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RECONSTRUCTION OF COMPLEX JACOBI MATRICES FROM SPECTRAL DATA

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Abstract

In this paper, we introduce spectral data for finite order complex Jacobi matrices and investigate the inverse problem of determining the matrix from its spectral data. Necessary and sufficient conditions for the solvability of the inverse problem are established. An explicit procedure of reconstruction of the matrix from the spectral data is given.

Keywords: Jacobi matrix, Difference equation, Spectral data, Inverse spectral problem.

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1. Introduction

Inverse spectral problems for Jacobi matrices play an important role in the study of nonlinear discrete dynamical systems such as Toda lattices [4, 14, 16, 17]. Also, Jacobi matrices are known to be a very useful tool in the study of the moment problem, orthogonal polynomials, Padé approximation, and Jacobi continued fractions [1, 3, 7, 15, 18].

Consider a general $N \times N$ complex, symmetric, tri-diagonal matrix – a Jacobi matrix:

$$J = \begin{bmatrix} b_0 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & b_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{N-3} & a_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-3} & b_{N-2} & a_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-2} & b_{N-1} \end{bmatrix}, \tag{1.1}$$

where for each n, a_n and b_n are arbitrary complex numbers such that a_n is different from zero:

$$a_n,\ b_n \in \mathbb{C},\ a_n \neq 0. \tag{1.2}$$

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Such a matrix can be viewed in relation to a boundary value problem for a symmetric, second order, linear difference equation. The general inverse spectral problem is to reconstruct the matrix (or equivalently, the coefficients in the related second order linear difference equation) given some of its spectral characteristics (spectral data).

In the real case

$$a_n,\ b_n \in \mathbb{R},\ a_n \neq 0, \tag{1.3}$$

the matrix J is selfadjoint and in this case many versions of the inverse spectral problem for J have been investigated in the literature, see [8] and references given therein. In connection with the inverse spectral problems for selfadjoint Jacobi operators with matrix-valued coefficients, see [3, 5, 11, 12]. In the complex case (1.2), the matrix J is in general non-selfadjoint and recently the author [10] introduced the concept of *generalized spectral function* for matrices J of the form (1.1) with the entries satisfying (1.2), and solved the inverse problem consisting in the recovering the matrix from its generalized spectral function.

In the case of the infinite complex Jacobi matrix

$$J_\infty = \begin{bmatrix} b_0 & a_0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & \cdots \\ 0 & 0 & a_2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},\ a_n, b_n \in \mathbb{C},\ a_n \neq 0,\ (n = 0, 1, 2, \ldots),$$

the Favard theorem (see [1, 2, 7, 9, 13, 18]) states that there is a unique linear functional $\Omega : \mathbb{C}[\lambda] \to \mathbb{C}$ on the linear space $\mathbb{C}[\lambda]$ of all polynomials in λ with complex coefficients such that

$$\langle \Omega, P_m P_n \rangle = \delta_{mn},\ m, n \in \{0, 1, 2, \ldots\},$$

where $\langle \Omega, G \rangle$ denotes the value of Ω on the element (polynomial) $G(\lambda)$, δ_{mn} is the Kronecker delta, and $\{P_n(\lambda)\}_{n=0}^{\infty}$ is the unique solution of the recursion relation

$$\begin{gathered} b_0 y_0 + a_0 y_1 = \lambda y_0, \\ a_{n-1} y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n\ (n = 1, 2, 3, \ldots), \end{gathered}$$

satisfying the initial condition $y_0 = 1$. The functional Ω is also called an *orthogonality functional* for the polynomials $P_n(\lambda)$, $n = 0, 1, 2, \ldots$, or a *generalized spectral function* (see [9]) for the matrix J.

It is substantial to note that in the case of the finite complex Jacobi matrix (1.1) our definition of a generalized spectral function (orthogonality functional) given below contains, besides (2.5), an extra condition of the form (2.6). This circumstance implies, in contrast to infinite complex Jacobi matrices, the extra condition (iii) in Theorem 2.3 and the extra condition $D_N = 0$ in Theorem 2.4.

In general, very little is known about the structure of generalized spectral functions. For the structure of generalized spectral functions for some classes of infinite Jacobi matrices see [6] and references given therein. In the present paper, we describe explicitly the structure of the generalized spectral function for an arbitrary finite complex Jacobi matrix and in this way we define the concept of spectral data for matrices (1.1). Following this, the inverse problem with respect to the spectral data for the matrix J is completely explored.

