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CERTAIN COEFFICIENT INEQUALITIES FOR SOME SUBCLASSES OF ANALYTIC FUNCTIONS

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Abstract

The purpose of this present paper is to derive certain coefficient estimates for a normalized analytic function defined in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. A certain application of our main result for a class of functions defined by a Hadamard product is given. As a special case of our result, we obtain the Fekete-Szegö inequality for a class of functions defined through fractional derivatives.

Keywords: Analytic functions, Starlike functions, Convex functions, Subordination, Fekete-Szegö inequality, Fractional derivatives, Hadamard product.

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1. Introduction

Let $\mathcal{A}$ denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\Delta := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and S be the subclass of $\mathcal{A}$ consisting of univalent functions.

Let f and g be functions analytic in Δ. Then we say that the function f is subordinate to g if there exists a Schwarz function $w(z)$, analytic in Δ with

$$w(0) = 0 \text{ and } |w(z)| < 1, \ (z \in \Delta),$$

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such that

$$f(z) = g(\omega(z)),\ (z \in \Delta).$$

We denote this subordination by

$$f \prec g \text{ or } f(z) \prec g(z),\ (z \in \Delta).$$

In particular, if the function g is univalent in Δ, the above subordination is equivalent to

$$f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta).$$

Let $\phi(z)$ be an analytic function with $\phi(0) = 1$, $\phi'(0) > 0$ and

$$\Re\,(\phi(z)) > 0,\ (z \in \Delta),$$

which maps the open unit disk Δ onto a region starlike with respect to 1 and is symmetric with respect to the real axis. By $S^*(\phi)$ and $C(\phi)$, respectively, we denote the subclasses of the normalized analytic function class $\mathcal{A}$ which satisfy the following subordination relations:

$$\frac{zf'(z)}{f(z)} \prec \phi(z) \text{ and } 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z),\ (z \in \Delta).$$

These classes were introduced and studied by Ma and Minda [3]. In particular, if we set

$$\phi(z) = \frac{1 + (1 - 2\beta)z}{1 - z},\ (z \in \Delta;\ 0 \leqq \beta < 1)$$

we get the well-known classes $S^*(\beta)$ $(0 \leqq \beta < 1)$ of starlike functions of order β and the class $C(\beta)$ of convex functions of order α, respectively.

In [3], the Fekete-Szegö inequality for functions in the class $C(\phi)$ was obtained and in view of the Alexander result between the class $S^*(\phi)$ and $C(\phi)$, the Fekete-Szegö inequality for functions in $S^*(\phi)$ was also obtained. For a brief history of the Fekete-Szegö problem for the class of starlike, convex and various other subclasses of analytic functions, we refer the interested reader to [10].

Let $\alpha \geqq 0$, $\lambda \geqq 0$, $0 \leqq \rho < 1$ and $f \in \mathcal{A}$. We say that $f \in M(\alpha, \lambda, \rho)$ if it satisfies the condition

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)} - 1\right)\right]\right\} > \rho.$$

The class $M(\alpha, \lambda, \rho)$ was introduced very recently by Guo and Liu [2].

Motivated essentially by the aforementioned works, we prove the Fekete-Szegö inequality in Theorem 2.1 for a more general class of analytic functions which we define below in Definition 1.1. Also we give applications of our results to certain functions defined through the Hadamard product and in particular we consider a class defined by fractional derivatives. The results obtained in this paper generalize the results given in [3] and [9].

Now, we define the following class $M_{\alpha,\,\lambda}(\phi)$ of functions which unifies the classes $S^*(\phi)$ and $C(\phi)$:

1.1. Definition. Let $\phi(z)$ be a univalent starlike function with respect to 1 which maps the open unit disk Δ onto a region in the right half plane and is symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f \in \mathcal{A}$ is in the class $M_{\alpha,\,\lambda}(\phi)$ if

$$\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)} - 1\right)\right]\right\} \prec \phi(z),$$

$$(\alpha \geqq 0,\ \lambda \geqq 0).$$

Note that $M_{0,0}(\phi) \equiv S^*(\phi)$ and $M_{0,1}(\phi) \equiv C(\phi)$ given in [3].

