

DI-EXTREMITIES ON TEXTURES[‡]

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Abstract

In this paper, the authors adapt the notion of proximity to the point free setting of textures and investigate the relationship with dimetrics and di-uniformities.

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1. Introduction

In a topological space X , the topology can be determined by the Kuratowski closure axioms. When x is a closure point of A , we may say that “ x is near A ” and then a continuous function $f : X \rightarrow Y$ may be described as one exhibiting the property “if x is near A , then $f(x)$ is near $f(A)$ ”.

For the case in which X is a pseudometric space with pseudometric d , this nearness can be defined in a natural way. Let $D(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}$ and define A to be near B if and only if $D(A, B) = 0$. Then the closure of A is $\text{cl}(A) = \{x \mid D(A, \{x\}) = 0\}$. Let (Y, e) be another pseudometric space, E be defined in a similar manner to D , and f a function from X to Y . Then f is uniformly continuous if and only if $D(A, B) = 0$ implies $E(f(A), f(B)) = 0$. Thus we see that this nearness relation between the subsets is closely connected with topology, continuity and uniformity. In 1951, by investigating the properties of this nearness relation, Efremovic generalized this notion to an arbitrary set X and introduced the proximity spaces. Later, several authors continued to study proximity spaces, quasi-proximity spaces and other generalized proximity spaces.

Let us recall the definition of quasi-proximity [11]. Let X be a set. A binary relation η on $\mathcal{P}(X)$ satisfying the following conditions

- (1) $A\eta B$ implies $A \neq \emptyset, B \neq \emptyset$,
- (2) $(A \cup B)\eta C$ iff $A\eta C$ or $B\eta C$,

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- (3) $A\eta(B \cup C)$ iff $A\eta B$ or $A\eta C$,
- (4) if $A \not\eta B$, there exists $E \subseteq X$ such that $A \not\eta E$ and $(X \setminus E) \not\eta B$,
- (5) $A \cap B \neq \emptyset$ implies $A\eta B$.

is called a *quasi-proximity* on X . If η satisfies the symmetry condition $A\eta B$ iff $B\eta A$, then it is called a *proximity* on X . The (P4) axiom is called the *strong axiom*, and it plays an important role in the theory of proximity spaces.

A quasi-proximity induces a topology on X via the closure operator

$$A \mapsto \text{cl}(A) = \{x \mid A\eta\{x\}\}.$$

This topology will be denoted by $\tau(\eta)$. If η is a proximity, then the induced topology $\tau(\eta)$ is completely regular and thus the space $(X, \tau(\eta))$ is uniformizable.

Ditopological spaces [1, 2] were introduced by L. M. Brown as a natural extension of the work of the second author on the representation of lattice valued topologies by bitopologies [9]. However, in place of the full lattice of subsets of some base set S , attention is focused on a suitable subfamily of subsets, called a *texturing* of S . In this setting bitopologies are replaced by dichotomous topologies, or ditopologies for short. Fuzzy sets can be represented as textures [4], and the concept of textures provides a complement-free framework for generalizing topology related structures such as uniformities and metrics [12]. Some results on ditopological texture spaces may be found in [3, 5, 6, 7].

Proximities and quasi-proximities constitute an important and intensely investigated area in the field of classical and fuzzy topological spaces, because they possess rich topological properties as well as characterizing totally bounded uniformity spaces. For this reason, the authors believe that adapting the notion of proximity to a textural setting and investigating its relation with dimetrics and uniformities will give important results in the theory of textures. However, textures are generally not closed under the operation of taking the complement of a set, so the usual proximity definition is not suitable for textures since the definition requires the complement. Although this problem can be avoided by considering generalized proximities having a weaker form of the strong axiom, such as in [10, 14]), in this paper we prefer to base our work on the classical definition of quasi-proximity. Hence we begin by giving an alternative description of a classical quasi-proximity by considering not the nearness of two sets A and B , but the nearness of A and $X \setminus B$. The axioms of the resulting structure, called here an *extremity* because if A is near to $X \setminus B$ it is on the “extremity” of B , do not involve the set complement explicitly and so can be generalized to the textural setting.

2. Preliminaries

Let S be a set. We recall [3] that a *texturing* on S is a point separating, complete, completely distributive lattice \mathcal{S} of subsets of S with respect to inclusion, which contains S, \emptyset , and for which meet \wedge coincides with intersection \cap and finite joins \vee with unions \cup . The pair (S, \mathcal{S}) is then known as a texture.

In a texture, arbitrary joins need not coincide with unions, and clearly this will be so if and only if \mathcal{S} is closed under arbitrary unions. In this case (S, \mathcal{S}) is said to be *plain*. In general, a texturing of S need not be closed under taking the set complement, but it may be that there exists a mapping $\sigma : \mathcal{S} \rightarrow \mathcal{S}$ satisfying $\sigma(\sigma(A)) = A, \forall A \in \mathcal{S}$ and $A \subseteq B \implies \sigma(B) \subseteq \sigma(A), \forall A, B \in \mathcal{S}$. In this case σ is called a *complementation* on (S, \mathcal{S}) and (S, \mathcal{S}, σ) is said to be a *complemented texture*.

For $s \in S$ the set P_s is defined by $P_s = \bigcap \{A \in \mathcal{S} \mid s \in A\}$, and is therefore the smallest element of \mathcal{S} containing s . The sets P_s play an important role in the study of textures, along with the sets $Q_s = \bigvee \{A \in \mathcal{S} \mid s \notin A\} = \bigvee \{P_u \mid u \in S, s \notin P_u\}$. The sets

P_s, Q_s are known as the p -sets and q -sets of (S, \mathcal{S}) , respectively. Together they enable the formulation of a powerful concept of duality.

2.1. Examples.

- (1) For any set X , $(X, \mathcal{P}(X), \pi_X), \pi_X(Y) = X \setminus Y$ for $Y \subseteq X$, is the complemented discrete texture representing the usual set structure of X . Clearly, $P_x = \{x\}, Q_x = X \setminus \{x\}$ for all $x \in X$. Hence, $(X, \mathcal{P}(X), \pi_X)$ is plain.
- (2) $S = \{a, b, c\}$ and $\mathcal{S} = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, S\}$. Clearly $P_a = \{a, b\}, P_b = \{b\}, P_c = \{b, c\}, Q_a = \{b, c\}, Q_b = \emptyset$, and $Q_c = \{a, b\}$. Thus (S, \mathcal{S}) is a plain texture. No complementation can be defined on this texture.

Let us recall that the core of a set A in \mathcal{S} is the set

$$A^b = \bigcap \{ \bigcup \{A_j \mid j \in J\} \mid \{A_j \mid j \in J\} \subseteq \mathcal{S}, A = \bigvee \{A_j \mid j \in J\} \}.$$

2.2. Theorem. [5] *If (S, \mathcal{S}) is a texture, then the following statements hold:*

- (1) $s \notin A \implies A \subseteq Q_s \implies s \notin A^b$ for all $s \in S, A \in \mathcal{S}$.
- (2) $A^b = \{s \in S \mid A \not\subseteq Q_s\}$ for all $A \in \mathcal{S}$.
- (3) $(\bigvee_{i \in I} A_i)^b = \bigcup_{i \in I} A_i^b$ for all $s \in S, A \in \mathcal{S}$.
- (4) A is the smallest element of \mathcal{S} containing A^b .
- (5) For $A, B \in \mathcal{S}$, if $A \not\subseteq B$ then there exists $s \in S$ with $A \not\subseteq Q_s$ and $P_s \not\subseteq B$.
- (6) $A = \bigcap \{Q_s \mid P_s \not\subseteq A\}$ for all $A \in \mathcal{S}$.
- (7) $A = \bigvee \{P_s \mid A \not\subseteq Q_s\}$ for all $A \in \mathcal{S}$.

We see from (6) and (7) that the duality mentioned above is between \cap and \vee , Q_s and P_s , and $P_s \not\subseteq A$ and $A \not\subseteq Q_s$. To emphasize this duality we normally write $P_s \not\subseteq A$ in place of $s \notin A$.

If $(S, \mathcal{S}), (T, \mathcal{T})$ are textures the product texturing $\mathcal{S} \otimes \mathcal{T}$ on $S \times T$ consists of arbitrary intersections of sets of the form $(A \times T) \cup (S \times B)$ for $A \in \mathcal{S}, B \in \mathcal{T}$ [4]. For $s \in S, t \in T$ we have $P_{(s,t)} = P_s \times P_t, Q_{(s,t)} = (S \times Q_t) \cup (Q_s \times T)$.

In the following definition we consider $(S \times T, \mathcal{P}(S) \otimes \mathcal{T})$ rather than $(S \times T, \mathcal{S} \otimes \mathcal{T})$, and to avoid confusion, we use the notation $\overline{P}_{(s,t)}, \overline{Q}_{(s,t)}$ for the p-sets and q-sets in this texture. Hence, $\overline{P}_{(s,t)} = \{s\} \times P_t$ and $\overline{Q}_{(s,t)} = (S \times Q_t) \cup ((S \setminus \{s\}) \times T)$. (Similarly, $\overline{P}_{(t,s)}$ and $\overline{Q}_{(t,s)}$ denote the p-sets and q-sets of $(T \times S, \mathcal{P}(T) \otimes \mathcal{S})$).