2. Generalized spectral function and the inverse problem

Given a matrix J of the form (1.1) with entries satisfying (1.2), consider the eigenvalue problem $Jy = \lambda y$ for a column vector $y = \{y_n\}_{n=0}^{N-1}$, that is equivalent to the second order linear difference equation

(2.1) $$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n,\ n \in \{0, 1, \ldots, N-1\},\ a_{-1} = a_{N-1} = 1,$$

for $\{y_n\}_{n=-1}^{N}$, with the boundary conditions

(2.2) $$y_{-1} = y_N = 0.$$

Denote by $\{P_n(\lambda)\}_{n=-1}^{N}$ the solution of equation (2.1) satisfying the initial conditions

(2.3) $$y_{-1} = 0,\ y_0 = 1.$$

Using (2.3), we can find recurrently from equation (2.1) the quantities $P_n(\lambda)$ for $n = 1, 2, \ldots,\ N;\ P_n(\lambda)$ is a polynomial in λ of degree n. It turns out that the equality

(2.4) $$\det(J - \lambda I) = (-1)^N a_0 a_1 \cdots a_{N-2} P_N(\lambda)$$

holds, so that the eigenvalues of the matrix J coincide with the zeros of the polynomial $P_N(\lambda)$.

For any nonnegative integer m, denote by $\mathbb{C}_m[\lambda]$ the ring of all polynomials in λ of degree $\leq m$ with complex coefficients. A mapping $\Omega : \mathbb{C}_m[\lambda] \to \mathbb{C}$ is called a *linear functional* if for any $G(\lambda), H(\lambda) \in \mathbb{C}_m[\lambda]$ and $\alpha \in \mathbb{C}$, we have

$$\langle \Omega, G + H \rangle = \langle \Omega, G \rangle + \langle \Omega, H \rangle \quad \text{and} \quad \langle \Omega, \alpha G \rangle = \alpha \langle \Omega, G \rangle,$$

where $\langle \Omega, G \rangle$ denotes the value of Ω on the element (polynomial) $G(\lambda)$.

In [10], the following theorem is proved.

2.1. Theorem. *There exists a unique linear functional* $\Omega : \mathbb{C}_{2N}[\lambda] \to \mathbb{C}$ *such that the relations*

(2.5) $$\langle \Omega, P_m P_n \rangle = \delta_{mn},\ m, n \in \{0, 1, \ldots, N-1\},$$

(2.6) $$\langle \Omega, P_m P_N \rangle = 0,\ m \in \{0, 1, \ldots, N\},$$

hold, where δ_{mn} *is the Kronecker delta.*

2.2. Definition. The linear functional Ω defined in Theorem 2.1 we call the generalized spectral function of the matrix J given in (1.1).

The inverse problem is stated as follows:

(1) To see if it is possible to reconstruct the matrix J, given its generalized spectral function Ω. If it is possible, to describe the reconstruction procedure.
(2) To find necessary and sufficient conditions for a given linear functional Ω on $\mathbb{C}_{2N}[\lambda]$ to be the generalized spectral function for some matrix J of the form (1.1) with entries belonging to the class (1.2).

This problem was solved by the author in [10] and the following results established.

2.3. Theorem. *In order for a given linear functional* Ω *defined on* $\mathbb{C}_{2N}[\lambda]$ *to be the generalized spectral function for some Jacobi matrix* J *of the form (1.1) with entries belonging to the class (1.2), it is necessary and sufficient that the following conditions be satisfied:*

(i) $\langle \Omega, 1 \rangle = 1$ *(normalization condition);*
(ii) *If, for some polynomial* $G(\lambda)$, $\deg G(\lambda) \leq N-1$,

$$\langle \Omega, GH \rangle = 0$$

for all polynomials $H(\lambda)$, $\deg H(\lambda) = \deg G(\lambda)$, *then* $G(\lambda) \equiv 0$*;*

(iii) *There exists a polynomial* $T(\lambda)$ *of degree* N *such that*

$$\langle \Omega, GT)\rangle = 0$$

for all polynomials $G(\lambda)$ *with* $\deg G(\lambda) \le N$.

