations in an infinite-dimensional space. In particular, by using a fixed point theorem the controllability of an integrodifferential neutral evolution system with a state-dependent delay, nonlocal and integrodifferential has been implemented by Radhakrishnan [20, 21, 22, 24]. Controllability of an integrodifferential evolution and neutral evolution system with nonlocal initial condition and infinite delay presented in[27]. Neutral differential equations arise in many areas of applied mathematics, and for this reason, this type of equation has received much attention in recent years. Theory of partial neutral integrodifferential with delay equations has been studied by several authors [13, 16] in Banach spaces.

Recently, the study of periodicity is one of the most attractive topics in mathematical analysis. The idea of almost periodicity, which generalizes the concept of periodicity, performs a significant role in several fields including harmonic analysis, physics, and dynamical system. Bochner [4] first introduced these functions, which begin with abstract differential equations with periodic functions. Zaidman [28] explained that the existence of almost periodic solution for abstract differential equation with C_0-Semi group. Later, Guerekata [12] a studied the existence and uniqueness of almost automorphic solutions of semilinear evolution equations. Almost periodic solutions of differential equations and their natures have been studied so many authors [2, 3, 8, 7, 14, 15, 9, 25] in the very beginning of this century. Almost periodic solutions of evolution problems and time dependent evolution equations has been discussed in Francois et al,[11]. Marko

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Kosti [17] discussed on generalized C^n almost periodic solutions of abstract Volterra integrodifferential differential equations. The existence and uniqueness of a compact almost automorphic solution for dissipative differential equations in Banach spaces were discussed by Drisi et al. [10, 26]. Cao [5] discussed new existence theorem for semilinear evolution equations of the form

$$x'(t) = A(t)x(t) + F(t, x(t))$$

with the asymptotically almost automorphic mild solutions. Ding [6] discussed existence and uniqueness of asymptotically almost automorphic solution for the following equation

$$z'(t) = A(t)z(t) + B(t)z([t]) + f(t, z(t), z([t]))$$

with piecewise constant argument. Controllability and almost periodic results for neutral impulsive evolution system, using analytic semigroup and fixed point principle has been studied by Radhakrishnan [23]. This fact that the present work is much more interested in authors.

Our main contributions are highlighted as follows:

- A new set of sufficient conditions are established for the periodicity results of the nonlinear neutral integrodifferential evolution system.
- Most of the available literature, for the first time controllability with periodicity results involving resolvent operator, are have been investigated.
- The Sadovskii's and Banach fixed point theorems are effectively used to establish the results.
- Controllability of the solution of the neutral integrodifferential evolution system has been proved.
- Subsequently, the mild solution of the control system has been shown to be compact Asymptotically Almost Automorphic ($\mathscr{A}\mathscr{A}\mathscr{A}_c$).

In this spirit, inspired by the efforts contributed up to here, we note the necessity and importance of the realization of a work that will contribute to the controllability and periodicity of an integrodifferential system. To the best of our knowledge, an investigation concerning the controllability with asymptotically almost automorphic solutions of an integrodifferential neutral evolution systems with bounded delay and non-local condition in Banach spaces has not been established yet. Thus, we will make an effort to analyze such results in this paper.
The current work is designed as follows, in the second section basic definitions, theorems are provided. In Section 3, the controllability result of the system (2.1) is established by Sadovskii Fixed-Point Theorem. Fourth section deals the basic definitions of periodicity results and theorems to the same system. Finally, the obtained theoretical result is validated through an example.

2. Preliminaries

In this paper, consider an integrodifferential neutral evolution system with bounded delay

$$\left.\begin{aligned} \mathbb{D}\left[x(t) - \int_{-r}^{t} G(t,s)x(s)ds\right] &= A(t)x(t) + Bu(t) + \int_{-r}^{t} \mathcal{K}(t,s)x(s)ds + h(t,x_t), \\ x(0) + g(x) &= x_0, \quad t \in I, \end{aligned}\right\} \tag{2.1}$$

where $\mathbb{D} = \frac{d}{dt}$ and the state $x(\cdot)$ takes values in a Banach space X. Here $G(t,s)$, $0 \le s \le t \le b$ are closed and bounded linear operators defined on the common domain $\mathscr{D}(A)$ which is dense in X. Also $A(t)$, $\mathcal{K}(t,s)$, $0 \le s \le t \le b$ are closed linear operators defined on a common domain $\mathscr{D} := \mathscr{D}(A(t))$, which is dense in X. B is a bounded linear operator from U into X and $h : I \times \mathscr{B} \to X$, g are appropriated functions. The history $x_t : (-r, 0] \to X$, $x_t(\theta) = x(t+\theta)$, belongs to an abstract phase space $\mathscr{B}$ defined axiomatically, and the control function $u(\cdot)$ is given in $\mathscr{L}^2(I, U)$, a Banach space of admissible control functions with U as a Banach space the interval $I = [0, b]$.
In this frame, some basic definitions and properties are in the sequel.
Let $(Z, \|\cdot\|_Z)$, $(X, \|\cdot\|)$ be Banach spaces. The notation $[\mathscr{D}A(t)]$ represents the domain of $A(t)$ endowed with the graph norm. The notation $\rho(P)$ stands for the resolvent set of P, and $\mathscr{R}(\lambda, P) = (\lambda I - P)^{-1}$ is the resolvent operator of P. If $f : \mathbb{R} \to Z$, denote $\|f\|_{Z,\infty} = \sup_{s\in\mathbb{R}} \|f(s)\|_Z$ or if $f : [0,\infty) \to Z$, denote $\|f\|_{Z,\infty} = \sup_{s\in[0,\infty)} \|f(s)\|_Z$.
Next, the phase space $\mathscr{B}$ will denote a vector space of functions defined from $(-r, 0]$ into X endowed with a semi-norm denoted by $\|\cdot\|_{\mathscr{B}}$ and such that the following axioms hold:

(d1) If $x : (-\infty, \sigma + b) \to X$ with $b > 0$ is continuous on $[\sigma, \sigma + b)$ and $x_\sigma \in \mathscr{B}$, then for each $t \in [\sigma, \sigma + b)$ the following conditions hold:

 (i) x_t is in $\mathscr{B}$,
 (ii) $\|x(t)\| \le H\|x_t\|_{\mathscr{B}}$,
 (iii) $\|x_t\|_{\mathscr{B}} \le P_1(t-\sigma)\sup\{\|x(s)\| : \sigma \le s \le t\} + P_2(t-\sigma)\|x_\sigma\|_{\mathscr{B}}$,

 where $H > 0$ is a constant, and $P_1, P_2 : [0,\infty) \mapsto [1,\infty)$ are functions such that $P_1(\cdot)$ and $P_2(\cdot)$ are respectively continuous and locally bounded, and H, P_1, P_2 are independent of $x(\cdot)$.

(d2) If $x(\cdot)$ is a function as in (d1), then x_t is a $\mathscr{B}$-valued continuous function on $[\sigma, \sigma + b)$.

(d3) The space $\mathscr{B}$ is complete.

Initially consider the linear integrodifferential equation,

$$\frac{dx(t)}{dt} = A(t)x(t) + \int_{-r}^{t} \mathcal{K}(t,s)x(s)ds, \tag{2.2}$$

$$x(0) = \delta, \quad \delta \in X. \tag{2.3}$$

It has an associated resolvent operator of bounded linear operators $\mathscr{R}(t,s)$ on X.