2.3. Definition. [5] Let (S, \mathcal{S}) and (T, \mathcal{T}) be textures. Then

- (1) $r \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *relation* from (S, \mathcal{S}) to (T, \mathcal{T}) if it satisfies
 - (R1) $r \not\subseteq \overline{Q}_{(s,t)}, P_{s'} \not\subseteq Q_s \implies r \not\subseteq \overline{Q}_{(s',t)}$,
 - (R2) $r \not\subseteq \overline{Q}_{(s,t)} \implies \exists s' \in S$ such that $P_s \not\subseteq Q_{s'}$ and $r \not\subseteq \overline{Q}_{(s',t)}$.
- (2) $R \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *corelation* from (S, \mathcal{S}) to (T, \mathcal{T}) if it satisfies
 - (CR1) $\overline{P}_{(s,t)} \not\subseteq R, P_s \not\subseteq Q_{s'} \implies \overline{P}_{(s',t)} \not\subseteq R$,
 - (CR2) $\overline{P}_{(s,t)} \not\subseteq R \implies \exists s' \in S$ such that $P_{s'} \not\subseteq Q_s$ and $\overline{P}_{(s',t)} \not\subseteq R$.
- (3) A pair (r, R) where r is a relation and R a corelation from (S, \mathcal{S}) to (T, \mathcal{T}) is called a *direlation* from (S, \mathcal{S}) to (T, \mathcal{T}) .

Normally relations will be denoted by lower case and corelations by upper case letters, as in the above definition. If $(r_1, R_1), (r_2, R_2)$ are both direlations from (S, \mathcal{S}) to (T, \mathcal{T}) we write $(r_1, R_1) \sqsubseteq (r_2, R_2)$ if $r_1 \subseteq r_2$ and $R_2 \subseteq R_1$.

For a general texture (S, \mathcal{S}) we define $i_{(S,S)} = \bigvee \{\overline{P}_{(s,s)} \mid s \in S\}$ and $I_{(S,S)} = \bigcap \{\overline{Q}_{(s,s)} \mid s \in S\}$. It is trivial to verify that $i_{(S,S)} \not\subseteq \overline{Q}_{(s,t)} \iff P_s \not\subseteq Q_t, \overline{P}_{(s,t)} \not\subseteq I_{(S,S)} \iff P_t \not\subseteq Q_s$, whence $i_{(S,S)}$ is a relation and $I_{(S,S)}$ a corelation on (S, \mathcal{S}) , that is from (S, \mathcal{S}) to (S, \mathcal{S}) .

2.4. Definition. [5]

- (1) The direlation $(i_{(S,S)}, I_{(S,S)})$ is called the *identity direlation* on (S, \mathcal{S}) . Where there can be no confusion it can be denoted by (i_S, I_S) , or even (i, I) .
- (2) A direlation (r, R) on (S, \mathcal{S}) is called *reflexive* if $(i, I) \sqsubseteq (r, R)$. In particular the identity direlation is reflexive.

2.5. Definition. [5]

- (1) Let $(S, \mathcal{S}), (T, \mathcal{T})$ be textures and (r, R) a direlation from (S, \mathcal{S}) to (T, \mathcal{T}) . Then the direlation $(r, R)^\leftarrow = (R^\leftarrow, r^\leftarrow)$ from (T, \mathcal{T}) to (S, \mathcal{S}) defined by

$$r^\leftarrow = \bigvee \{ \overline{Q}_{(t,s)} \mid r \not\subseteq \overline{Q}_{(s,t)} \}, \quad R^\leftarrow = \bigcap \{ \overline{P}_{(t,s)} \mid \overline{P}_{(s,t)} \not\subseteq R \}$$

is called the *inverse* of (r, R) . Likewise, r^\leftarrow is called the *inverse* of r and R^\leftarrow the *inverse* of R .

- (2) A direlation (r, R) on (S, \mathcal{S}) is called *symmetric* if $(r, R)^\leftarrow = (r, R)$. For any texture (S, \mathcal{S}) , the identity direlation is clearly symmetric.

In the case of discrete textures we call an element φ of $\mathcal{P}(S \times T) = \mathcal{P}(S) \otimes \mathcal{P}(T)$, that is a binary relation from S to T in the ordinary sense, a *point relation* from S to T . Clearly, φ is both a relation and a corelation from the texture $(S, \mathcal{P}(S))$ to the texture $(T, \mathcal{P}(T))$. Under both interpretations the inverse is given by $\varphi^\leftarrow = (\varphi^{-1})^c = (\varphi^c)^{-1}$; where $\varphi^{-1} = \{(t, s) \mid (s, t) \in \varphi\}$ is the usual point inverse, and c denotes set complementation.

2.6. Definition. [5] Let $(S, \mathcal{S}), (T, \mathcal{T})$ be textures, (r, R) a direlation from (S, \mathcal{S}) to (T, \mathcal{T}) and $A \in \mathcal{S}, B \in \mathcal{T}$.

- (1) The *A-section* of r is the element $r^\rightarrow A$ of \mathcal{T} defined by

$$r^\rightarrow A = \bigvee \{ Q_t \mid \forall s, r \not\subseteq \overline{Q}_{(s,t)} \implies A \subseteq Q_s \}.$$

- (2) The *A-section* of R is the element $R^\rightarrow A$ of \mathcal{T} defined by

$$R^\rightarrow A = \bigcap \{ P_t \mid \forall s, \overline{P}_{(s,t)} \not\subseteq R \implies P_s \subseteq A \}.$$

- (3) The *B-presection* of r is the element $r^\leftarrow B$ of \mathcal{S} defined by

$$r^\leftarrow B = (r^\leftarrow)^\rightarrow B = \bigvee \{ P_s \mid \forall t, r \not\subseteq \overline{Q}_{(s,t)} \implies P_t \subseteq B \}.$$

- (4) The *B-presection* of R is the element $R^\leftarrow B$ of \mathcal{S} defined by

$$R^\leftarrow B = (R^\leftarrow)^\rightarrow B = \bigcap \{ Q_s \mid \forall t, \overline{P}_{(s,t)} \not\subseteq R \implies B \subseteq Q_t \}.$$

Clearly, the sections and the presections preserve inclusion.

2.7. Definition. [5] Let $(S, \mathcal{S}), (T, \mathcal{T}), (M, \mathcal{M})$ be textures.

- (1) If r is a relation from (S, \mathcal{S}) to (T, \mathcal{T}) and u a relation from (T, \mathcal{T}) to (M, \mathcal{M}) then their *composition* is the relation $u \circ r$ from (S, \mathcal{S}) to (M, \mathcal{M}) defined by

$$u \circ r = \bigcap \{ \overline{P}_{(s,m)} \mid \exists t \in T \text{ with } r \not\subseteq \overline{Q}_{(s,t)} \text{ and } u \not\subseteq \overline{Q}_{(t,m)} \}.$$

- (2) If R is a corelation from (S, \mathcal{S}) to (T, \mathcal{T}) and U a corelation from (T, \mathcal{T}) to (M, \mathcal{M}) then their *composition* is the corelation $U \circ R$ from (S, \mathcal{S}) to (M, \mathcal{M}) defined by

$$U \circ R = \bigvee \{ \overline{Q}_{(s,m)} \mid \exists t \in T \text{ with } \overline{P}_{(s,t)} \not\subseteq R \text{ and } \overline{P}_{(t,m)} \not\subseteq U \}.$$

- (3) With u, r, U, R as above, the *composition* of the direlations $(r, R), (u, U)$ is the direlation $(u, U) \circ (r, R) = (u \circ r, U \circ R)$.

- (4) A direlation (r, R) on (S, \mathcal{S}) is called *transitive* if $(r, R) \sqsubseteq (r, R) \circ (r, R)$. In this case r and R are also said to be transitive.

No confusion will be caused by the use of the single symbol \circ to denote these various compositions. Composition combines with sections and presections as one would expect.

2.8. Definition. [12] Let $(r, R), (u, U)$ be direlations from (S, \mathcal{S}) to (T, \mathcal{T}) . Then we set

$$\begin{aligned} r \sqcap u &= \bigvee \{ \overline{P}_{(s,t)} \mid \exists s' \in S \text{ with } P_s \not\subseteq Q_{s'} \text{ and } r, u \not\subseteq \overline{Q}_{(s',t)} \}, \\ R \sqcup U &= \bigcap \{ \overline{Q}_{(s,t)} \mid \exists s' \in S \text{ with } P_{s'} \not\subseteq Q_s \text{ and } \overline{P}_{(s',t)} \not\subseteq R, U \}, \text{ and} \\ (r, R) \sqcap (u, U) &= (r \sqcap u, R \sqcup U). \end{aligned}$$

Note that $r \sqcap u$ is the greatest lower bound of r and u in the set of all relations on (S, \mathcal{S}) to (T, \mathcal{T}) , and $R \sqcup U$ the least upper bound of R and U in the set of all corelations on (S, \mathcal{S}) to (T, \mathcal{T}) , ordered by inclusion.

2.9. Definition. [13] Let (r, R) be a direlation between the complemented textures (S, \mathcal{S}, σ) and (T, \mathcal{T}, θ) .