To prove our main result, we need the following:

1.2. Lemma. [3] *If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part in Δ, then*

$$|c_2 - v c_1^2| \leqq \begin{cases} -4v+2, & \text{if } v \leqq 0, \\ 2, & \text{if } 0 \leqq v \leqq 1, \\ 4v-2, & \text{if } v \geqq 1. \end{cases}$$

When $v < 0$ or $v > 1$, equality holds if and only if $p_1(z)$ is $\dfrac{1+z}{1-z}$, or one of its rotations. If $0 < v < 1$, then equality holds if and only if $p_1(z)$ is $\dfrac{1+z^2}{1-z^2}$ or one of its rotations. If $v = 0$, equality holds if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\eta\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\eta\right)\frac{1-z}{1+z}, \ (0 \leqq \eta \leqq 1)$$

or one of its rotations. If $v = 1$, equality holds if and only if p_1 is the reciprocal of one of the functions such that equality holds in the case of $v = 0$.

Although the above upper bound is sharp, when $0 < v < 1$, it can be improved as follows:

$$|c_2 - v c_1^2| + v|c_1|^2 \leqq 2, \ (0 < v \leqq 1/2)$$

and

$$|c_2 - v c_1^2| + (1-v)|c_1|^2 \leqq 2, \ (1/2 < v \leqq 1).$$

We also need the following:

1.3. Lemma. [7] *If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part in Δ, then*

$$|c_2 - v c_1^2| \leqq 2 \max(1, |2v - 1|).$$

The result is sharp for the functions

$$p(z) = \frac{1+z^2}{1-z^2}, \ p(z) = \frac{1+z}{1-z}.$$

2. A coefficient estimate

By making use of Lemma 1.2, we prove the following:

2.1. Theorem. *Let $0 \leqq \mu \leqq 1$, $\alpha \geqq 0$ and $\lambda \geqq 0$. Further, let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, where the B_n's are real with $B_1 > 0$, $B_2 \geqq 0$. If $f(z)$ given by (1.1) belongs to $M_{\alpha,\lambda}(\phi)$, then*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \dfrac{1}{2\xi}\left(2B_2 - \dfrac{B_1^2\gamma}{2\tau^2}\right), & \text{if } \mu \leqq \sigma_1, \\ \dfrac{B_1}{\xi}, & \text{if } \sigma_1 \leqq \mu \leqq \sigma_2, \\ \dfrac{1}{2\xi}\left(-2B_2 + \dfrac{B_1^2\gamma}{2\tau^2}\right), & \text{if } \mu \geqq \sigma_2, \end{cases}$$

where, for convenience,

$$\sigma_1 := \frac{2\tau^2(B_2 - B_1) + ((\alpha+3)\lambda - \rho)B_1^2}{2\xi B_1^2},$$

$$\sigma_2 := \frac{2\tau^2(B_2 + B_1) + ((\alpha+3)\lambda - \rho)B_1^2}{2\xi B_1^2},$$

$$\sigma_3 := \frac{2\tau^2 B_2 - (\rho^2 - (\alpha+3)\lambda)B_1^2}{2\xi B_1^2},$$

$$\gamma := \rho - (\alpha+3)\lambda + 2\mu\xi, \tag{2.1}$$

$$\rho := \alpha^2 + \alpha - 2, \tag{2.2}$$

$$\xi := (\alpha+2)(1+2\lambda), \text{ and} \tag{2.3}$$

$$\tau := (1+\alpha)(1+\lambda). \tag{2.4}$$

Further, if $\sigma_1 \leqq \mu \leqq \sigma_3$*, then*

$$|a_3 - \mu a_2^2| + \frac{\tau^2}{\xi B_1}\left(1 - \frac{B_2}{B_1} + \frac{\gamma B_1}{2\tau^2}\right)|a_2|^2 \leqq \frac{B_1}{\xi}.$$

If $\sigma_3 \leqq \mu \leqq \sigma_2$*, then*

$$|a_3 - \mu a_2^2| + \frac{\tau^2}{\xi B_1}\left(1 + \frac{B_2}{B_1} - \frac{\gamma B_1}{2\tau^2}\right)|a_2|^2 \leqq \frac{B_1}{\xi}.$$

These results are sharp.