Definition 2.1. *[19] A two-parameter family of bounded linear operators $\mathscr{R}(t,s)$, $0 \le s \le t \le b$ on X is called a resolvent operator of (2.2) - (2.3) if the following conditions are satisfied:*

(a) $\mathscr{R}(t,t) = I$, for every $t \in [0,b]$ and $\mathscr{R}(t,\cdot) \in \mathscr{C}([0,t],X)$, $\mathscr{R}(\cdot,s) \in \mathscr{C}([s,b],X)$, for every $x \in X$, $t \in [0,b]$, $s \in [0,b]$.

$$\|\mathscr{R}(t,s)\| \le M_1 e^{\beta(t-s)},$$

for some constants M_1 and β.

(b) For $x \in \mathscr{D}(A)$, $\mathscr{R}(t,s)x \in \mathscr{C}\big([0,\infty[,\ [\mathscr{D}(A)]\big) \cap \mathscr{C}'(]0,\infty[,X)$.

(c) For each $x \in X$, $\mathscr{R}(t,s)x$ is continuously differentiable in $s \in I$

$$\frac{\partial \mathscr{R}}{\partial s}(t,s)x = -\mathscr{R}(t,s)A(s)x - \int_s^t \mathscr{R}(t,\tau)\mathcal{K}(\tau,s)x d\tau.$$

(d) For each $x \in X$, $s \in I$, $\mathscr{R}(t,s)x$ is continuously differentiable in $t \in [s,b]$ and

$$\frac{\partial \mathscr{R}}{\partial t}(t,s)x = A(t)\mathscr{R}(t,s)x + \int_s^t \mathcal{K}(\tau,s)\mathscr{R}(t,\tau)x d\tau,$$

with $\frac{\partial \mathscr{R}}{\partial s}$ and $\frac{\partial \mathscr{R}}{\partial t}$ are strongly continuous on $0 \le s \le t \le b$. And $\mathscr{R}(t,s)$ can be extracted from the evolution operator of the generator $A(t)$.

Inspired by the theory of resolvent operator, further discuss the following concept of mild solution for (2.1).

Definition 2.2. *Let $\mathcal{E}_i : [0,b] \to X,\ i = 1,2$ will be the functions defined by*

$$\mathcal{E}_1(t) = \int_{-r}^{0} G(t,s)x(s)ds$$

and

$$\mathcal{E}_2(t) = \int_{-r}^{0} \mathcal{K}(t,s)x(s)ds$$

in (2.1) and a function $x : [0,b] \to X$ is called a mild solution of an integrodifferential neutral system (2.1) on $[0,b]$ if and only if $x(0) = x_0 - g(x)$, $\mathcal{E}_1$ is differentiable on $[0,b]$, $\mathcal{E}_1', \mathcal{E}_2 \in \mathscr{L}^1([0,b],X)$ and the following integral equation

$$\left.\begin{aligned} x(t) &= \mathscr{R}(t,0)\left[x_0 - g(x)\right] + \int_0^t \mathscr{R}(t,s)\left[\mathcal{E}_1'(s) + \mathcal{E}_2(s)\right]ds \\ &\quad + \int_0^t \mathscr{R}(t,s)\left[Bu(s) + h(s,x_s)\right]ds, \quad t \in I, \end{aligned}\right\} \tag{2.4}$$

is satisfied.

The following theorem gives a formula for a control transferring the initial state x_0 to final state x_b at time b.

Theorem 2.3. *For $x_b \in X$, define the control*

$$\left.\begin{aligned} u(t) &= W^{-1}\Big\{x_b - \mathscr{R}(b,0)[x_0 - g(x)] - \int_0^b \mathscr{R}(t,s)[\mathcal{E}_1'(s) + \mathcal{E}_2(s)]ds \\ &\quad - \int_0^b \mathscr{R}(b,s)h(s,x_s)ds\Big\}(t) \end{aligned}\right\} \tag{2.5}$$

transfers initial state x_0 to final state

$$\left.\begin{aligned} x(b) &= \mathscr{R}(b,0)[x_0 - g(x)] + \int_0^b \mathscr{R}(b,s)[\mathcal{E}_1'(s) + \mathcal{E}_2(s)]ds \\ &\quad + \int_0^b \mathscr{R}(b,s)\left[Bu(s) + h(s,x_s)\right]ds, \quad t \in I, \end{aligned}\right\} \tag{2.6}$$

at time t=b.

Proof. By substituting this control (2.5) in equation (2.6), then

$$\begin{aligned} x(b) &= \mathscr{R}(b,0)[x_0 - g(x)] + \int_0^b \mathscr{R}(b,s))[\mathcal{E}_1'(s) + \mathcal{E}_2(s)]ds \\ &\quad + \int_0^b \mathscr{R}(b,s)BW^{-1}\Big\{x_b - \mathscr{R}(b,0)[x_0 + g(x)] - \int_0^b \mathscr{R}(b,s)[\mathcal{E}_1'(s) + \mathcal{E}_2(s)]ds \\ &\quad - \int_0^b \mathscr{R}(b,s)h(s,x_s)\Big\}(s)ds + \int_0^b \mathscr{R}(b,s)h(s,x_s)ds \\ &= \mathscr{R}(b,0)[x_0 - g(x)] + \int_0^b \mathscr{R}(b,s)[\mathcal{E}_1'(s) + \mathcal{E}_2(s)]ds \\ &\quad + WW^{-1}\Big\{x_b - \mathscr{R}(b,0)[x_0 + g(x)] - \int_0^b \mathscr{R}(b,s)[\mathcal{E}_1'(s) + \mathcal{E}_2(s)]ds \\ &\quad - \int_0^b \mathscr{R}(b,s)h(s,x_s)\Big\} + \int_0^b \mathscr{R}(b,s)h(s,x_s)ds = x_b. \qquad \square \end{aligned}$$

□

Definition 2.4. *[1] The system (2.1) is said to be controllable on the interval I iff, for every $x_0, x_b \in X$, there exists a control $u \in \mathscr{L}^2(I,U)$ such that the solution $x(\cdot)$ of the system (2.1) satisfies $x(0) = x_0,\ x(b) = x_b$.*

Lemma 2.5. *(Sadovskii's Fixed-Point Theorem) Let $\mathcal{N}$ be the condensing operator on a Banach space X. If $\mathcal{N}(S) \subset S$ for a convex, closed and bounded set S of X, then $\mathcal{N}$ has fixed point in S.*

3. Controllability Result

In this section, we study the controllability results for the neutral system (2.1), there exists a positive constant M_1 such that $\|\mathscr{R}(t,s)\| \leq M_1$, for every $t \in I$. The following assumptions are needed to establish our controllability results.

(H1) The linear operator $W : \mathscr{L}^2(I,U) \to X$ is defined by

$$Wu = \int_0^b \mathscr{R}(b,s)Bu(s)ds,$$

has an induced inverse operator W^{-1} which takes values in $\mathscr{L}^2(I,U)/kerW$ and there exists a positive constant N such that $\|BW^{-1}\| \leq$ N.