- (1) The *complement* r' of the relation r is the co-relation $r' = \bigcap \{ \overline{Q}_{(s,t)} \mid \exists s_1, t_1 \text{ with } r \not\subseteq \overline{Q}_{(s_1, t_1)}, \sigma(Q_s) \not\subseteq Q_{s_1} \text{ and } P_{t_1} \not\subseteq \theta(P_t) \}$.
- (2) The *complement* R' of the corelation R is the relation $R' = \bigvee \{ \overline{P}_{(s,t)} \mid \exists s_1, t_1 \text{ with } \overline{P}_{(s_1, t_1)} \not\subseteq R, P_{s_1} \not\subseteq \sigma(P_s) \text{ and } \theta(Q_t) \not\subseteq Q_{t_1} \}$.
- (3) The *complement* $(r, R)'$ of the direlation (r, R) is the direlation $(r, R)' = (R', r')$. The direlation (r, R) is said to be *complemented* if $(r, R)' = (r, R)$.

2.10. Definition. [5] Let (f, F) be a direlation from (S, \mathcal{S}) to (T, \mathcal{T}) . Then (f, F) is called a *difunction* from (S, \mathcal{S}) to (T, \mathcal{T}) if it satisfies the following two conditions:

- (DF1) For $s, s' \in S, P_s \not\subseteq Q_{s'} \implies \exists t \in T$ with $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F$.
- (DF2) For $t, t' \in T$ and $s \in S, f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t')} \not\subseteq F \implies P_{t'} \not\subseteq Q_t$.

It is clear that (i_S, I_S) is a difunction on (S, \mathcal{S}) , in which case it is called the identity difunction. In the particular case of discrete textures $(X, \mathcal{P}(X)), (Y, \mathcal{P}(Y))$, the pair (φ, ψ) of point relations from X to Y is a difunction if and only if φ is a point function $\varphi : X \rightarrow Y$ and $\psi = \varphi^c$.

2.11. Theorem. [5] For a direlation (f, F) from (S, \mathcal{S}) to (T, \mathcal{T}) the following are equivalent:

- (1) (f, F) is a difunction.
- (2) The following inclusions hold:
 - (i) $f^{\leftarrow}(F^{\rightarrow}A) \subseteq A \subseteq F^{\leftarrow}(f^{\rightarrow}A), \forall A \in \mathcal{S}$, and
 - (ii) $f^{\rightarrow}(F^{\leftarrow}B) \subseteq B \subseteq F^{\rightarrow}(f^{\leftarrow}B), \forall B \in \mathcal{T}$.
- (3) $f^{\leftarrow}B = F^{\leftarrow}B, \forall B \in \mathcal{T}$.

2.12. Definition. [1, 3] A *dichotomous* topology, or *ditopology* for short, on a texture (S, \mathcal{S}) is a pair (τ, κ) of subsets of \mathcal{S} , where the set τ of open sets satisfies

- (1) $S, \emptyset \in \tau$, (2) $G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau$ and (3) $G_i \in \tau, i \in I \implies \bigvee_i G_i \in \tau$,

and the set κ of closed sets satisfies

- (1) $S, \emptyset \in \kappa$, (2) $K_1, K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa$ and (3) $K_i \in \kappa, i \in I \implies \bigcap_i K_i \in \kappa$.

Hence a ditopology is essentially a "topology" for which there is no priori relation between the open and closed sets. When a complementation σ is given, the ditopology (τ, κ) is said to be *complemented* if $\kappa = \sigma(\tau)$.

For $A \in \mathcal{S}$ the closure of A is the set $[A] = \bigcap \{ K \in \kappa \mid A \subseteq K \}$, and the interior of A the set $]A[= \bigvee \{ G \in \tau \mid G \subseteq A \}$. As in the classical case, the operator $\text{int} : \mathcal{S} \rightarrow \mathcal{S}$, $A \mapsto]A[$ and the operator $\text{cl} : \mathcal{S} \rightarrow \mathcal{S}$, $A \mapsto [A]$ have the following properties

- (I1) $\text{int}(S) = S$.
- (I2) $\text{int}(A) \subseteq A$.
- (I3) $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$.
- (I4) $\text{int}(\text{int}(A)) = A$.

and

- (C1) $\text{cl}(\emptyset) = \emptyset$.
- (C2) $A \subseteq \text{cl}(A)$.
- (C3) $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$.
- (C4) $\text{cl}(\text{cl}(A)) = A$.

Conversely, given a texture (S, \mathcal{S}) and operators $\text{int} : \mathcal{S} \rightarrow \mathcal{S}$ satisfying (I1) through (I4) and $\text{cl} : \mathcal{S} \rightarrow \mathcal{S}$ satisfying (C1) through (C4), if we define $\tau = \{G \in \mathcal{S} \mid G = \text{int}(G)\}$ and $\kappa = \{K \in \mathcal{S} \mid K = \text{cl}(K)\}$ then we obtain a ditopology (τ, κ) on (S, \mathcal{S}) whose closure and interior operators are just the operators int and cl . More details about products of textures, direlations, difunctions and ditopologies can be found in [2, 3, 4, 5, 6, 7, 12].

3. Di-extremities

In this section we will adapt classical quasi-proximities to a textural setting. To do that, we will first give an alternative description of a classical quasi-proximity as outlined in the introduction.

Let η be a quasi-proximity on X and define a binary relation $\bar{\delta}_\eta$ on $\mathcal{P}(X)$ by

$$A\bar{\delta}_\eta B \iff A\eta(X \setminus B).$$

Then it is straightforward to check that $\bar{\delta} = \bar{\delta}_\eta$ satisfies the following properties:

- (1) $A\bar{\delta}B$ implies $A \neq \emptyset$ and $B \neq S$.
- (2) $(A \cup B)\bar{\delta}C$ iff $A\bar{\delta}C$ or $B\bar{\delta}C$.
- (3) $A\bar{\delta}(B \cap C)$ iff $A\bar{\delta}B$ or $A\bar{\delta}C$.
- (4) If $A\bar{\delta}B$, there exists $E \subseteq X$ such that $A\bar{\delta}E$ and $E\bar{\delta}B$.
- (5) $A\bar{\delta}B$ implies $A \subseteq B$.

We will call $\bar{\delta}_\eta$ the *extremity on X corresponding to η* . Conversely if the binary relation $\bar{\delta}$ on $\mathcal{P}(X)$ satisfies (1) through (5) then by setting $A\eta B \iff A\bar{\delta}(X \setminus B)$, one obtains a quasi-proximity η on X whose corresponding extremity $\bar{\delta}_\eta$ is just $\bar{\delta}$. Thus we see that extremities on X correspond precisely to quasi-proximities on X .

We note that we can also get a dual structure $\underline{\delta}_\eta$ by setting $A\underline{\delta}_\eta B \iff B\eta(X \setminus A)$. This is related to $\bar{\delta}_\eta$ by the “symmetry relation” $A\bar{\delta}_\eta B \iff B\underline{\delta}_\eta A$.

We observe that the set complement is not directly involved in the axioms for $\bar{\delta}_\eta$ (or those for $\underline{\delta}_\eta$). This suggests the following definition for the textural case.

3.1. Definition. Let (S, \mathcal{S}) be a texture, $\bar{\delta}, \underline{\delta}$ two binary relations on \mathcal{S} . Then $\delta = (\bar{\delta}, \underline{\delta})$ is called a *di-extremity* on (S, \mathcal{S}) if

- (E1) $A\bar{\delta}B$ implies $A \neq \emptyset, B \neq S$.
- (E2) $(A \cup B)\bar{\delta}C$ iff $A\bar{\delta}C$ or $B\bar{\delta}C$.
- (E3) $A\bar{\delta}(B \cap C)$ iff $A\bar{\delta}B$ or $A\bar{\delta}C$.
- (E4) If $A\bar{\delta}B$, there exists $E \in \mathcal{S}$ such that $A\bar{\delta}E$ and $E\bar{\delta}B$.
- (E5) $A\bar{\delta}B$ implies $A \subseteq B$.
- (DE) $A\underline{\delta}B \iff B\bar{\delta}A$.
- (CE1) $A\underline{\delta}B$ implies $A \neq S, B \neq \emptyset$.
- (CE2) $A\underline{\delta}(B \cup C)$ iff $A\underline{\delta}B$ or $A\underline{\delta}C$.
- (CE3) $(A \cap B)\underline{\delta}C$ iff $A\underline{\delta}C$ or $B\underline{\delta}C$.

(CE4) If $A \underline{\delta} B$, there exists $E \in \mathcal{S}$ such that $A \underline{\delta} E$ and $E \underline{\delta} B$.

(CE5) $A \underline{\delta} B$ implies $B \subseteq A$.

In this case it is said that $\bar{\delta}$ is the *extremity* and $\underline{\delta}$ the *co-extremity* of δ . Also, (S, \mathcal{S}, δ) is known as a *di-extremial* texture space.

Note that when giving examples it will clearly suffice to give only $\bar{\delta}$ satisfying the extremity conditions, since DE may then be used to define $\underline{\delta}$, which will automatically satisfy the co-extremity conditions.

Let η be a quasi-proximity on X . Then $\delta_\eta = (\bar{\delta}_\eta, \underline{\delta}_\eta)$ is the di-extremity on the discrete texture $(X, \mathcal{P}(X))$ corresponding to η . In fact, every di-extremity on $(X, \mathcal{P}(X))$ arises in this way from some quasi-proximity, so there is a bijection between the quasi-proximities on X and di-extremities on $(X, \mathcal{P}(X))$. Moreover we should note that the proximities on X will be characterized in terms of the corresponding di-extremities later on when complemented textures are considered.