Proof. If $f \in M_{\alpha,\lambda}(\phi)$, then there is a Schwarz function $w(z)$, analytic in Δ, with $w(0) = 0$ and $|w(z)| < 1$ in Δ such that

$$\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^\alpha + \lambda\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)} - 1\right)\right]\right\} = \phi(w(z)). \tag{2.5}$$

Define the function $p_1(z)$ by

$$p_1(z) := \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots. \tag{2.6}$$

Since $w(z)$ is a Schwarz function, we see that $\Re(p_1(z)) > 0$ and $p_1(0) = 1$. Let us define the function $p(z)$ by

$$\begin{aligned} p(z) &:= \left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^\alpha + \lambda\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)} - 1\right)\right]\right\} \\ &= 1 + b_1 z + b_2 z^2 + \cdots. \end{aligned} \tag{2.7}$$

In view of the equations (2.5), (2.6), (2.7), we have

$$p(z) = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right). \tag{2.8}$$

Using (2.6) in (2.8), we get,

$$b_1 = \frac{1}{2}B_1 c_1 \text{ and } b_2 = \frac{1}{2}B_1\left(c_2 - \frac{1}{2}c_1^2\right) + \frac{1}{4}B_2 c_1^2.$$

A computation shows that

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (2a_3 - a_2^2)z^2 + (3a_4 + a_2^3 - 3a_3 a_2)z^3 + \cdots.$$

Similarly we have

$$1 + \frac{zf''(z)}{f'(z)} = 1 + 2a_2 z + (6a_3 - 4a_2^2)z^2 + \cdots.$$

An easy computation shows that

$$\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha}+\lambda\left[1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}+\alpha\left(\frac{zf'(z)}{f(z)}-1\right)\right]\right\}$$
$$=1+(1+\alpha)(1+\lambda)a_2z+(\alpha+2)(1+2\lambda)a_3z^2$$
$$+\left(\frac{\alpha^2+\alpha}{2}-(\alpha+3)\lambda-1\right)a_2^2z^2+\cdots.$$

In view of equation (2.7), we see that

$$b_1=(1+\alpha)(1+\lambda)a_2$$
$$b_2=(\alpha+2)(1+2\lambda)a_3+\left(\frac{\alpha^2+\alpha}{2}-(\alpha+3)\lambda-1\right)a_2^2$$

or equivalently, we have

$$a_2=\frac{B_1c_1}{2(1+\alpha)(1+\lambda)},$$
$$a_3=\frac{B_1}{2\left((\alpha+2)(1+2\lambda)\right)}\left(c_2-\frac{1}{2}\left(1-\frac{B_2}{B_1}+B_1\Lambda_0\right)c_1^2\right),$$

where

$$\Lambda_0=\left((\alpha+3)\lambda+1-\frac{\alpha^2+\alpha}{2}\right)\frac{1}{4\left((1+\alpha)(1+\lambda)\right)^2}.$$

Therefore, we have

$$a_3-\mu a_2^2=\frac{B_1}{2\left((\alpha+2)(1+2\lambda)\right)}\left(c_2-vc_1^2\right)$$

where

$$v:=\frac{1}{2}\left(1-\frac{B_2}{B_1}+\frac{\alpha^2+\alpha-2+2\mu(\alpha+2)(1+2\lambda)-(\alpha+3)\lambda}{2\left((1+\alpha)(1+\lambda)\right)^2}B_1\right).$$

The assertion of Theorem 2.1 now follows by an application of Lemma 1.2.

To show that the bounds are sharp, we define the functions K_{ϕ_n}, $(n=2,3,\ldots)$ with $K_{\phi_n}(0)=0=[K_{\phi_n}]'(0)-1$, by

$$\frac{z(K_{\phi_n})'(z)}{K_{\phi_n}(z)}\left(\frac{K_{\phi_n}(z)}{z}\right)^{\alpha}$$
$$+\lambda\left[1+\frac{z(K_{\phi_n})''(z)}{(K_{\phi_n})'(z)}-\frac{z(K_{\phi_n})'(z)}{K_{\phi_n}(z)}+\alpha\left(\frac{z(K_{\phi_n})'(z)}{K_{\phi_n}(z)}-1\right)\right]=\phi(z^{n-1}),$$