(H2) The function $g : X \to X$ is a continuous and there exist a constants $K_g, M_2 > 0$ such that

$$\|g(v_1) - g(v_2)\| \leq K_g\|v_1 - v_2\|, \text{ for } v_1,\ v_2 \in X,$$

and $\|g(0)\| \leq M_2$.

(H3) The function $h : [0,b] \times \mathscr{B} \to X$ satisfies the following conditions:

(i) For each $t \in I$, the function $h(t,\cdot) : \mathscr{B} \to X$ is continuous.

(ii) The function h is continuous and there exists a constant $K_h \in \mathscr{L}^1([0,b],\mathbb{R}^+)$ such that

$$\|h(t,\eta_1) - h(t,\eta_2)\| \leq K_h\|\eta_1 - \eta_2\|,\ t \in [0,b].$$

(iii) There exists a continuous function $h_g \in \mathscr{C}([0,b],[0,\infty])$ and a non-decreasing continuous function $L_g : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|h(t,y)\| \leq h_g(t)L_g(\|y\|_{\mathscr{B}}).$$

Theorem 3.1. *If the assumptions $(H1)-(H3)$ are holds, $\mathcal{E}_1, \mathcal{E}_2 \in \mathscr{L}^1([0,b],X)$, and $x(\cdot)$ is a mild solution of the system (2.1), then the system is controllable on I provided*

$$(1+tN)\Big[M_1[\|x_0\| + K_g + M_2]\Big] + [1 + M_1Nb]\Big[tM_1\widehat{K}_2 + M_1h_g(t)L_g\|x\|\Big] + \widehat{K_3} < 1.$$

Proof. By means of the assumption (H_1), for arbitrary function $x(\cdot)$, we define the control in (2.5). Consider the space $\mathscr{S}(b) = \{x \in \mathscr{C}(I,X) : x(0) = x_0 - g(x)\}$ and define the operator $\Psi : \mathscr{S}(b) \to \mathscr{S}(b)$ by

$$\begin{aligned}\Psi x(t) \quad &= \quad \mathscr{R}(t,0)[x_0 - g(x)] + \int_0^t \mathscr{R}(t,s)[\mathcal{E}_1^{'}(s) + \mathcal{E}_2(s)]ds \\ &+ \int_0^t \mathscr{R}(t,s)BW^{-1}\Big\{x_b - \mathscr{R}(b,0)\,[x_0 + g(x)] \\ &- \int_0^b \mathscr{R}(b,s)[\mathcal{E}_1^{'}(s) + \mathcal{E}_2(s)]ds - \int_0^b \mathscr{R}(b,s)h(x,x_s)\Big\}(s)ds \\ &+ \int_0^t \mathscr{R}(t,s)h(s,x_s)ds, \quad t \in I,\end{aligned}$$

as a fixed point $x(\cdot)$. Then from Definition 2.2, $x(\cdot)$ is a integral solution of system (2.1). This fixed point is then a solution of the control problem (2.1). Clearly, it is verified that $x(b) = (\Psi x)(b) = x_b$, which implies that the system is controllable.
Now to prove that the operator Ψ is a completely continuous operator. Set

$$\mathbb{B}_n = \{x \in \mathscr{S}(b) : \|x\| < n\}.$$

Step 1: We claim that there exists a positive constant $n > 0$ such that

$$\Psi\,[\mathbb{B}_n(x,\mathscr{S}(b))] \subseteq \mathbb{B}_n(x,\mathscr{S}(b)).$$

If this property fails, then for each positive number n, there exists a function $x^n \in \mathbb{B}_n$, but $\Psi(\mathbb{B}_n)$ does not belong to $\mathbb{B}_n$ such that

$n < \|\Psi(x^n)(t) - x(0)\|$

$$\begin{aligned}
\|\Psi(x^n)(t)-x(0)\| &\leq \|\mathscr{R}(t,0)[x_0-g(x^n)]\| + \int_0^t \|\mathscr{R}(t,s)\|\|\mathcal{E}_1'(s)+\mathcal{E}_2(s)\|ds \\
&+ \int_0^t \|\mathscr{R}(t,s)\|\|BW^{-1}\|\Big[\|x_b\| - \|\mathscr{R}(t,0)\|\|x_0+g(x^n)\| \\
&- \int_0^b \|\mathscr{R}(b,s)\|\|\mathcal{E}_1'(s)+\mathcal{E}_2(s)\|ds + \int_0^b \|\mathscr{R}(b,s)\|\|h(s,x_s)\|ds\Big](s)ds \\
&+ \int_0^t \|\mathscr{R}(t,s)\|\|h(s,x_s)\|ds - \|x_0\| \\
&\leq M_1[\|x_0\| - \|g(x^n)-g(0)\| + \|g(0)\|] + \int_0^t M_1\|\mathcal{E}_1'(s)+\mathcal{E}_2(s)\|ds \\
&+ \int_0^t M_1 N\Big[[\|x_b\| - M_1\|x_0\| + \|g(x^n)-g(0)\| + \|g(0)\|] \\
&- \int_0^b M_1\|\mathcal{E}_1'(s)+\mathcal{E}_2(s)\|ds + \int_0^b M_1\|h(s,x_s)\|ds\Big](s)ds \\
&+ \int_0^t M_1\|h(s,x_s)\|ds - \|x_0\| - \|g(x)\| \\
&\leq M_1[\|x_0\| - K_g(x) + \|g(0)\|] + M_1 t\widehat{K}_2 + tM_1N\Big\{\|x_b\| - M_1\|x_0\| \\
&+ K_g(x) + \|g(0)\| - bM_1\widehat{K}_2 + bM_1h_g(t)L_g\|x\|\Big\} + tM_1h_g(t)L_g\|x\| \\
&- \|x_0\| + \|g(x)-g(0)\| + \|g(0)\| \\
&\leq (1+tN)\Big[M_1[\|x_0\| + K_g + M_2]\Big] + [1+M_1Nb] \\
&\times\Big[tM_1\widehat{K}_2 + M_1h_g(t)L_g\|x\|\Big] + \widehat{K_3}
\end{aligned}$$

where $\widehat{K_2} = [\mathcal{E}_1'(s)+\mathcal{E}_2(s)]_{\mathscr{L}^1[0,b]}$, $\widehat{K_3} = tM_1N\|x_b\| + \|x_0\| + K_g + M_2$. Therefore

$$1 \leq (1+tN)\Big[M_1[\|x_0\| + K_g + M_2]\Big] + [1+M_1Nb]\Big[tM_1\widehat{K}_2 + M_1h_g(t)L_g\|x\|\Big] + \widehat{K_3}$$

which is contradicts our assumption. Hence for some positive number n,

$$\Psi[\mathbb{B}_n(x,\mathscr{S}(b))] \subseteq \mathbb{B}_n(x,\mathscr{S}(b)).$$