3.2. Lemma. Let $\delta = (\bar{\delta}, \underline{\delta})$ be a di-extremity on (S, \mathcal{S}) . Then,

- (1) $A \bar{\delta} B, A \subseteq C, D \subseteq B \implies C \bar{\delta} D$.
- (2) If there exists $s \in S$ such that $A \bar{\delta} Q_s$ and $P_s \bar{\delta} B$, then $A \bar{\delta} B$.
- (3) $A \underline{\delta} B, C \subseteq A, B \subseteq D \implies C \underline{\delta} D$.
- (4) If there exists $s \in S$ such that $A \underline{\delta} P_s$ and $Q_s \underline{\delta} B$, then $A \underline{\delta} B$.
- (5) (E5) is equivalent to $(A \not\subseteq Q_s \text{ implies } A \bar{\delta} Q_s)$.
- (6) (CE5) is equivalent to $(P_s \not\subseteq A \text{ implies } A \underline{\delta} P_s)$.

Proof. It is straightforward to verify (1), (3), (5) and (6), and (4) is similar to (2).

(2) Let $A \bar{\delta} Q_s$ and $P_s \bar{\delta} B$ for some $s \in S$, and suppose $A \not\subseteq B$. Then by (E4), there exists $E \in \mathcal{S}$ such that $A \bar{\delta} E$ and $E \bar{\delta} B$. We have either $P_s \subseteq E$ or $P_s \not\subseteq E$. If $P_s \subseteq E$, then $E \bar{\delta} B$ by (1) and this is a contradiction. If $P_s \not\subseteq E$ then $E \subseteq Q_s$ so $A \bar{\delta} E$ by (1) and again we have a contradiction. That is, if $A \bar{\delta} Q_s$ and $P_s \bar{\delta} B$ then $A \bar{\delta} B$. \square

Our aim now is to define and investigate the ditopology induced by a di-extremity. Let (S, \mathcal{S}) be a texture, δ a di-extremity on (S, \mathcal{S}) and for any $A \in \mathcal{S}$, put

$$\text{int}(A) = \bigcap \{Q_s \mid P_s \bar{\delta} A\} \text{ and } \text{cl}(A) = \bigvee \{P_s \mid Q_s \underline{\delta} A\}.$$

3.3. Lemma. The functions $\text{int} : (S, \mathcal{S}) \rightarrow (S, \mathcal{S})$ and $\text{cl} : (S, \mathcal{S}) \rightarrow (S, \mathcal{S})$ have the following properties:

- (1) $A \not\subseteq \text{int}(B)$ implies $\exists s \in S$ such that $P_s \bar{\delta} B$ and $A \not\subseteq Q_s$.
- (2) $P_s \bar{\delta} B$ implies $\text{int}(B) \subseteq Q_s$.
- (3) $A \bar{\delta} B$ implies $A \subseteq \text{int}(B)$.
- (4) $\text{cl}(A) \not\subseteq B$ implies $\exists s \in S$ such that $Q_s \underline{\delta} A$ and $P_s \not\subseteq B$.
- (5) $Q_s \underline{\delta} B$ implies $P_s \subseteq \text{cl}(B)$.
- (6) $A \underline{\delta} B$ implies $\text{cl}(B) \subseteq A$.
- (7) $\text{int}(A) = \bigvee \{P_s \mid P_s \bar{\delta} A\}$.
- (8) $\text{cl}(A) = \bigcap \{Q_s \mid Q_s \underline{\delta} A\}$.
- (9) $P_s \bar{\delta} B$ implies $P_s \subseteq \text{int}(B)$.
- (10) $Q_s \underline{\delta} B$ implies $\text{cl}(B) \subseteq Q_s$.

Proof. We will prove (3) and leave the other results to the interested reader.

(3) Suppose $A \bar{\delta} B$ and $A \not\subseteq \text{int}(B)$. Then by (1), there exists $s \in S$ such that $P_s \bar{\delta} B$ and $A \not\subseteq Q_s$. By (E5), $A \bar{\delta} Q_s$ and by lemma 3.2(2), $A \bar{\delta} B$ which is a contradiction. \square

3.4. Theorem. Let $\delta = (\bar{\delta}, \underline{\delta})$ be a di-extremity on (S, S) . The function $\text{int} : S \rightarrow S$, $\text{int}(A) = \bigcap \{Q_s \mid P_s \bar{\delta}A, s \in S\}$ satisfies the interior axioms and the function $\text{cl} : S \rightarrow S$, $\text{cl}(A) = \bigvee \{P_s \mid Q_s \underline{\delta}A, s \in S\}$ satisfies the closure axioms.

Proof. We will show the interior axioms, leaving the closure axioms to the interested reader.

(I1) Clear by the definition.

(I2) Take $P_s \not\subseteq A$. Then there exists $y \in S$ such that $P_s \not\subseteq Q_y$ and $P_y \not\subseteq A$. $P_y \bar{\delta}A$ by (E5) and $\text{int}(A) \subseteq Q_y$ by the definition of int . Therefore $P_s \not\subseteq \text{int}(A)$. Hence we get $\text{int}(A) \subseteq A$.

(I3) It is easy to observe that for any $A, B \in S$, $A \subseteq B \implies \text{int}(A) \subseteq \text{int}(B)$. From this observation we get $\text{int}(A \cap B) \subseteq \text{int}(A) \cap \text{int}(B)$. To show the reverse inclusion, take $P_s \not\subseteq \text{int}(A \cap B)$. Then there exists $y \in S$ such that $P_y \bar{\delta}(A \cap B)$ and $P_s \not\subseteq Q_y$. By (E3), $P_y \bar{\delta}A$ or $P_y \bar{\delta}B$ and we get $\text{int}(A) \subseteq Q_y$ or $\text{int}(B) \subseteq Q_y$. Therefore we get $\text{int}(A) \cap \text{int}(B) \subseteq Q_y$. Thus $P_s \not\subseteq \text{int}(A) \cap \text{int}(B)$. Hence $\text{int}(A) \cap \text{int}(B) = \text{int}(A \cap B)$.

(I4) It is clear that $\text{int}(\text{int}(A)) \subseteq \text{int}(A)$ by (I2). Now take $\text{int}(A) \not\subseteq Q_x$ for any $x \in X$. Then there exists $y \in S$ such that $\text{int}(A) \not\subseteq Q_y$ and $P_y \not\subseteq Q_x$. Now $P_y \bar{\delta}A$ since $\text{int}(A) \not\subseteq Q_y$, and by (E4) there exists $E \in S$ such that $P_y \bar{\delta}E$ and $E \bar{\delta}A$. We have $E \subseteq \text{int}(A)$ by Lemma 3.3(3) and we obtain $P_y \bar{\delta}\text{int}(A)$ by Lemma 3.2(1). If we use Lemma 3.3(3) again, we get $P_y \subseteq \text{int}(\text{int}(A))$. Therefore $\text{int}(\text{int}(A)) \not\subseteq Q_x$. Hence we get $\text{int}(\text{int}(A)) = \text{int}(A)$. \square

3.5. Definition. Let (S, S) be a texture and $\delta = (\bar{\delta}, \underline{\delta})$ a di-extremity on (S, S) . Then the topology induced by δ is denoted by $\tau(\delta)$ and consists of all the sets $A \in S$ such that $A = \text{int}(A)$. Similarly the cotopology induced by δ is denoted by $\kappa(\delta)$ and consists of all the sets $A \in S$ such that $A = \text{cl}(A)$. The ditopology induced by $\delta = (\bar{\delta}, \underline{\delta})$ is denoted by $(\tau(\delta), \kappa(\delta))$.

The following proposition and definition are needed to characterize proximities on X in terms of the corresponding di-extremities on $(X, \mathcal{P}(X), \pi_X)$.

3.6. Proposition. Let $\delta = (\bar{\delta}, \underline{\delta})$ be a di-extremity on a complemented texture (S, S, σ) . Define $\delta' = \sigma(\delta) = (\bar{\delta}', \underline{\delta}')$ where for all $A, B \in S$,

$$A \bar{\delta}'B \iff \sigma(A) \underline{\delta}\sigma(B) \text{ and } A \underline{\delta}'B \iff \sigma(A) \bar{\delta}\sigma(B).$$

Then δ' is a di-extremity on (S, S, σ) .

Proof. We will prove that the (E4) axiom holds. The other axioms can be shown in a similar manner and the details are omitted. Let $A \not\delta' B$. Then by the definition, $\sigma(A) \underline{\delta}\sigma(B)$. Since $\underline{\delta}$ satisfies (CE4), there exists $F \in S$ such that $\sigma(A) \underline{\delta}F$ and $F \underline{\delta}\sigma(B)$. Now by setting $E = \sigma(F)$ we get $\sigma(A) \underline{\delta}\sigma(E)$ and $\sigma(E) \underline{\delta}\sigma(B)$. Again by the definition, $A \bar{\delta}'E$ and $E \bar{\delta}'B$. \square

3.7. Definition. The di-extremity δ' defined in proposition 3.6 is said to be complement of δ . The di-extremity δ is said to be complemented if $\delta = \delta'$.

3.8. Theorem. Let δ be a complemented di-extremity on (S, S, σ) . Then ditopology induced by δ is a complemented ditopology.