and the functions F_η and G_η $(0\leqq\eta\leqq1)$, respectively, with $F_\eta(0)=0=F_\eta'(0)-1$ and $G_\eta(0)=0=G_\eta'(0)-1$ by

$$\frac{z(F_\eta)'(z)}{F_\eta(z)}\left(\frac{F_\eta(z)}{z}\right)^{\alpha}$$
$$+\lambda\left[1+\frac{z(F_\eta)''(z)}{(F_\eta)'(z)}-\frac{z(F_\eta)'(z)}{F_\eta(z)}+\alpha\left(\frac{z(F_\eta)'(z)}{F_\eta(z)}-1\right)\right]=\phi\left(\frac{z(z+\eta)}{1+\eta z}\right),$$

and

$$\frac{z(G_\eta)'(z)}{G_\eta(z)}\left(\frac{G_\eta(z)}{z}\right)^{\alpha}$$
$$+\lambda\left[1+\frac{z(G_\eta)''(z)}{(G_\eta)'(z)}-\frac{z(G_\eta)'(z)}{G_\eta(z)}+\alpha\left(\frac{z(G_\eta)'(z)}{G_\eta(z)}-1\right)\right]=\phi\left(-\frac{z(z+\eta)}{1+\eta z}\right),$$

respectively. Clearly the functions $K_{\phi_n}, F_\eta, G_\eta\in M_{\alpha,\lambda}(\phi)$. Also we write $K_\phi:=K_{\phi_2}$.

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then equality holds if and only if f is K_ϕ or one of its rotations.
When $\sigma_1 < \mu < \sigma_2$, then equality holds if and only if f is K_{ϕ_3} or one of its rotations.
If $\mu = \sigma_1$ then equality holds if and only if f is F_η or one of its rotations.
If $\mu = \sigma_2$ then equality holds if and only if f is G_η or one of its rotations. □

By making use of Lemma 1.3, we immediately obtain the following:

2.2. Theorem. *Let* $0 \leqq \alpha \leqq 1,\ 0 < \beta \leqq 1,$ *and* $0 \leqq \lambda \leqq 1$. *Further, let*

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots,$$

where the B_n*'s are real with* $B_1 > 0$ *and* $B_2 \geqq 0$. *If* $f \in M_{\alpha,\beta,\lambda}(\phi)$, *then for complex* μ, *we have*

$$|a_3 - \mu a_2^2| = \frac{B_1}{\xi} \max\left\{1,\ \left|-\frac{B_2}{B_1} + \frac{\gamma}{2\tau^2} B_1\right|\right\},$$

where γ *is as defined in* (2.1). *The result is sharp.*

2.3. Remark. The coefficient bounds for $|a_2|$ and $|a_3|$ are special cases of our Theorem 2.1.

2.4. Remark. For the choice $\lambda = 0$, Theorem 2.1 reduces to the result obtained in [7].

For the choices $\alpha = 0$ and $\lambda = 0$, Theorem 2.1 reduces to the following result for the class $S^*(\phi)$.

2.5. Corollary. *If* f *given by* (1.1) *belongs to* $S^*(\phi)$, *then*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \dfrac{B_2}{2} - \mu B_1^2 + \dfrac{1}{2} B_1^2 & \text{if } \mu \leqq \sigma_1 \\ \dfrac{B_1}{2} & \text{if } \sigma_1 \leqq \mu \leqq \sigma_2 \\ -\dfrac{B_2}{2} + \mu B_1^2 - \dfrac{1}{2} B_1^2 & \text{if } \mu \geq \sigma_2 \end{cases}$$

where,

$$\sigma_1 := \frac{(B_2 - B_1) + B_1^2}{2B_1^2}$$

$$\sigma_2 := \frac{(B_2 + B_1) + B_1^2}{2B_1^2}.$$

The result is sharp.

For the choices $\alpha = 0$ and $\lambda = 1$, Theorem 2.1 coincides with the following result obtained for the class $C(\phi)$ by Ma and Minda [3].