Let us set

$$s_l = \langle \Omega, \lambda^l \rangle,\ l = 0, 1, \ldots, 2N, \tag{2.7}$$

(the "power moments" of the functional Ω), and introduce the determinants

$$D_n = \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n} \end{vmatrix},\ n = 0, 1, \ldots, N. \tag{2.8}$$

If we express the conditions (ii) and (iii) of Theorem 2.3 in terms of the coefficients of the polynomials $G(\lambda)$, $H(\lambda)$ and $T(\lambda)$, then we get that Theorem 2.3 is equivalent to the following theorem.

2.4. Theorem. *In order for a given linear functional* Ω *defined on* $\mathbb{C}_{2N}[\lambda]$ *to be the generalized spectral function for some Jacobi matrix* J *of the form* (1.1)*, with entries belonging to the class* (1.2)*, it is necessary and sufficient that*

$$D_0 = 1,\ D_n \neq 0\ (n = 1, 2, \ldots, N-1),\ \textit{and}\ D_N = 0, \tag{2.9}$$

where D_n *is defined by* (2.8) *and* (2.7)*.*

If the conditions of Theorem 2.3 or, equivalently, the conditions of Theorem 2.4 are satisfied, then the entries a_n, b_n of the matrix J for which Ω is the generalized spectral function, are recovered by the formulas

$$a_n = \pm \left(D_{n-1} D_{n+1}\right)^{1/2} D_n^{-1},\ n \in \{0, 1, \ldots, N-2\},\ D_{-1} = 1, \tag{2.10}$$

$$b_n = \Delta_n D_n^{-1} - \Delta_{n-1} D_{n-1}^{-1},\ n \in \{0, 1, \ldots, N-1\},\ \Delta_{-1} = 0,\ \Delta_0 = s_1, \tag{2.11}$$

where D_n is defined by (2.8) and (2.7), and Δ_n is the determinant obtained from the determinant D_n by replacing in D_n the last column by the column with the components $s_{n+1}, s_{n+2}, \ldots, s_{2n+1}$.

2.5. Remark. It follows from the above solution of the inverse problem that the matrix (1.1) is not uniquely restored from the generalized spectral function. This is linked with the fact that the a_n are determined from (2.10) uniquely up to a sign. To ensure that the inverse problem is uniquely solvable, we have to specify additionally a sequence of signs $+$ and $-$. Namely, let $\{\sigma_1, \sigma_2, \ldots, \sigma_{N-1}\}$ be a given finite sequence, where for each $n \in \{1, 2, \ldots, N-1\}$ the σ_n is $+$ or $-$. We have 2^{N-1} such different sequences. Now to determine a_n uniquely from (2.10) for $n \in \{0, 1, \ldots, N-2\}$ we can choose the sign σ_n when extracting the square root. In this way we get precisely 2^{N-1} distinct Jacobi matrices possessing the same generalized spectral function. The inverse problem is solved uniquely from the data consisting of Ω and a sequence $\{\sigma_1, \sigma_2, \ldots, \sigma_{N-1}\}$ of signs $+$ and $-$.

3. Structure of the generalized spectral function, spectral data

Given a Jacobi matrix J of the form (1.1) with entries (1.2), consider the second order linear difference equation

$$a_{n-1} y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n,\ n \in \{0, 1, \ldots, N-1\},\ a_{-1} = a_{N-1} = 1, \tag{3.1}$$

where $\{y_n\}_{n=-1}^N$ is a desired solution. Denote by $\{P_n(\lambda)\}_{n=-1}^N$ and $\{Q_n(\lambda)\}_{n=-1}^N$ the solutions of this equation satisfying the initial conditions

$$P_{-1}(\lambda) = 0,\ P_0(\lambda) = 1; \tag{3.2}$$

$$Q_{-1}(\lambda) = -1,\ Q_0(\lambda) = 0. \tag{3.3}$$

For each $n \geq 0$, $P_n(\lambda)$ is a polynomial of degree n and is called a *polynomial of first kind* (note that $P_n(\lambda)$ is the same polynomial as above in Section 2), and $Q_n(\lambda)$ is a polynomial of degree $n-1$ and is known as a *polynomial of second kind.*