Proof. Let $\delta = \delta'$ and take $G \in \tau(\delta)$. Suppose $\text{cl}(\sigma(G)) \not\subseteq \sigma(G)$. Then there exists $s \in S$ such that $\text{cl}(\sigma(G)) \not\subseteq \sigma(P_s)$ and $\sigma(Q_s) \not\subseteq \sigma(G)$ by [5, Lemma 2.19(2)]. From $\text{cl}(\sigma(G)) \not\subseteq \sigma(P_s)$ we get $\sigma(P_s) \underline{\delta}\sigma(G)$ by Lemma 3.3(6). On the other hand, $\sigma(Q_s) \not\subseteq \sigma(G) \implies \text{int}(G) = G \not\subseteq Q_s \implies P_s \bar{\delta}G$ by Lemma 3.3(2), and by the assumption

$\delta = \delta'$, we get $P_s \overline{\delta'}G$, or equivalently $\sigma(P_s) \underline{\delta}\sigma(G)$, which is a contradiction. Thus $\text{cl}(\sigma(G)) = \sigma(G)$ or equivalently $\sigma(G) \in \kappa$.

Similarly it can be shown that if $K \in \kappa$ then $\sigma(K) \in \tau$. \square

3.9. Examples. (1) Let X be a set and η a proximity on X in the usual sense. Define $A\overline{\delta}B \iff A\eta(X - B)$ and $A\underline{\delta}B \iff B\eta(X - A)$. Then $\delta = (\overline{\delta}, \underline{\delta})$ is a complemented di-extremity on $(X, \mathcal{P}(X), \pi_X)$. At the beginning of this section, we have already mentioned that δ satisfies the di-extremity axioms. Now let us see that δ is complemented. Indeed, $A\overline{\delta'}B \iff (X \setminus A)\underline{\delta}(X \setminus B) \iff (X \setminus B)\eta A \iff A\eta(X \setminus B) \iff A\overline{\delta}B$, and similarly $A\underline{\delta'}B \iff A\underline{\delta}B$. Moreover $\kappa(\eta) = \{x \mid A\eta\{x\}\} = \{x \mid (X \setminus \{x\})\underline{\delta}A\} = \bigvee\{P_x \mid Q_x \underline{\delta}A\} = \kappa(\delta)$ and $\tau(\eta) = \pi_X(\kappa(\eta)) = \pi_X(\kappa(\delta)) = \tau(\delta)$ since δ is complemented.

Conversely let δ be any complemented di-extremity on $(X, \mathcal{P}(X), \pi_X)$. Define $A\eta B \iff A\overline{\delta}(X \setminus B) \iff B\underline{\delta}(X \setminus A)$. We already know that η is a quasi-proximity on X in the usual sense. Now let us see that η satisfies the symmetry axiom. $A\eta B \iff A\overline{\delta}(X \setminus B) \iff A\overline{\delta'}(X \setminus B) \iff (X \setminus A)\underline{\delta}B \iff B\overline{\delta}(X \setminus A) \iff B\eta A$. Moreover $\kappa(\delta) = \bigvee\{P_x \mid A\overline{\delta}Q_x\} = \{x \mid A\eta\{x\}\} = \kappa(\eta)$. Thus we see that complemented di-extremities on $(X, \mathcal{P}(X), \pi_X)$ correspond exactly to the classical proximities on X .

(2) Let (S, \mathcal{S}) be a texture. The extremity $\overline{\delta}$ defined by $A\overline{\delta}B \iff A \not\subseteq B$ is called the *discrete extremity* on (S, \mathcal{S}) , and the co-extremity $\underline{\delta}$ defined by $A\underline{\delta}B \iff B \not\subseteq A$ is called the *discrete co-extremity* on (S, \mathcal{S}) . Clearly, $\tau(\delta)$ and $\kappa(\delta)$ are the discrete topology and discrete cotopology on (S, \mathcal{S}) , respectively.

(3) Let (S, \mathcal{S}) be a texture. The extremity $\overline{\delta}$ defined by $A\overline{\delta}B \iff A \neq \emptyset, B \neq S$ is called the *trivial extremity* on (S, \mathcal{S}) , and the co-extremity $\underline{\delta}$ defined by $A\underline{\delta}B \iff B \neq \emptyset, A \neq S$ is called the *trivial co-extremity* on (S, \mathcal{S}) . Clearly, $\tau(\delta)$ and $\kappa(\delta)$ are the trivial topology and trivial cotopology on (S, \mathcal{S}) , respectively.

(4) Let (S, \mathcal{S}) be the texture of Example 2.1(2). Define for each $A, B \in \mathcal{S}$, $\emptyset \overline{\delta} B$, $A \overline{\delta} S$, $\{b\} \overline{\delta} \{b, c\}$ and $\{b, c\} \overline{\delta} \{b, c\}$, all other pairs being related under $\overline{\delta}$. Then $\overline{\delta}$ is an extremity on (S, \mathcal{S}) . It is easy to verify that $\tau(\delta) = \{\emptyset, \{b, c\}, S\}$ and $\kappa(\delta) = \{\emptyset, \{b, c\}, S\}$.

3.10. Proposition. *Let $\delta = (\overline{\delta}, \underline{\delta})$ be a di-extremity and $(\tau(\delta), \kappa(\delta))$ the ditopology induced by δ . Then the following statements hold:*

- (1) $A\overline{\delta}B$ if and only if $A\overline{\delta}\text{int}(B)$.
- (2) $A\underline{\delta}B$ if and only if $A\underline{\delta}\text{cl}(B)$.

Proof. We will prove (1) and leave (2) to the interested reader.

(1) The “only if” part is clear by Lemma 3.2(1) since $\text{int}(B) \subseteq B$. For the “if” part, let $A\overline{\delta}B$. Then there exists $E \in \mathcal{S}$ such that $A\overline{\delta}E$ and $E\overline{\delta}B$. Then $E \subseteq \text{int}(B)$ by Lemma 3.3(3), and $A\overline{\delta}\text{int}(B)$ by Lemma 3.2(1). \square

The following definition is a natural extension of proximal continuity.

3.11. Definition. Let $(S, \mathcal{S}, \delta_1)$ and $(T, \mathcal{T}, \delta_2)$ be di-extremial texture spaces, and $(f, F) : (S, \mathcal{S}) \rightarrow (T, \mathcal{T})$ a difunction. Then (f, F) is called *extremial bicontinuous* if it satisfies one, and hence both, of the following equivalent conditions:

- (1) $A\overline{\delta}_1 B$ implies $f \rightarrow A\overline{\delta}_2 F \rightarrow B$ for all $A, B \in \mathcal{S}$.
- (2) $A\underline{\delta}_1 B$ implies $F \rightarrow A\underline{\delta}_2 f \rightarrow B$ for all $A, B \in \mathcal{S}$.

Note that the equivalence of these conditions comes from the (DE) axiom. Moreover, the following lemma holds as in the classical case and will be used in later proofs.

3.12. Lemma. *Let $(S, \mathcal{S}, \delta_1)$ and $(T, \mathcal{T}, \delta_2)$ be di-extremial texture spaces, and $(f, F) : (S, \mathcal{S}) \rightarrow (T, \mathcal{T})$ a difunction. Then (f, F) is extremial bicontinuous if and only if it satisfies one, and hence both, of the following equivalent conditions:*

- (1) $C \overline{\delta}_2 D$ implies $f^{\leftarrow} C \overline{\delta}_1 f^{\leftarrow} D$ for all $C, D \in \mathcal{T}$.
- (2) $C \underline{\delta}_2 D$ implies $f^{\leftarrow} C \underline{\delta}_1 f^{\leftarrow} D$ for all $C, D \in \mathcal{T}$.

Proof. We will show that (1) is equivalent to Definition 3.11 (1). The equivalence of (2) and Definition 3.11 (2) can be shown in a similar way.

Let (f, F) be extremial bicontinuous and suppose there exists $C, D \in \mathcal{T}$ such that $C \overline{\delta}_2 D$ and $f^{\leftarrow} C \overline{\delta}_1 f^{\leftarrow} D$. Then $f^{\rightarrow} (f^{\leftarrow} C) \overline{\delta}_2 F^{\rightarrow} (f^{\leftarrow} D)$ by extremial bicontinuity. But $f^{\rightarrow} (f^{\leftarrow} C) \subseteq C$ and $D \subseteq F^{\rightarrow} (f^{\leftarrow} D)$ by Theorem 2.11(2), so by Lemma 3.2(1) we get $C \overline{\delta}_2 D$, which is a contradiction.