2.6. Corollary. (Ma and Minda [3]) *Let* $0 \leqq \mu \leqq 1$. *Further, let*

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots,$$

where the B_n*'s are real with* $B_1 > 0$, $B_2 \geq 0$ *and*

$$\sigma_1 := \frac{2(B_2 - B_1 + B_1^2)}{3B_1^2},$$

$$\sigma_2 := \frac{2(B_2 - B_1 + B_1^2)}{3B_1^2}.$$

$$\sigma_3 := \frac{2(B_2 + B_1^2)}{3B_1^2}.$$

If $f(z)$ given by (1.1) belongs to $C(\phi)$, then

$$|a_3-\mu a_2^2| \leqq \begin{cases} \frac{1}{6}\left(B_2-\frac{3}{2}\mu B_1^2+B_1^2\right), & \text{if } \mu \leqq \sigma_1, \\ \frac{1}{6}B_1, & \text{if } \sigma_1 \leqq \mu \leqq \sigma_2, \\ -\frac{1}{6}\left(B_2-\frac{3}{2}\mu B_1^2+B_1^2\right), & \text{if } \mu \geq \sigma_2. \end{cases}$$

Further, if $\sigma_1 \leqq \mu \leqq \sigma_3$, then

$$|a_3-\mu a_2^2| + \frac{2}{3}B_1^2\left(\frac{3}{2}\mu B_1^2+B_1-B_2-B_1^2\right)|a_2|^2 \leqq \frac{B_1}{6}.$$

If $\sigma_3 \leqq \mu \leqq \sigma_2$, then

$$|a_3-\mu a_2^2| + \frac{2}{3}B_1^2\left(-\frac{3}{2}\mu B_1^2+B_1+B_2+B_1^2\right)|a_2|^2 \leqq \frac{B_1}{6}.$$

These results are sharp.

For the choice of $\alpha = 0$, Theorem 2.1 at once reduces to the following result.

2.7. Corollary. *Let $0 \leqq \mu \leqq 1$, and $0 \leqq \lambda \leqq 1$. Further, let*

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots,$$

where the B_n's are real with $B_1 > 0$, $B_2 \geqq 0$ and

$$\sigma_1 := \frac{2(1+\lambda)^2(B_2-B_1) - [(1+\lambda)^2 - 3(1+3\lambda)]B_1^2}{4(1+2\lambda)B_1^2},$$

$$\sigma_2 := \frac{2(1+\lambda)^2(B_2+B_1) - [(1+\lambda)^2 - 3(1+3\lambda)]B_1^2}{4(1+2\lambda)B_1^2},$$

$$\sigma_3 := \frac{2(1+\lambda)^2 B_2 - [(1+\lambda)^2 - 3(1+3\lambda)]B_1^2}{4(1+2\lambda)B_1^2}.$$

If $f(z)$ given by (1.1) belongs to $M_{0,\lambda}(\phi)$, then

$$|a_3-\mu a_2^2| \leqq \begin{cases} \frac{1}{4(1+2\lambda)}\left[2B_2 - \frac{B_1^2}{(1+\lambda)^2}\gamma_2\right], & \text{if } \mu \leqq \sigma_1, \\ \frac{1}{2(1+2\lambda)}B_1, & \text{if } \sigma_1 \leqq \mu \leqq \sigma_2, \\ \frac{1}{4(1+2\lambda)}\left[-2B_2 + \frac{B_1^2}{(1+\lambda)^2}\gamma_2\right], & \text{if } \mu \geqq \sigma_2, \end{cases}$$

where, for convenience,

$$\gamma_2 := (1+\lambda)^2 - 3(1+3\lambda) + 4\mu(1+2\lambda).$$

Further, if $\sigma_1 \leqq \mu \leqq \sigma_3$, then

$$|a_3-\mu a_2^2| + \frac{(1+\lambda)^2}{2(1+2\lambda)B_1}\left[1 - \frac{B_2}{B_1} + \frac{\gamma_2 B_1}{2(1+\lambda)^2}\right]|a_2|^2 \leqq \frac{B_1}{2(1+2\lambda)}.$$

If $\sigma_3 \leqq \mu \leqq \sigma_2$, then

$$|a_3-\mu a_2^2| + \frac{(1+\lambda)^2}{2(1+2\lambda)B_1}\left[1 + \frac{B_2}{B_1} - \frac{\gamma_2 B_1}{2(1+\lambda)^2}\right]|a_2|^2 \leqq \frac{B_1}{2(1+2\lambda)}.$$

These results are sharp.