Next to prove that Ψ is a condensing operator and introduce the decomposition $\Psi=\Psi_1+\Psi_2$, where

$$\begin{aligned}
\Psi_1 x(t) &= \mathscr{R}(t,0)[x_0-g(x)] + \int_0^t \mathscr{R}(t,s)[\mathcal{E}_1'(s)+\mathcal{E}_2(s)]ds \\
&+ \int_0^t \mathscr{R}(t,s)BW^{-1}[x_b - \mathscr{R}(b,0)[x_0+g(x)] \\
&- \int_0^b \mathscr{R}(b,s)[\mathcal{E}_1'(s)+\mathcal{E}_2(s)]ds - \int_0^b \mathscr{R}(b,s)h(s,x_s)](s)ds, \\
\Psi_2 x(t) &= \int_0^t \mathscr{R}(t,s)h(s,x_s)ds, \quad t \in I.
\end{aligned}$$

Step 2: To claim that Ψ_1 is contraction on $\mathbb{B}_n$. Let $x,y \in \mathbb{B}_n(\mathscr{S}(b))$ and $t \in [0,b]$, we see that

$$\begin{aligned}
\|\Psi_1(x)(t)-\Psi_1(y)(t)\| &\leq \|\mathscr{R}(t,0)\|[\|x_0-y_0\| + \|g(x)-g(y)\|] + \int_0^t \|\mathscr{R}(t,s)\|\|BW^{-1}\| \\
&\times\Big\{\|x_b-y_b\| + \|\mathscr{R}(b,0)[\|x_0-y_0\| + \|g(x)-g(y)\|] \\
&+ \int_0^b \|\mathscr{R}(t,s)\|\|h(s,x_s)-h(s,y_s)\|\Big\}(s)ds \\
&\leq M_1[\|x_0-y_0\| + K_g\|x-y\|] + tM_1N\Big\{\|x_b-y_b\| \\
&+ M_1[\|x_0-y_0\| + K_g\|x-y\|] + bM_1K_h\|x-y\|\Big\} \\
&\leq [1+M_1Nb]\big[M_1\|x_0-y_0\| + K_g\|x-y\|\big] \\
&+ M_1b\big[N\|x_b-y_b\| + K_h\|x-y\|\big] \\
&\leq \big[M_1[1+K_g+tM_1NK_g] + tM_1NM_1(1+bK_h)\big]\|x-y\| \\
&\leq \Delta\|x-y\|,
\end{aligned}$$

where $\Delta = M_1[1+K_g+tM_1NK_g] + tM_1NM_1(1+bK_h) < 1$, which implies that $\Psi_1(\cdot)$ is a contraction on $\mathbb{B}_n(\mathscr{S}(b))$. It remains to show that $\Psi_2(\cdot)$ is completely continuous on $\mathbb{B}_n(\mathscr{S}(b))$.

Step 3: First, we prove that the set $\Psi_2(\mathbb{B}_n(\mathscr{S}(b))$ is equicontinuous on $[0,b]$.

Let $0<\tau<t<b$ such that $\|\mathscr{R}(t+\alpha,s')-\mathscr{R}(t,s')\|\leq\tau$, for every $s'\in[\tau,b]$. Under the conditions, $x\in(\mathbb{B}_n(x,\mathscr{S}(b))$ and $t+\alpha\in[0,b]$. We get

$$\begin{aligned}\|\Psi_2(x)(t+\alpha)-\Psi_2(x)(t)\| &\leq \Big\|\int_0^{t+\alpha}\mathscr{R}(t+\alpha,s)h(s,x_s)ds\Big\|-\Big\|\int_0^t\mathscr{R}(t,s)h(s,x_s)ds\Big\|\\ &\leq \Big\|\int_0^{t+\alpha}\mathscr{R}(t+\alpha,s')\mathscr{R}(s',s)h(s,x_s)ds\Big\|\\ &\quad -\Big\|\int_0^t\mathscr{R}(t,s')\mathscr{R}(s',s)h(s,x_s)ds\Big\|\\ &\leq \Big\|\int_0^t\mathscr{R}(t+\alpha,s')\mathscr{R}(s',s)h(s,x_s)ds\Big\|\\ &\quad +\Big\|\int_t^{t+\alpha}\mathscr{R}(t+\alpha,s')\mathscr{R}(s',s)h(s,x_s)ds\Big\|\\ &\quad -\Big\|\int_0^t\mathscr{R}(t,s')\mathscr{R}(s',s)h(s,x_s)ds\Big\|\\ &\leq \Big\|\int_0^t\mathscr{R}(s',s)\left[\mathscr{R}(t+\alpha,s')-\mathscr{R}(t,s')\right]h(s,x_s)ds\Big\|\\ &\quad +\Big\|\int_t^{t+\alpha}\mathscr{R}(t+\alpha,s')\mathscr{R}(s',s)h(s,x_s)ds\Big\|\\ &\leq M_1\tau h_g(t)L_g\|x_s\|[t+\alpha],\end{aligned}$$

which shows that the set of functions $\Psi_2(\mathbb{B}_n\mathscr{S}(b))$ is right-equicontinuous at $t\in]0,b[$. A similar procedure proves the right-equicontinuity at zero and left equi-continuity at $t\in(0,b]$. Thus $\Psi_2(\mathbb{B}_n(\mathscr{S}(b)))$ is equi-continuous on I.
Further more, we have to prove that the set $\Psi_2(\mathbb{B}_n(\mathscr{S}(b))$ is relatively compact on X, for every $t\in[0,b]$. From the assumptions, fix $0=t_0<t_1<\ldots<t_n=t-\tau$ if $s\in[t_i,t_{i+1}]$, $i=0,1,2...,n-1$. Let $x\in\mathbb{B}_n$, by the mean value theorem for the Bochner integral (see [?]),

$$\begin{aligned}\Psi_2x(t) &= \sum_{i=1}^n\int_{t_{i-1}}^{t_i}[\mathscr{R}(t,s)-\mathscr{R}(t,t_i)]h(s,x_s)ds+\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\mathscr{R}(t,t_i)h(s,x_s)ds\\ &\quad +\int_{t_n}^t\mathscr{R}(t,s)h(s,x_s)ds\\ &\in \tau h_g(t)L_g\|x\|_{\mathscr{B}}+(t_i-t_{i-1})\overline{Co[\mathscr{R}(t,t_i)h(s,y):y\in\mathbb{B}_n(0,\mathscr{B}),s\in[0,b]]}\\ &\quad +M_1(t-t_n)h_g(t)x_g(\eta)\\ &\in \tau\mathbb{B}_n(0,X)+(t_i-t_{i-1})\overline{Co[\mathscr{R}(t,t_i)h(s,y):y\in\mathbb{B}_n(0,\mathscr{B}),s\in[0,b]]}+E_\tau\end{aligned}$$

hence this prove that $\Psi_2x(t)$ is totally bounded according to $diam(E_\tau)\to0$ when $\tau\to0$. Therefore $\Psi_2x(t)$ is relatively compact in X, for every $t\in[0,b]$.
Finally, to show that the map $\Psi_2(\cdot)$ is continuous on $\mathbb{B}_n$. Let (x^n) be a sequence in $\mathbb{B}_n$ and $x\in\mathbb{B}_n$ such that $x^n\to x$ in $\mathscr{S}(b)$,

$$\|\Psi_2(x^n)-\Psi_2(x)\|\leq\int_0^t\|\mathscr{R}(t,s)\|\|h(s,x_s^n)-h(s,x_s)\|ds\leq M_1tK_h\|x_s^n-x_s\|,$$

which proves that $x^n\to x$ in $\mathscr{B}$ as $n\to\infty$. Thus Ψ_2 is continuous on $\mathscr{B}_n$. Now the assumption $(H3)$ and the Lebesgue dominated convergence theorem permit us to assert that $\Psi(x^n)\to\Psi x$ in $\mathscr{S}(b)$. Thus $\Psi(\cdot)$ is continuous, which completes the proof that $\Psi_2(\cdot)$ is completely continuous.
Hence these arguments enable us to conclude that $\Psi(\cdot)$ is a condensing map on $\mathscr{S}(b)$ and by the theorem of Sadovskii, there exists a fixed point $x(\cdot)$ for Ψ on $\mathscr{S}(b)$, which is the expected mild solution of the system (2.1) satisfying $x(b)=x_b$. This completes the proof. □