Now let $C \overline{\delta}_2 D \implies f^{\leftarrow} C \overline{\delta}_1 f^{\leftarrow} D$ for all $C, D \in \mathcal{T}$ and suppose there exists $A, B \in \mathcal{S}$ such that $A \overline{\delta}_1 B$ and $f^{\rightarrow} A \overline{\delta}_2 F^{\rightarrow} B$. Then we have $f^{\leftarrow} (f^{\rightarrow} A) \overline{\delta}_2 f^{\leftarrow} (F^{\rightarrow} B)$ by hypothesis, and since $A \subseteq f^{\leftarrow} (f^{\rightarrow} A)$ and $f^{\leftarrow} (F^{\rightarrow} B) \subseteq B$ by Theorem 2.11(2), we get $A \overline{\delta}_1 B$ by Lemma 3.2(1). This result contradicts $A \overline{\delta}_1 B$. □

3.13. Theorem. *Let δ_1 be a di-extremity on (S, \mathcal{S}) , δ_2 a di-extremity on (T, \mathcal{T}) and $(f, F) : (S, \mathcal{S}) \rightarrow (T, \mathcal{T})$ an extremial bicontinuous difunction. Then (f, F) is also bicontinuous with respect to the induced ditopologies.*

Proof. Take $G \in \tau(\delta_2)$, equivalently $G = \text{int}_{\delta_2}(G)$. To show $f^{\leftarrow} G \in \tau(\delta_1)$, we will show $\text{int}_{\delta_1}(f^{\leftarrow} G) = f^{\leftarrow} G$. Suppose $f^{\leftarrow} G \not\subseteq \text{int}_{\delta_1}(f^{\leftarrow} G)$. Then there exists $s \in S$ such that $f^{\leftarrow} G \not\subseteq Q_s$ and $P_s \not\subseteq \text{int}_{\delta_1}(f^{\leftarrow} G)$. By the definition of presection, there exists $t \in T$ such that $\overline{P}_{(s,t)} \not\subseteq F$ and $G \not\subseteq Q_t$. From $\overline{P}_{(s,t)} \not\subseteq F$, and using [5, Lemma 2.6], we get $F^{\leftarrow} P_t \not\subseteq Q_s$ and so $f^{\leftarrow} P_t \overline{\delta}_1 Q_s$ by (E5). From $P_s \not\subseteq \text{int}_{\delta_1} f^{\leftarrow} G$ and using Lemma 3.3(3), we get $P_s \overline{\delta}_1 f^{\leftarrow} G$. Thus by Lemma 3.2(2), $f^{\leftarrow} P_t \overline{\delta}_1 f^{\leftarrow} G$. On the other hand, $P_t \overline{\delta}_2 G$ since $G = \text{int}_{\delta_2}(G) \not\subseteq Q_t$. Therefore $f^{\leftarrow} P_t \overline{\delta}_1 f^{\leftarrow} G$ since (f, F) is extremial bicontinuous. This is a contradiction.

The cotopology part is similar and is omitted. □

3.14. Proposition. *Let (S, \mathcal{S}, δ) be a di-extremial texture space. Then the following hold:*

- (1) *The identity difunction (S, \mathcal{S}, δ) is extremial bicontinuous.*
- (2) *The composition of extremial bicontinuous difunctions is extremial bicontinuous.*

Proof. (1) Clear by the definition of the identity difunction.

(2) Let $(S, \mathcal{S}, \delta_1)$, $(T, \mathcal{T}, \delta_2)$, $(M, \mathcal{M}, \delta_3)$ be di-extremial texture spaces and $(f, F) : (S, \mathcal{S}, \delta_1) \rightarrow (T, \mathcal{T}, \delta_2)$, $(g, Q) : (T, \mathcal{T}, \delta_2) \rightarrow (M, \mathcal{M}, \delta_3)$ extremial bicontinuous difunctions. Suppose $A \overline{\delta}_1 B$. Then by the extremial bicontinuity of (f, F) , $f^{\rightarrow} A \overline{\delta}_2 F^{\rightarrow} B$. Since $f^{\rightarrow} A, F^{\rightarrow} B \in \mathcal{T}$, by the extremial bicontinuity of (g, Q) , we get $q^{\rightarrow} (f^{\rightarrow} A) \overline{\delta}_3 Q^{\rightarrow} (F^{\rightarrow} B)$, or equivalently $(q \circ f)^{\rightarrow} A \overline{\delta}_3 (Q \circ F)^{\rightarrow} B$. Hence $(q \circ f, Q \circ F)$ is extremial bicontinuous. □

The following proposition verifies that extremial bicontinuity really does generalize proximal continuity.

3.15. Proposition. *Let (X, η_1) and (Y, η_2) be quasi-proximity spaces. Then $f : (X, \eta_1) \rightarrow (Y, \eta_2)$ is proximal continuous if and only if $(f, f^c) : (X, \mathcal{P}(X), \delta_{\eta_1}) \rightarrow (Y, \mathcal{P}(X), \delta_{\eta_2})$ is extremial bicontinuous.*

Proof. First recall that $f^{-1}(A) = f^{\leftarrow} A$ in a discrete texture $(X, \mathcal{P}(X))$, and that $f : X \rightarrow Y$ is proximal continuous if and only if $A \eta_1 B$ implies $f(A) \eta_2 f(B)$ for all $A, B \subseteq X$ if and only if $C \eta_2 D$ implies $f^{-1}(C) \eta_1 f^{-1}(D)$ for all $C, D \subseteq Y$.

Let f be proximal continuous. To show that (f, f^c) is extremial bicontinuous, take $C, D \in \mathcal{P}(Y)$ with $C \overline{\delta}_2 D$. Then by using the definitions and the proximal continuity of f ,

$$\begin{aligned} C \overline{\delta}_2 D &\implies C \eta_2(Y \setminus D) \implies f^{-1}(C) \eta_1 f^{-1}(Y \setminus D) \\ &\implies f^{-1}(C) \eta_1(X \setminus f^{-1}(D)) \implies f^{\leftarrow} C \eta_1(X \setminus f^{\leftarrow} D) \implies f^{\leftarrow} C \overline{\delta} f^{\leftarrow} D. \end{aligned}$$

Hence, (f, f^c) is extremial bicontinuous.

Now Let (f, f^c) be extremial bicontinuous. To show that f is proximal continuous, take $C, D \subseteq Y$ with $C \eta_2 D$. Then using the definitions and the extremial bicontinuity of (f, f^c) ,

$$\begin{aligned} C \eta_2 D &\implies C \overline{\delta}(Y \setminus D) \implies f^{\leftarrow} C \overline{\delta} f^{\leftarrow}(Y \setminus D) \\ &\implies f^{\leftarrow} C \overline{\delta}(X \setminus f^{\leftarrow} D) \implies f^{-1}(C) \overline{\delta}(X \setminus f^{-1}(D)) \implies f^{-1}(C) \eta_2 f^{-1}(D). \end{aligned}$$

Hence f is proximal continuous. \square

4. Di-extremities and di-uniformities

Two characterizations of di-uniformities on textures, namely direlational uniformities and dicovering uniformities, are given in [12]. The former is defined using direlations and the latter using dicoverings. In [12], it is also shown that these two notions are equivalent. We will use direlational uniformities since it is easier to work with them in the di-extremity context.

4.1. Definition. [12] Let (S, \mathcal{S}) be a texture and \mathcal{U} a family of direlations on (S, \mathcal{S}) . If \mathcal{U} satisfies the conditions

- (U1) $(i, I) \sqsubseteq (u, U)$ for all $(u, U) \in \mathcal{U}$.
- (U2) If $(u, U) \in \mathcal{U}$ and (r, R) is a direlation on (S, \mathcal{S}) such that $(u, U) \sqsubseteq (r, R)$, then $(r, R) \in \mathcal{U}$.
- (U3) $(u_1, U_1), (u_2, U_2) \in \mathcal{U}$ implies $(u_1, U_1) \sqcap (u_2, U_2) \in \mathcal{U}$.
- (U4) Given $(u, U) \in \mathcal{U}$ there exists $(r, R) \in \mathcal{U}$ satisfying $(r, R) \circ (r, R) \sqsubseteq (u, U)$.
- (U5) Given $(u, U) \in \mathcal{U}$ there exists $(r, R) \in \mathcal{U}$ satisfying $(r, R)^{\leftarrow} \sqsubseteq (u, U)$.

then \mathcal{U} is called a *direlational uniformity* on (S, \mathcal{S}) , and $(S, \mathcal{S}, \mathcal{U})$ is known as a direlational uniform texture space.

4.2. Definition. [13] For a given direlational uniformity \mathcal{U} on the complemented texture (S, \mathcal{S}, σ) , the direlational uniformity $\mathcal{U}' = \{(u, U)' \mid (u, U) \in \mathcal{U}\}$ is called the *the complement of \mathcal{U}* . The di-uniformity \mathcal{U} is said to be *complemented* if $\mathcal{U} = \mathcal{U}'$.

The uniform ditopology $(\tau_{\mathcal{U}}, \kappa_{\mathcal{U}})$ induced by a di-uniformity is characterized by the following lemma. In this lemma $u[s]$ denotes $u^{\rightarrow} P_s$ and $U[s]$ denotes $U^{\rightarrow} P_s$. (See [12, 13] for further details about the uniform ditopology).

4.3. Lemma. [12] *Let $(S, \mathcal{S}, \mathcal{U})$ be a direlational uniform texture space with uniform ditopology $(\tau_{\mathcal{U}}, \kappa_{\mathcal{U}})$. Then*

- (1) $G \in \tau_{\mathcal{U}} \iff (G \not\subseteq Q_s \implies \exists (u, U) \in \mathcal{U} \text{ with } u[s] \subseteq G)$.
- (2) $K \in \kappa_{\mathcal{U}} \iff (P_s \not\subseteq K \implies \exists (u, U) \in \mathcal{U} \text{ with } K \subseteq U[s])$.

As in stated [13],

- 1) \mathcal{U} is complemented if and only if it has a base of complemented direlations,
- 2) The uniform ditopology of a complemented direlational uniformity is a complemented ditopology.