2.8. Remark. For the choices $\lambda = 1$, $\alpha = 0$ and $\beta = 1$, Theorem 2.1 reduces to a known result of Ma and Minda [3].

3. Application to functions defined by convolution

For an application of the results given in the previous section, we define the class $M^{\delta}_{\alpha,\,\beta,\,\lambda}(\phi)$. This will require the following concept:

3.1. Definition. (see [5, 6]; see also [11, 12]) Let f be analytic in a simply connected region of the z-plane containing the origin. The *fractional derivative* of f of order δ is defined by

$$D^{\delta}_{z}f(z) := \frac{1}{\Gamma(1-\delta)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{\delta}}d\zeta,\ (0 \leqq \delta < 1),$$

where the multiplicity of $(z-\zeta)^{\delta}$ is removed by requiring that $\log(z-\zeta)$ is real for $z-\zeta > 0$.

Using the above Definition 3.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [5] introduced the operator $\Omega^{\delta} : \mathcal{A} \to \mathcal{A}$ defined by

$$(\Omega^{\delta} f)(z) = \Gamma(2-\delta)z^{\delta}D^{\delta}_{z}f(z),\ (\delta \neq 2, 3, 4, \cdots).$$

We define the class $M^{\delta}_{\alpha,\,\lambda}(\phi)$ in the following way:

$$M^{\delta}_{\alpha,\,\lambda}(\phi) := \{f \in \mathcal{A} \text{ and } \Omega^{\delta} f \in M_{\alpha,\,\lambda}(\phi)\}$$

where $M_{\alpha,\,\lambda}(\phi)$ is given by Defintion 1.1. Note that $M^{\delta}_{\alpha,\,\lambda}(\phi)$ is the special case of the class $M^{g}_{\alpha,\,\lambda}(\phi)$ when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)} z^n.$$

Let

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n,\ (g_n > 0).$$

Since

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in M^{g}_{\alpha,\,\lambda}(\phi) \iff (f*g) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in M_{\alpha,\beta,\lambda}(\phi),$$

we obtain the coefficient estimate for functions in the class $M^{g}_{\alpha,\,\lambda}(\phi)$, from the corresponding estimate for functions in the class $M_{\alpha,\,\lambda}(\phi)$. Applying Theorem 2.1 for the function $(f*g)(z) = z + g_2a_2z^2 + g_3a_3z^3 + \cdots$, we get the following Theorem 3.2 after an obvious change of the parameter μ.

3.2. Theorem. *Let $0 \leqq \mu \leqq 1$, $\alpha \geqq 0$ and $\lambda \geqq 0$. Further, let*

$$\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots,$$

where the B_n's are real with $B_1 > 0$, $B_2 \geqq 0$, $g_n > 0$. If $f(z)$ given by (1.1) *belongs to $M^{g}_{\alpha,\,\lambda}(\phi)$, then*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \dfrac{1}{2g_3\xi}\left(2B_2 - \dfrac{B_1^2\gamma}{2\tau^2}\right), & \text{if } \mu \leqq \sigma_1, \\ \dfrac{B_1}{g_3\xi}, & \text{if } \sigma_1 \leqq \mu \leqq \sigma_2, \\ [1.8ex]\dfrac{1}{2g_3\xi}\left(-2B_2 + \dfrac{B_1^2\gamma}{2\tau^2}\right), & \text{if } \mu \geqq \sigma_2, \end{cases}$$

where, for convenience,

$$\sigma_1 := \frac{g_3}{g_2^2}\frac{2\tau^2(B_2-B_1)+((\alpha+3)\lambda-\rho)B_1^2}{2\xi B_1^2},$$
$$\sigma_2 := \frac{g_3}{g_2^2}\frac{2\tau^2(B_2+B_1)+((\alpha+3)\lambda-\rho)B_1^2}{2\xi B_1^2},$$
$$\gamma_2 := \rho-(\alpha+3)\lambda+2\mu\frac{g_3}{g_2^2}\xi,$$

and ρ, ξ, τ *are as defined in* (2.2), (2.3) *and* (2.4). *These results are sharp.*

Since, $(\Omega^\delta f)(z) = z+\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}a_n z^n$, we have

$$g_2 := \frac{\Gamma(3)\Gamma(2-\delta)}{\Gamma(3-\delta)} = \frac{2}{2-\delta} \tag{3.1}$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(2-\delta)}{\Gamma(4-\delta)} = \frac{6}{(2-\delta)(3-\delta)}. \tag{3.2}$$

For g_2 and g_3 given by (3.1) and (3.2), Theorem 3.2 reduces to the following.