4. Periodicity Results

Asymptotically Almost Automorphic Functions:

Definition 4.1. *[12] A function $\mathscr{G}\in\mathscr{C}(\mathbb{R},Z)$ is almost periodic if for every $\tau>0$ there exists a relatively dense subset of $\mathbb{R}$, denoted by $\mathfrak{I}(\tau,\mathscr{G},Z)$, such that*

$$\|\mathscr{G}(t+\mu)-\mathscr{G}(t)\|_Z<\tau,$$

$t\in\mathbb{R},\ \mu\in\mathfrak{I}(\tau,\mathscr{G},Z)$.

Definition 4.2. *[12] A function $\mathscr{G}\in\mathscr{C}([0,\infty),Z)$ is asymptotically almost periodic if there exists an almost periodic function $v_1(\cdot)$ and $v_2\in\mathscr{C}_0([0,\infty),Z)$ such that $\mathscr{G}(\cdot)=v_1(\cdot)+v_2(\cdot)$.*

Definition 4.3. *[12] A function $\mathscr{G}\in\mathscr{C}(\mathbb{R},X)$ is said to be almost automorphic $(\mathscr{A}\mathscr{A})$ if for every sequence of real numbers $(s_n')_{n\in\mathbb{N}}$, there exists a sub-sequence $(s_n)_{n\in\mathbb{N}}\subset(s_n')_{n\in\mathbb{N}}$ such that*

$$v_1(t):=\lim_{n\to\infty}\mathscr{G}(t+s_n)$$

is well defined for each $t\in\mathbb{R}$, and

$$\mathscr{G}(t)=\lim_{n\to\infty}v_1(t-s_n),\ \textit{for all}\ t\in\mathbb{R}.$$

It is associated that the range of an almost automorphic function is relatively compact on X, and therefore it is bounded. Furthermore, the space of all almost automorphic functions, expressed at $\mathscr{A}\mathscr{A}(X)$, provided by the norm of the uniform convergence is a Banach space.

Definition 4.4. *[2] A function $\mathscr{G} \in C([0,\infty),Z)$ is said to be asymptotically almost automorphic if it can be written as $\mathscr{G} = v_1 + v_2$ where $v_1 \in \mathcal{AA}(Z)$ and $v_2 \in \mathscr{C}_0([0,\infty),Z)$. Denote by $\mathscr{A}\mathscr{A}\mathscr{A}(Z)$ the set of all such functions.*

Compact Asymptotically Almost Automorphic:

Definition 4.5. *[2] A function $\mathscr{G} \in C(\mathbb{R},Z)$ is said to be compact almost automorphic if for every sequence of real numbers $(\sigma_n)_{n\in\mathbb{N}}$ there exists a sub sequence $(s_n)_{n\in\mathbb{N}} \subset (\sigma_n)_{n\in\mathbb{N}}$ such that*

$$v_1(t) := \lim_{n\to\infty} \mathscr{G}(t+s_n),$$
$$\mathscr{G}(t) = \lim_{n\to\infty} v_1(t-s_n)$$

uniformly on compact subsets of $\mathbb{R}$. It is denoted by $\mathcal{AA}_c(Z)$.

Lemma 4.6. *[7] If $x \in \mathscr{A}\mathscr{A}_c(X)$, then the function $s \to x_s$ belongs to $\mathscr{A}\mathscr{A}_c(\mathscr{B})$. Moreover, if $\mathscr{B}$ is a fading memory space and $x \in \mathscr{C}(\mathbb{R},X)$ is such that $x_0 \in \mathscr{B}$ and $x_{[0,\infty)} \in \mathscr{A}\mathscr{A}\mathscr{A}_c(X)$, then $t \mapsto x_t \in \mathscr{A}\mathscr{A}\mathscr{A}_c(\mathscr{B})$.*

To establish our existence result, motivated by the previous facts the following assumption is needed:

(H4) There exists a Banach space $(Y,\|\cdot\|_Y)$ continuously included in X such that the following conditions are verified.
(i) For every $t,s \in (0,\infty)$, $\mathscr{R}(t,s) \in \mathscr{L}(X) \cap \mathscr{L}(Y,[D(A(t))])$ and $\mathcal{K}(t,s) \in \mathscr{L}(Y,X)$. In addition, $A(t)\mathscr{R}(\cdot,\cdot)x,\ \ \mathcal{K}(\cdot,\cdot)x \in \mathscr{C}((0,\infty),X)$, for every $x \in Y$.

Lemma 4.7. *[7] Suppose that assumptions are holds and $h \in \mathscr{A}\mathscr{A}\mathscr{A}_c(X)$. If $\mathcal{F}$ is the function defined by*

$$\mathcal{F}(t) := \int_0^t \mathscr{R}(t,s)h(s)ds, \quad t \geq 0$$

then $\mathcal{F} \in \mathscr{A}\mathscr{A}\mathscr{A}_c(X)$.

The following frames are gives the existence result of periodicity:

Theorem 4.8. *If $\mathscr{Z}(\cdot) : [0,\infty) \to X$ is the function defined by*

$$\mathscr{Z}(t) = \int_0^t \mathscr{R}(t,s) \int_0^s \mathcal{K}(t,s)x(\tau)\, d\tau ds = \int_0^t \mathscr{R}(t,s)\mathscr{E}_2(s)ds$$

$t \geq 0$, then $\mathscr{Z}(t)$ is almost periodic function.