4.4. Definition. [12] Let (S, \mathcal{S}) , (T, \mathcal{T}) be textures, $(r, R) : (S, \mathcal{S}) \rightarrow (T, \mathcal{T})$ a direlation. Then

$$\begin{aligned} (f, F)^{-1}(r) &= \bigvee \{ \overline{P}_{(s_1, s_2)} \mid \exists P_{s_1} \not\subseteq Q_{s'_1} \text{ so that } \overline{P}_{(s'_1, t_1)} \not\subseteq F, f \not\subseteq \overline{Q}_{(s_2, t_2)} \\ &\quad \implies r \not\subseteq \overline{Q}_{(t_1, t_2)} \} \\ (f, F)^{-1}(R) &= \bigcap \{ \overline{Q}_{(s_1, s_2)} \mid P_{s'_1} \not\subseteq Q_{s_1} \text{ so that } f \not\subseteq \overline{Q}_{(s'_1, t_1)}, \overline{P}_{(s_2, t_2)} \not\subseteq F \\ &\quad \implies \overline{P}_{(t_1, t_2)} \not\subseteq R \} \\ (f, F)^{-1}((r, R)) &= ((f, F)^{-1}(r), (f, F)^{-1}(R)) \end{aligned}$$

4.5. Definition. [12] Let \mathcal{U} be a direlational uniformity on (S, \mathcal{S}) , \mathcal{V} a direlational uniformity on (T, \mathcal{T}) and $(f, F) : (S, \mathcal{S}) \rightarrow (T, \mathcal{T})$ a difunction. If

$$(v, V) \in \mathcal{V} \implies (f, F)^{-1}(v, V) \in \mathcal{U},$$

the difunction (f, F) is said to be *uniformly bicontinuous*.

We now show that, as in the classical case, a di-uniformity induces a di-extremity in a natural way.

4.6. Proposition. Let \mathcal{U} be a direlational uniformity on the texture (S, \mathcal{S}) . Define

$$A \overline{\delta} B \iff u^{-} A \not\subseteq B \quad \forall (u, U) \in \mathcal{U} \quad \text{and} \quad A \underline{\delta} B \iff B \not\subseteq U^{-} A \quad \forall (u, U) \in \mathcal{U}.$$

Then $\delta = (\overline{\delta}, \underline{\delta})$ is a di-extremity on (S, \mathcal{S}) .

Proof. We will prove (E4) and the (DE) condition. The proof of (CE4) is similar to that of (E4), and the other axioms are straightforward.

To show (E4), suppose $A \overline{\delta} B$. Then there exists $(r, R), (u, U) \in \mathcal{U}$ such that $u^{-} A \subseteq B$ and $(r, R) \circ (r, R) \subseteq (u, U)$. By applying r^{-} to both sides of $u^{-} A \subseteq B$ we obtain $r^{-}(u^{-} A) \subseteq r^{-} B$. Let $E = r^{-} B$. Then we have $r^{-}(u^{-} A) \subseteq E$, and $u^{-} A \subseteq E$ since r is reflexive. Thus $A \overline{\delta} E$. On the other hand, $E \overline{\delta} B$ since $r^{-} E = r^{-}(r^{-} B) \subseteq B$ by [5, Lemma 2.9(1)].

Now let us see that the (DE) axiom holds. Let $A \overline{\delta} B$, or equivalently $u^{-} A \not\subseteq B$ for all $(u, U) \in \mathcal{U}$. Suppose $B \underline{\delta} A$. Then there exists $(r, R) \in \mathcal{U}$ such that $A \subseteq R^{-} B$. By applying R^{-} to both sides, we get $R^{-} A \subseteq R^{-}(R^{-} B)$ and so $R^{-} A \subseteq B$ since $R^{-} R^{-} B \subseteq B$ by [5, Lemma 2.9(2)]. Now setting $(u, U) = (r, R)^{-}$, we get $u^{-} A \subseteq B$. This contradicts $A \overline{\delta} B$. Hence $A \overline{\delta} B$ implies $B \underline{\delta} A$.

The other direction can be shown in a similar way. \square

4.7. Definition. The di-extremity $(\overline{\delta}, \underline{\delta})$ defined in Proposition 4.6 is called the *di-extremity induced on (S, \mathcal{S}) by \mathcal{U}* , or the *induced di-extremity* for short, and is denoted by $\delta_{\mathcal{U}} = (\overline{\delta}_{\mathcal{U}}, \underline{\delta}_{\mathcal{U}})$.

4.8. Theorem. Let \mathcal{U} be direlational uniformity on (S, \mathcal{S}) and $\delta = \delta_{\mathcal{U}}$. Then $\tau(\mathcal{U}) = \tau(\delta)$ and $\kappa(\mathcal{U}) = \kappa(\delta)$.

Proof. To show $\tau(\mathcal{U}) = \tau(\delta)$, first take $G \in \tau(\delta)$ and $G \not\subseteq Q_s$. Then $P_s \overline{\delta} G$ by Lemma 3.3(2), so there exists $(u, U) \in \mathcal{U}$ such that $u^{-} P_s \subseteq G$. Therefore $G \in \tau(\mathcal{U})$ by Lemma 4.3(1). Now take $G \in \tau(\mathcal{U})$ and suppose $G \not\subseteq \text{int}(G)$. Then there exists $s \in S$ such that $G \not\subseteq Q_s$ and $P_s \not\subseteq \text{int}(G)$. From $G \in \tau(\mathcal{U})$ and by Lemma 4.3(1), there exists $(u, U) \in \mathcal{U}$ such that $u^{-} P_s \subseteq G$. Therefore $P_s \overline{\delta} G$ by Definition 4.7, and so $P_s \subseteq \text{int}(G)$ by Lemma 3.3(3). But this contradicts $P_s \not\subseteq \text{int}(G)$. Hence $G = \text{int}(G)$, that is $G \in \tau(\delta)$.

The cotopology part can be shown similarly. \square

4.9. Theorem. Let \mathcal{U} be a direlational uniformity on (S, \mathfrak{S}) , \mathcal{V} a direlational uniformity on (T, \mathfrak{T}) and $(f, F) : (S, \mathfrak{S}) \rightarrow (T, \mathfrak{T})$ a uniformly bicontinuous difunction. Then (f, F) is also extremial bicontinuous with respect to the induced di-extremities

Proof. Let (f, F) be uniformly bicontinuous. We will show that (f, F) is extremial continuous. To do this, take $C \bar{\rho}_{\mathcal{V}} D$. Then there exists $(r, R) \in \mathcal{V}$ such that $r \rightarrow C \subseteq D$. Let $(u, U) = (f, F)^{-1}((r, R))$. Since (f, F) is uniformly bicontinuous, $(u, U) \in \mathcal{U}$. We claim that $u \rightarrow (f \leftarrow C) \subseteq f \leftarrow D$.

Suppose $u \rightarrow (f \leftarrow C) \not\subseteq f \leftarrow D$. Then there exists $s \in S$ such that $u \rightarrow (f \leftarrow C) \not\subseteq Q_s$ and $P_s \not\subseteq f \leftarrow D$. Since $u \rightarrow (f \leftarrow C) \not\subseteq Q_s$ there exists $s' \in S$ such that $u \not\subseteq \bar{Q}_{(s', s)}$, $f \leftarrow C \not\subseteq Q_{s'}$. By the definition of $(f, F)^{-1}((r, R))$, we have $s'_1, s_2 \in S$ with $P_{s_2} \not\subseteq Q_s$, $P_{s'} \not\subseteq Q_{s'_1}$ such that

$$(*) \quad \bar{P}_{(s'_1, t_1)} \not\subseteq F, f \not\subseteq \bar{Q}_{(s_2, t_2)} \implies r \not\subseteq \bar{Q}_{(t_1, t_2)} \text{ for all } t_1, t_2 \in T.$$

Firstly, since $f \leftarrow C \not\subseteq Q_{s'}$ and $P_{s'} \not\subseteq Q_{s'_1}$, we have $f \leftarrow C \not\subseteq Q_{s'_1}$, and by using the fact that $f \leftarrow C = F \leftarrow C$, we get $t'_1 \in T$ with $\bar{P}_{(s'_1, t'_1)} \not\subseteq F$ and $C \not\subseteq Q_{t'_1}$. Thus we have

$$P_{t'_1} \subseteq C \implies r \rightarrow P_{t'_1} \subseteq r \rightarrow C.$$

Secondly, $P_s \not\subseteq f \leftarrow D \implies P_{s_2} \not\subseteq f \leftarrow D \implies \exists t'_2 \in T, f \not\subseteq \bar{Q}_{(s_2, t'_2)}$ and $P_{t'_2} \not\subseteq D$. Now, if we take $t_1 = t'_1$ and $t_2 = t'_2$ in $(*)$, we get $r \not\subseteq \bar{Q}_{(t'_1, t'_2)}$. But

$$r \not\subseteq \bar{Q}_{(t'_1, t'_2)} \implies r \rightarrow P_{t'_1} \not\subseteq Q_{t'_2} \implies P_{t'_2} \subseteq r \rightarrow P_{t'_1},$$

and since $r \rightarrow P_{t'_1} \subseteq r \rightarrow C$, we get $r \rightarrow C \not\subseteq D$ which contradicts $r \rightarrow C \subseteq D$. Therefore $u \rightarrow (f \leftarrow C) \subseteq f \leftarrow D$. Hence $f \leftarrow C \bar{\rho}_{\mathcal{U}} f \leftarrow D$.