3.3. Theorem. *Let* $0 \leqq \mu \leqq 1$, $\alpha \geqq 0$ *and* $\lambda \geqq 0$. *Further, let*

$$\phi(z) = 1+B_1 z+B_2 z^2+B_3 z^3+\cdots,$$

where the B_n*'s are real with* $B_1>0$, $B_2 \geqq 0$ *and*$g_n>0$. *If* $f(z)$ *given by* (1.1) *belongs to* $M^g_{\alpha,\lambda}(\phi)$, *then*

$$|a_3-\mu a_2^2| \leqq \begin{cases} \dfrac{(2-\delta)(3-\delta)}{12\xi}\left(2B_2-\dfrac{B_1^2\gamma}{2\tau^2}\right), & \text{if } \mu \leqq \sigma_1, \\ \dfrac{(2-\delta)(3-\delta)B_1}{6\xi}, & \text{if } \sigma_1 \leqq \mu \leqq \sigma_2, \\ \dfrac{(2-\delta)(3-\delta)}{12\xi}\left(-2B_2+\dfrac{B_1^2\gamma}{2\tau^2}\right), & \text{if } \mu \geqq \sigma_2, \end{cases}$$

where, for convenience,

$$\sigma_1 := \frac{2(3-\delta)}{3(2-\delta)}\frac{2\tau^2(B_2-B_1)+((\alpha+3)\lambda-\rho)B_1^2}{2\xi B_1^2},$$
$$\sigma_2 := \frac{2(3-\delta)}{3(2-\delta)}\frac{2\tau^2(B_2+B_1)+((\alpha+3)\lambda-\rho)B_1^2}{2\xi B_1^2},$$
$$\gamma_3 := \rho-(\alpha+3)\lambda+2\mu\frac{2(3-\delta)}{3(2-\delta)}\xi,$$

and ρ, ξ, τ *are as defined in* (2.2), (2.3) *and* (2.4). *These results are sharp.*

For the choices $\alpha=0$, $\lambda=0$, $B_1=\frac{8}{\pi^2}$ and $B_2=\frac{16}{3\pi^2}$, Theorem 3.3 coincides with the following result obtained in [9] for which $\Omega^\lambda f(z)$ is a parabolic starlike function [1, 8].

3.4. Theorem. [9] *Let* $0 \leqq \mu \leqq 1$. *Let*

$$\sigma_1 := \frac{(3-\delta)}{(2-\delta)}\left(\frac{1}{3}+\frac{5\pi^2}{72}\right) \text{ and } \sigma_2 := \frac{(3-\delta)}{(2-\delta)}\left(\frac{1}{3}-\frac{\pi^2}{72}\right).$$

If $f(z)$ given by (1.1) belongs to $M^{\delta}_{0,0}(\phi)$, then

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \dfrac{4}{3\pi^2}(3-\delta)(2-\delta)\left(\dfrac{12(2-\delta)\mu}{(3-\delta)\pi^2} - \dfrac{4}{\pi^2} - \dfrac{1}{3}\right), & \text{if } \mu \leqq \sigma_1, \\ \dfrac{2}{3\pi^2}(3-\delta)(2-\delta), & \text{if } \sigma_1 \leqq \mu \leqq \sigma_2, \\ \dfrac{4}{3\pi^2}(3-\delta)(2-\delta)\left(\dfrac{1}{3} + \dfrac{4}{\pi^2} - \dfrac{12(2-\delta)\mu}{(3-\delta)\pi^2}\right), & \text{if } \mu \geqq \sigma_2. \end{cases}$$

These results are sharp.

3.5. Remark. For the choices $\alpha = 0$, $\lambda = 0$, $\delta = 1$, $B_1 = \dfrac{8}{\pi^2}$ and $B_2 = \dfrac{16}{3\pi^2}$, Theorem 3.3 coincides with the result obtained in [4].

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