Proof. Using the assumptions,

$$\begin{aligned}
\mathscr{Z}(t) &= \int_0^t \mathscr{R}(t,s)[\mathcal{K}(t,\tau)x(\tau)]_{\mathscr{L}^1} ds \\
\|\mathscr{Z}(t+\xi) - \mathscr{Z}(t)\| &\leq \left\| \int_0^{t+\xi} \mathscr{R}(t+\xi,s)[\mathcal{K}(t+\xi,\tau)x(\tau)]_{\mathscr{L}^1} ds - \int_0^t \mathscr{R}(t,s)[\mathcal{K}(t,\tau)x(\tau)]_{\mathscr{L}^1} ds \right\| \\
&\leq \int_0^{t+\xi} \|\mathscr{R}(t+\xi,s)\| \|\mathcal{K}(t+\xi,\tau)x(\tau)\|_{\mathscr{L}^1} ds - \int_0^t \|\mathscr{R}(t,s)\| \|\mathcal{K}(t,\tau)x(\tau)\|_{\mathscr{L}^1} ds \\
&\leq M_1 \sup \|\mathcal{K}(t)\|_{\mathscr{L}^1} \|x(\tau)\|_{\mathscr{L}^1} (t+\xi) - M_1 \sup \|\mathcal{K}(t)\|_{\mathscr{L}^1} \|x(\tau)\|_{\mathscr{L}^1} (t) \\
&\leq M_1 \|x(\tau)\|_{\mathscr{L}^1} \xi\ k_0 \\
&\leq \varepsilon,
\end{aligned}$$

where $\varepsilon = M_1 \|x(\tau)\|_{\mathscr{L}^1} \xi\ k_0,\ \ k_0 = \sup \|\mathcal{K}(t)\|_{\mathscr{L}^1}$. Hence the function $\mathscr{Z}(t) = \int_0^t \mathscr{R}(t,s)\mathscr{E}_2(s)ds$ is almost periodic. This completes the proof. ☐

Theorem 4.9. *Suppose that the assumptions $(H1)$ and $(H4)$ are satisfied, then mild solution $x(\cdot)$ of the system (2.1) is asymptotically almost periodic function.*

Proof. Assume that the mild solution $x(t) = \mathscr{Z}(t) + \mathscr{Y}(t)$, where

$$\begin{aligned}
\mathscr{Z}(t) &= \int_0^t \mathscr{R}(t,s) \int_0^s \mathcal{K}(t,\tau)x(\tau)\, d\tau ds = \int_0^t \mathscr{R}(t,s)\mathscr{E}_2(s)ds \\
\mathscr{Y}(t) &= \mathscr{R}(t,0)[x_0 - g(x)] + \int_0^t \mathscr{R}(t,s)\mathscr{E}_1'(s)ds \\
&\quad + \int_0^t \mathscr{R}(t,s)\,[Bu(s) + h(s,x_s)]\,ds.
\end{aligned}$$

By the Theorem 4.8 the defined function $\mathscr{Z}(t)$ is almost periodic function and $\mathscr{Y}(t) \in \mathscr{C}_0([0,\infty),X)$. Hence by the Definition 4.2, the mild solution $x(t)$ is asymptotically almost periodic function. ☐

Theorem 4.10. *Assume the conditions $(H4)$ is satisfied. Further, suppose that $x \in \mathscr{A}\mathscr{A}\mathscr{A}_c(Y)$ and $\mathscr{Z}(\cdot) : [0,\infty) \to X$, be the function defined by*

$$\mathscr{Z}(t) = \int_0^t \mathscr{R}(t,s) \int_0^s \mathcal{K}(t,\tau)x(\tau)\, d\tau ds, \quad t \geq 0,$$

then $\mathscr{Z}(\cdot) \in \mathscr{A}\mathscr{A}\mathscr{A}_c(X)$.

Proof. Using the assumption $(H4)$ and $x \in \mathscr{A}\mathscr{A}\mathscr{A}_c(Y)$ we defined the function

$$\Psi(t) = \int_0^t \mathcal{K}(t,s)x(s)ds.$$

Since $\mathscr{Z}$ is a continuous function and x has a decomposition of ϕ_1 and ϕ_2, where $\phi_1 \in \mathcal{AA}_c(Y)$ and $\phi_2 \in \mathscr{C}_0(Y)$. Then

$$\begin{aligned}\Psi(t) &= \int_0^t \mathcal{K}(t,s)\phi_1(s)ds + \int_0^t \mathcal{K}(t,s)\phi_2(s)ds \\ &= \int_{-\infty}^t \mathcal{K}(t,s)\phi_1(s)ds - \int_{-\infty}^0 \mathcal{K}(t,s)\phi_1(s)ds + \int_0^t \mathcal{K}(t,s)\phi_2(s)ds.\end{aligned}$$

Let

$$\begin{aligned}\mathcal{P}_1(t) &= \int_{-\infty}^t \mathcal{K}(t,s)\phi_1(s)ds, \\ \mathcal{P}_2(t) &= \int_0^t \mathcal{K}(t,s)\phi_2(s)ds - \int_{-\infty}^0 \mathcal{K}(t,s)\phi_1(s)ds.\end{aligned}$$

By using the Lemma 4.7, it is sufficient to prove $\Psi(t) \in \mathscr{A}\mathscr{A}\mathscr{A}_c$. For a given sequence $(\alpha_n)_{n\in\mathbb{N}}$ of real numbers, fix a sub sequence $(r_n)_{n\in\mathbb{N}}$, and a continuous functions $y \in \mathscr{C}_b(\mathbb{R},Y)$ such that

$$\phi_1(t+r_n) = \lim_{t\to\infty} y(t)$$
$$\lim_{t\to\infty} y(t-r_n) = \phi_1(t)$$

uniformly on compact sets of $\mathbb{R}$. By the Bochner's criterion related to integrable functions and the measure

$$\left.\begin{aligned}\|\mathcal{K}(t,s)_{\mathscr{L}(Y,X)}\ \phi_1(s)\| &= \|\mathcal{K}(t,s)\|\|\phi_1(s)\|_Y \\ &\leq k(s)\|\phi_1(s)\|_Y\end{aligned}\right\} \tag{4.1}$$

consequently the function $s \to \mathcal{K}(t,s)\phi_1(s)$ is integrable over $(-\infty,t)$ for each $t \in \mathbb{R}$. Moreover,

$$\lambda(t+r_n) = \int_{-\infty}^t \mathcal{K}(t,s)\phi_1(s+r_n)ds, \quad t \in \mathbb{R},\ n \in \mathbb{N}$$

adopting the Lebesgue Dominated Convergence Theorem and using estimate (4.1), it follows that $\lambda(t+r_n)$ converges to $\mathcal{S}(t)$, where

$$\mathcal{S}(t) = \int_{-\infty}^t \mathcal{K}(t,s)y(s)ds, \text{ for each } t \in \mathbb{R}.$$

Hence $\mathcal{P}_1(\cdot) \in \mathscr{A}\mathscr{A}\mathscr{A}_c$. Next to show that the convergence is uniform on all compact subsets of $\mathbb{R}$ and that $\mathcal{P}_2(\cdot) \in \mathscr{C}_0([0,\infty),X)$.
Let $F \subset \mathbb{R}$ be an arbitrary compact and for each $t \in F$, there exist $\gamma > 0$. Since $\phi_1 \in \mathcal{AA}_c(Y)$ and $\phi_2 \in \mathscr{C}_0([0,\infty),Y)$, there exists a constants β and M such that $F \subset [\frac{-\beta}{2}, \frac{\beta}{2}]$ with

$$\int_{\frac{\beta}{2}}^\infty k(s)ds < \gamma$$

$$\begin{aligned}\|\phi_2(s)\|_Y &\leq \gamma, \quad s \geq \beta \\ \|\phi_1(s+r_n) - y(s)\|_Y &\leq \gamma, \quad n \geq M,\ s \in [-\beta,\beta].\end{aligned}$$

For all $t \in F$, then

$$\begin{aligned}\|\mathcal{P}_1(t+r_n) - \mathcal{S}(t)\| &\leq \int_{-\infty}^t \|\mathcal{K}(t,s)\|_{\mathscr{L}(Y,X)}\|\phi_1(s+r_n) - y(s)\|_Y ds \\ &\leq \int_{-\infty}^{-\beta} k(t,s)\|\phi_1(s+r_n) - y(s)\|_Y ds \\ &\quad + \int_{-\beta}^t k(t,s)\|\phi_1(s+r_n) - y(s)\|_Y ds.\end{aligned}$$