Cocontinuity can be shown similarly. \square

4.10. Theorem. The induced di-extremity δ of a complemented diuniformity \mathcal{U} is complemented.

Proof. Let $\mathcal{U} = \mathcal{U}'$ and suppose $A \bar{\delta} B$, that is $u \rightarrow A \not\subseteq B$ for every $(u, U) \in \mathcal{U}$. We will show that $A \underline{\delta}' B$. Let $(r, R) \in \mathcal{U}$ and set $(u, U) = (r, R)'$. Then by [13, Proposition 2.2 (i)],

$$\begin{aligned} A \bar{\delta} B &\implies u \rightarrow A \not\subseteq B \implies \sigma(B) \not\subseteq \sigma(u \rightarrow A) \implies \sigma(B) \not\subseteq (u') \rightarrow \sigma(A) \\ &\implies \sigma(B) \not\subseteq R \rightarrow \sigma(A) \implies \sigma(A) \underline{\delta} \sigma(B) \implies A \underline{\delta}' B. \end{aligned}$$

The same can be done to show $A \underline{\delta}' B \implies A \bar{\delta} B$. \square

5. Di-extremities and dimetrics

5.1. Definition. [12] Let (S, \mathfrak{S}) be a texture, $\bar{\rho}, \underline{\rho} : S \times S \rightarrow [0, \infty)$ two point functions. Then $\rho = (\bar{\rho}, \underline{\rho})$ is called a *pseudo dimetric* on (S, \mathfrak{S}) if

- (M1) $\bar{\rho}(s, t) \leq \bar{\rho}(s, u) + \underline{\rho}(u, t) \forall s, u, t \in S$,
- (M2) $P_s \not\subseteq Q_t$ implies $\bar{\rho}(s, t) = 0 \forall s, t \in S$,
- (DM) $\bar{\rho}(s, t) = \underline{\rho}(t, s) \forall s, t \in S$,
- (CM1) $\underline{\rho}(s, t) \leq \underline{\rho}(s, u) + \underline{\rho}(u, t) \forall s, u, t \in S$,
- (CM2) $P_t \not\subseteq Q_s$ implies $\underline{\rho}(s, t) = 0 \forall s, t \in S$,

in this case $\bar{\rho}$ is called the *pseudo metric*, $\underline{\rho}$ the *pseudo cometric* of ρ .

If ρ is a pseudo dimetric which satisfies the conditions

- (M3) $P_s \not\subseteq Q_u, \bar{\rho}(u, v) = 0, P_v \not\subseteq Q_t$ implies $P_s \not\subseteq Q_t \forall s, t, u, v \in S$
- (CM3) $P_u \not\subseteq Q_s, \underline{\rho}(u, v) = 0, P_t \not\subseteq Q_v$ implies $P_t \not\subseteq Q_s \forall s, t, u, v \in S$

it is called a *dimetric*.

When giving examples it will clearly suffice to give $\bar{\rho}$ satisfying the metric conditions, since DM may then be used to define $\underline{\rho}$.

5.2. Proposition. [12] Let ρ be a pseudo dimetric on (S, \mathcal{S}) and for $s \in S^b, \epsilon > 0$ define

$$N_\epsilon^\rho(s) = \bigvee \{P_t \mid \exists u \in S \text{ with } P_s \not\subseteq Q_u, \bar{\rho}(u, t) < \epsilon\},$$

$$M_\epsilon^\rho(s) = \bigcap \{Q_t \mid \exists u \in S \text{ with } P_u \not\subseteq Q_s, \underline{\rho}(u, t) < \epsilon\},$$

Then $\beta_\rho = \{N_\epsilon^\rho(s) \mid s \in S^b, \epsilon > 0\}$ is a base and $\gamma_\rho = \{M_\epsilon^\rho(s) \mid s \in S^b, \epsilon > 0\}$ a cobase for a ditopology $(\tau(\bar{\rho}), \kappa(\underline{\rho}))$ on (S, \mathcal{S}) .

5.3. Theorem. Let (S, \mathcal{S}) be a texture and $\rho = (\bar{\rho}, \underline{\rho})$ a pseudo dimetric on (S, \mathcal{S}) . For all $A, B \in \mathcal{S}$ define

$$\overline{D}(A, B) = \inf \{\bar{\rho}(s, t) \mid A \not\subseteq Q_s, P_t \not\subseteq B\},$$

$$\underline{D}(A, B) = \inf \{\underline{\rho}(s, t) \mid A \not\subseteq Q_s, P_t \not\subseteq B\},$$

and set

$$A\overline{\delta}B \iff \overline{D}(A, B) = 0 \text{ and } A\underline{\delta}B \iff \underline{D}(A, B) = 0.$$

Then $\delta = (\overline{\delta}, \underline{\delta})$ is a di-extremity on (S, \mathcal{S}) . Furthermore, the ditopology induced by δ and that induced by ρ are the same.

Proof. Since $\overline{D}(A, B) = \underline{D}(B, A)$ by the [DM] condition, $A\overline{\delta}B \iff B\underline{\delta}A$. Therefore the DE condition holds and it is enough to show that $\overline{\delta}$ is an extremity.

E1, E2 and E3 are clear.

For (E4), Let $A\overline{\delta}B$. Then $\overline{D}(A, B) = \epsilon > 0$. Set $E = \bigvee \{P_s \mid \overline{D}(P_s, B) > \frac{\epsilon}{2}\}$.

Firstly, to show $E\overline{\delta}B$, take $E \not\subseteq Q_s$ and $P_t \not\subseteq B$. By the definition of E and \overline{D} , there exists $x \in S$ such that $P_x \not\subseteq Q_s$ and $\bar{\rho}(y, z) > \frac{\epsilon}{2}$ for all $y, z \in S, P_x \not\subseteq Q_y$ and $P_z \not\subseteq B$. Now, since $P_t \not\subseteq B$ and $P_x \not\subseteq Q_s$, we get $\bar{\rho}(s, t) > \frac{\epsilon}{2}$. Thus

$$\overline{D}(E, B) = \inf \{\bar{\rho}(s, t) \mid E \not\subseteq Q_s, P_z \not\subseteq B\} \geq \frac{\epsilon}{2} > 0.$$

Hence $E\overline{\delta}B$.

Secondly, to show $A\overline{\delta}E$, observe that by the triangle inequality for $\bar{\rho}, \bar{\rho}(s, z) \leq \bar{\rho}(s, t) + \bar{\rho}(t, y) + \bar{\rho}(y, z)$ for all $s, t, y, z \in S$. Set s, t as constants for now and take the infimum of both side for all y, z such that $P_t \not\subseteq Q_y, P_z \not\subseteq B$ to give

$$(**) \quad \inf_{y,z} (\bar{\rho}(s, z) \leq \inf_{y,z} \bar{\rho}(s, t) + \bar{\rho}(t, y) + \bar{\rho}(y, z)).$$

By (M2), $\bar{\rho}(t, y) = 0$. Therefore $\inf_{y,z} \bar{\rho}(s, z) \leq \bar{\rho}(s, t) + \inf_{y,z} \bar{\rho}(y, z)$. On the other hand, by the definition of E, if $P_t \not\subseteq E$ then $\overline{D}(P_t, B) = \lambda_t \leq \frac{\epsilon}{2}$. Thus $\inf_{y,z} \bar{\rho}(s, z) \leq \bar{\rho}(s, t) + \frac{\epsilon}{2}$. Now if we take the infimum of both side of (**) for all s, t such that $A \not\subseteq Q_s, P_t \not\subseteq E$, then we get

$$\inf_{s,t} \inf_{y,z} \bar{\rho}(s, z) \leq \inf_{s,t} \bar{\rho}(s, t) + \frac{\epsilon}{2}.$$

From this we obtain $\epsilon \leq \overline{D}(A, E) + \frac{\epsilon}{2}$. Hence $\overline{D}(A, E) \neq 0$, that is $A\overline{\delta}E$.

For (E5), Let $A\overline{\delta}B$, which means $\overline{D}(A, B) \neq 0$. Suppose $A \not\subseteq B$. Then there exists $s, t \in S$ such that $A \not\subseteq Q_s, P_s \not\subseteq Q_t, P_t \not\subseteq B$. But $P_s \not\subseteq Q_t$ implies $\bar{\rho}(s, t) = 0$ implies $\overline{D}(A, B) = 0$, which is a contradiction.

To show the induced topologies are the same, it is enough to show that β_ρ is also a base of $\tau(\delta)$. Let $U \in \tau(\delta)$ and $U \not\subseteq Q_s$. By the definition of interior, $P_s \overline{\delta}U$, that is

$\overline{D}(P_s, U) = \epsilon > 0$. Now take $x \in S$ such that $N_\epsilon^p(s) \not\subseteq Q_x$. By the definition of $N_\epsilon^p(s)$, there exists $u, t \in S$ such that $P_s \not\subseteq Q_u, \overline{p}(u, t) < \epsilon$ and $P_t \not\subseteq Q_x$. But $\overline{p}(u, t) < \epsilon$ and $P_s \not\subseteq Q_u$ implies $P_t \subseteq U$. Therefore, $U \not\subseteq Q_x$. Thus $N_\epsilon^p(s) \subseteq U$.

The equality of the cotopologies can be shown similarly. \square

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