By the Kranosielski-krein lemma, for fixed γ in the half line $\mathbb{R}_+$, choose $t_0 > 0$, such that

$$\begin{aligned}2\int_{t_0}^\infty k(s)ds &< \frac{\gamma}{2}, \quad t_0 > 0, \text{ it follows that} \\ \sup_{s\in[0,t_0]} \|\mathcal{K}(s,\mu_1(s)) - \mathcal{K}(s,\mu_2(s))\| &< \frac{\gamma}{2t_0}, \quad t \in \mathbb{R}, \quad \mu_1,\mu_2 \in X.\end{aligned}$$

Since $F \subset \mathbb{R}$ be an arbitrary compact and $\gamma > 0$,

$$\begin{aligned}\sup_{s\in\mathbb{R}} \|\mathcal{K}(s)\|_w &= \|\mathcal{K}\|_{w,\infty} \\ \|\mathcal{P}_1(t+r_n) - \mathcal{S}(t)\| &\leq 2\|\phi_1\|_Y \int_{t+\beta}^\infty k(s)ds + \gamma\int_0^\infty k(s)ds \\ &\leq 2\|\phi_1\|_Y [\gamma\int_{\beta/2}^\infty k(s)ds + \gamma\int_0^\infty k(s)ds] \\ &= \gamma\Big[2\|\phi_1\|_{Y,\infty} + \int_0^\infty k(s)ds\Big],\end{aligned}$$

hence for given $\gamma > 0$, the above inequality is independent of t such that

$$\|\mathcal{P}_1(t+r_n) - \mathcal{S}(t)\| \leq \gamma,$$

for all $t \in F$. Which shows that the convergence is uniform on F. Proceeding the same procedure, one can achieve $\mathcal{S}(t-r_n)$ converges to $\mathcal{P}_1$ uniformly on all compact subsets of $\mathbb{R}$. Next, let us show that $\mathcal{P}_2(\cdot) \in C_0([0,\infty), X)$. For all $t \geq 2\beta$ we obtain,

$$\begin{aligned}
\|\mathcal{P}_2(t)\| \leq & \int_{-\infty}^{0} \|\mathcal{K}(t,s)\|_{\mathscr{L}(Y,X)} \|\phi_1(s)\|_Y ds + \int_0^t \|\mathcal{K}(t,s)\|_{\mathscr{L}(Y,X)} \|\phi_2(s)\|_Y ds \\
& \leq \int_{-\infty}^{0} k(t,s)\|\phi_1(s)\|_Y ds + \int_{t/2}^{t} k(t,s)\|\phi_2(s)\|_Y ds + \int_0^{t/2} k(t,s)\|\phi_2(s)\|_Y ds \\
& \leq \int_{\frac{\beta}{2}}^{\infty} k(t,s)ds \quad \|\phi_1\|_{Y,\infty} + \varepsilon \int_{t/2}^{t} k(t,s)ds + \int_{\frac{\beta}{2}}^{\infty} k(t,s)ds \quad \|\phi_2\|_{Y,\infty} \\
& \leq \varepsilon(\|\phi_2\|_{Y,\infty} + \int_0^{\infty} k(t,s)ds + \|\phi_2\|_{Y,\infty}).
\end{aligned}$$

This gives that $\mathcal{P}_2(\cdot) \in C_0([0,\infty), X)$. From the Lemma 4.6 and Lemma 4.7, the defined function $\mathscr{Z}(\cdot) \in \mathscr{A}\mathscr{A}\mathscr{A}_c(X)$. Hence the mild solution of the system (2.1) is compact asymptotically almost automorphic ($\mathscr{A}\mathscr{A}\mathscr{A}_c$). This completes the proof. □

The following theorem discuss the unique $\mathscr{A}\mathscr{A}\mathscr{A}_c$ solution for the system (2.1).

Theorem 4.11. *Assume that the assumptions $(H1)-(H4)$ are satisfied. Then there exists an unique asymptotically almost automorphic mild solution to (2.1), provided*

$$[M_1 + K_g + M_1 tN[1 + M_1 + K_g + M_1 bK_h] + tM_1 K_h] < 1.$$

Proof. Let $\mathcal{J} = \{x \in \mathscr{A}\mathscr{A}\mathscr{A}(\mathbb{R}^+, X)\}$. Then $\mathcal{J}$ is a closed subspace of $\mathscr{A}\mathscr{A}\mathscr{A}_c$. We define an operator Ω as,

$$\begin{aligned}
(\Omega x)(t) = & \mathscr{R}(t,0)\left[x_0 + g(x)\right] + \int_0^t \mathscr{R}(t,s)\left[\mathcal{E}_1'(s) + \mathcal{E}_2(s)\right] ds \\
& + \int_0^t \mathscr{R}(t,s)\left[Bu(s) + h(s,x_s)\right] ds, \quad t \in I.
\end{aligned}$$

First let us check $\Omega(\mathscr{A}\mathscr{A}\mathscr{A}_c(\mathbb{R},X)) \subset \mathscr{A}\mathscr{A}\mathscr{A}_c(\mathbb{R},X)$. Take $x \in \mathscr{A}\mathscr{A}\mathscr{A}_c(\mathbb{R}^+,X)$. Since Ωx is continuous. Using $(H4)$ and Lemma 4.10

$$\mathscr{Z}(t) = \int_0^t \mathscr{R}(t,s) \int_0^s \mathcal{K}(t,s)x(\tau)\, d\tau ds$$

is compact asymptotically almost automorphic function ($\mathscr{A}\mathscr{A}\mathscr{A}_c$).
Thus $\Omega x(t)$ is compact asymptotically almost automorphic on X. Set $\mathcal{B}_n = \{x \in \mathcal{J} : \|x\| < n\}$. Now,

$$\begin{aligned}
\|(\Omega x)(t)\| \leq & \|\mathscr{R}(t,0)\|\|x_0 + g(x)\| + \int_0^t \|\mathscr{R}(t,s)\|\|\mathcal{E}_1'(s) + \mathcal{E}_2(s)\|ds + \int_0^t \|\mathscr{R}(t,s)\|\|BW^{-1}\| \\
& \times\left\{\|x_b\| - \|\mathscr{R}(b,0)\left[x_0 + g(x)\right]\| - \int_0^b \|\mathscr{R}(b,s)\|\|\mathcal{E}_1'(s) + \mathcal{E}_2(s)\|ds \right.\\
& \left. + \int_0^b \|\mathscr{R}(b,s)\|\|h(s,x_s)\|ds\right\}(s)ds + \int_0^t \|\mathscr{R}(t,s)\|\|h(s,x_s)\|ds. \\
\leq & M_1\|x_0\| + \|g(x) - g(0)\| + \|g(0)\| + M_1 b\tilde{K}_2 + M_1 Nb\big[\|x_b \\
& - M_1\tilde{K}_2 + M_1 h_g(t)x_g(\eta)\big] + M_1 h_g(t)x_g(\eta) \\
\leq & M_1\|x_0\| + K_g(x) + M_1 b\widehat{K_2}\left[1 + N\right] + M_1 Nb\|x_b\| + M_1 h_g(t)x_g(\eta)\left[1 + Nb\right] < n.
\end{aligned}$$

Thus Ω maps $\mathcal{B}_n$ into itself. Next to show that Ω is a contraction on $\mathcal{B}_n$.
Finally, for $x, y \in \mathcal{B}_n$ we get,

$$\begin{aligned}
\|\Omega(x)(t) - \Omega(y)(t)\| \leq & \|\mathscr{R}(t,0)\|[\|x_0 - y_0\| + \|g(x) - g(y)\|] + \int_0^t \|\mathscr{R}(t,s)\|\|BW^{-1}\| \\
& \times\Big\{\|x_b - y_b\| - \|\mathscr{R}(b,0)\|[\|x_0 - y_0\| + \|g(x) - g(y)\|] \\
& + \int_0^b \|\mathscr{R}(b,s)\|\|h(s,x_s) - h(s,y_s)\|ds\Big\}(s)ds \\
& + \int_0^t \|\mathscr{R}(t,s)\|\|h(s,x_s) - h(s,y_s)\|ds \\
\leq & [M_1 + K_g + M_1 tN[1 + M_1 + K_g + M_1 bK_h] + tM_1 K_h]\|x - y\| \\
\leq & \Theta\|x - y\|,
\end{aligned}$$

where $\Theta = M_1 + K_g + M_1 tN[1 + M_1 + K_g + M_1 bK_h] + tM_1 K_h \geq 1$, which implies that the operator $\Omega(\cdot)$ is a contraction to our assumption, hence Ω has a unique fixed point in I which means there exist an compact asymptotically almost automorphic mild solution to the system (2.1). □

5. Application

Consider the partial neutral integrodifferential equation of the form

$$\left.\begin{array}{rcl}\frac{\partial}{\partial t}\left[z(t,y)-\int_{-r}^{t}a_1(t,s)z(s,y)ds\right] & = & \frac{\partial^2}{\partial y^2}z(t,y)+a_2(t,y)z(t,y)+\int_{-r}^{t}a_3(t,s)z(s,y)ds \\ & & +\mu(t,\varepsilon)+a_4(t,y)z_t(t,y)ds \\ z(t,0) & = & z(t,1)=0, \quad t\in I=[0,1], \\ z(0,y)+\sum_{i=1}^{n}e_i\phi(s_i,y) & = & z_0(\varepsilon), \quad 0\le\varepsilon\le\pi,\end{array}\right\} \tag{5.1}$$

where $a_k:\mathbb{R}\to\mathbb{R}, k=1,2,3,4$ are continuous and e_i is a small constant. Let us take $X=U=\mathscr{L}^2[0,\pi]$ endowed with the usual norm $|\cdot|_{\mathscr{L}^2}$. Put $x(t)=z(t,y)$ and $u(t)=\mu(t,y)$ where $\mu(t,y):I\times[0,\pi]\to[0,\pi]$ is continuous. Define the operators h,g by

$$\begin{aligned} h(t,\omega_t)(y) &= a_4(t,y)\ \omega(t,y) \\ g(\omega(t,y)) &= \sum_{i=1}^{n}e_i\ \ \phi(s_i,y). \end{aligned}$$

In particular, set $X=\mathbb{R}^+$, $I=[0,1]$.

$$\begin{aligned} h(t,x_t) &= a_4=\frac{e^{2t}}{t+9}\left(\frac{\sec x}{5+\cos t}\right), \\ g(x(t)) &= \sum_{i=1}^{n}e_i\phi=\frac{2e^{\sqrt{2t}}+4e^{\sqrt{t}}}{(4+e^t)}\cos x. \end{aligned}$$

Let $A(t):\mathscr{D}(A(t))\subseteq X\to X$ be the operator defined by $Az=z''$ with the domain $\mathscr{D}(A(t))=\{z\in X: z,z'$ are absolutely continuous, $z''\in X, z(0)=z(1)=0\}$.
Let $x,y\in X$ and $t=1$ we have,

$$\begin{aligned} \left\|h(t,x_t)-h(t,y_t)\right\| &= \left\|\frac{e^{2t}}{t+9}\left(\frac{\sec x}{5+\cos t}\right)-\frac{e^{2t}}{t+9}\left(\frac{\sec y}{5+\cos t}\right)\right\| \\ &= \frac{e^{2t}}{10}\left\|\frac{1}{0.540}(\sec x-\sec y)\right\| \\ &\le 1.36\|x-y\| \end{aligned}$$

and

$$\begin{aligned} \|g(x)-g(y)\| &= \left\|\frac{10e^{\sqrt{2t}}+4e^{\sqrt{t}}}{(4+e^{3t})}\cos x\right\|-\left\|\frac{10e^{\sqrt{2t}}+4e^{\sqrt{t}}}{(4+e^{3t})}\cos y\right\| \\ &\le \frac{2e^{\sqrt{2t}}+4e^{\sqrt{t}}}{(4+e^t)}\|\cos x-\cos y\| \\ &\le 2.76\|x-y\|. \end{aligned}$$

Hence, the condition $(H2)$ and $(H3)(i)$ are holds with $K_g=2.76$ and $K_h=1.366$. With this choice $A(t)$, $h(t,x_t)$ and $g(x(t))$ we see that the system (5.1) can be written in the abstract form of the system (2.1). Assume that the operator $W:\mathscr{L}^2(I,U)\to X$ defined by

$$Wu=\int_0^b\mathscr{R}(b,s)Bu(s)ds,$$

induces an invertible operator W^{-1} on $\mathscr{L}^2(I,U)/kerW$. Choose $b=1, M_1=1, N=-1$ in such a way that $M_1[1+K_g+tM_1NK_g]+tM_1NM_1(1+bK_h)<1$. Indeed

$$M_1[1+K_g+tM_1NK_g]+tM_1NM_1(1+bK_h) \quad = \quad -2.36<1,$$

furthermore, all the conditions stated in Theorem 3.1 is satisfied. Hence, the system (5.1) is controllable on I. Next the compact asymptotically almost automorphic ($\mathscr{A}\mathscr{A}\mathscr{A}_c$) case has been showed. Indeed

$$M_1+K_g+M_1tN[1+M_1+K_g+M_1bK_h]+tM_1K_h=-0.94<1.$$

Hence all the conditions stated in Theorem 4.11 is satisfied. Hence, the system (5.1) has unique compact asymptotically almost automorphic mild solution.

6. Conclusion

In this paper, the authors demonstrated the controllability results of an integrodifferential system in Banach space by using semi-group theory and Sadovskii fixed point theorem. Moreover, the existence of the compact asymptotically almost a ($\mathscr{A}\mathscr{A}\mathscr{A}_c$) solution has been performed. Further, the uniqueness result is proved for the same system by using the Banach contraction principle. Additionally, an example has been formed to illustrate the theory. Newly, the fractional-order integration and differentiation serve a fast-growing field both in theory and in applications to mathematical models. Based on this way, this study can be continued to obtain the solution by using fractional differential equations in the future.

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Conflict of interest:

There is no conflict of interest for all authors of this paper.

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