Let us set

$$M(\lambda) = -\frac{Q_N(\lambda)}{P_N(\lambda)}. \tag{3.4}$$

3.1. Lemma. *The entries $R_{nm}(\lambda)$ of the matrix $R(\lambda) = (J - \lambda I)^{-1}$ (resolvent of J) are of the form*

$$R_{nm}(\lambda) = \begin{cases} P_n(\lambda)[Q_m(\lambda) + M(\lambda)P_m(\lambda)], & 0 \leq n \leq m \leq N-1, \\ P_m(\lambda)[Q_n(\lambda) + M(\lambda)P_n(\lambda)], & 0 \leq m \leq n \leq N-1. \end{cases} \tag{3.5}$$

Proof. It is straightforward to check that the quantities $R_{nm} = R_{nm}(\lambda)$ defined by (3.5) satisfy the equations

$$\begin{aligned} &b_0 R_{0m} + a_0 R_{1m} - \lambda R_{0m} = \delta_{0m}, \\ &a_{n-1}R_{n-1,m} + b_n R_{nm} + a_n R_{n+1,m} - \lambda R_{nm} = \delta_{nm},\ n = 1, 2, \ldots, N-2, \\ &a_{N-2}R_{N-2,m} + b_{N-1}R_{N-1,m} - \lambda R_{N-1,m} = \delta_{N-1,m}, \\ &m = 0, 1, \ldots, N-1, \end{aligned}$$

where δ_{mn} is the Kronecker delta. Hence it follows that the column vector $y = \{y_n\}_{n=0}^{N-1}$ defined by the formula

$$y_n = \sum_{m=0}^{N-1} R_{nm}(\lambda) f_m$$

for any given column vector $f = \{f_n\}_{n=0}^{N-1}$, satisfies the equation

$$Jy = \lambda y + f.$$

This completes the proof. □

3.2. Lemma. *For any vector $f = \{f_n\}_{n=0}^{N-1} \in \mathbb{C}^N$ and any $n \in \{0, 1, \ldots, N-1\}$, the representation*

$$\sum_{n=0}^{N-1} R_{nm}(\lambda) f_m = -\frac{f_n}{\lambda} + r_n(\lambda) \tag{3.6}$$

holds and there exist sufficiently large positive constants Λ and C such that

$$|r_n(\lambda)| \leq \frac{C}{|\lambda|^2} \tag{3.7}$$

for all $n \in \{0, 1, \ldots, N-1\}$ and all $\lambda \in \mathbb{C}$ with $|\lambda| \geq \Lambda$.

Proof. Let us supply the space $\mathbb{C}^N$ with the norm

$$\|f\| = \max_{0 \leq n \leq N-1} |f_n|,$$

and let $\|J\|$ denote the matrix norm of J corresponding to that vector norm. Since

$$J - \lambda I = -\lambda \left(I - \frac{1}{\lambda} J \right),$$

we have

$$(J-\lambda I)^{-1} = -\frac{1}{\lambda}\left(I-\frac{J}{\lambda}\right)^{-1} = -\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{J}{\lambda}\right)^{k} = -\frac{I}{\lambda}+K(\lambda),$$

where

$$K(\lambda) = -\sum_{k=1}^{\infty}\frac{J^k}{\lambda^{k+1}},$$

and hence, for $|\lambda| \geq \|J\|+1$,

$$\begin{aligned}\|K(\lambda)\| &\leq \sum_{k=1}^{\infty}\frac{\|J\|^k}{|\lambda|^{k+1}}\\ &= \frac{\|J\|}{|\lambda|^2}\sum_{k=0}^{\infty}\frac{\|J\|^k}{|\lambda|^k}\\ &\leq \frac{\|J\|}{|\lambda|^2}\sum_{k=0}^{\infty}\frac{\|J\|^k}{(\|J\|+1)^k}\\ &= \frac{1}{|\lambda|^2}\|J\|(\|J\|+1).\end{aligned}$$

Now taking in the equality

$$(J-\lambda I)^{-1}f = -\frac{f}{\lambda}+K(\lambda)f$$

the n th component of both sides, we get

$$\sum_{m=0}^{N-1}R_{nm}(\lambda)f_m = -\frac{f_n}{\lambda}+[K(\lambda)f]_n$$

while, for the n th component $[K(\lambda)f]_n$ of the vector $K(\lambda)f$ we have

$$|[K(\lambda)f]_n| \leq \|K(\lambda)f\| \leq \|K(\lambda)\|\,\|f\| \leq \frac{1}{|\lambda|^2}\|J\|(\|J\|+1)\,\|f\|.$$

Thus, the lemma is true with the constants $\Lambda = \|J\|+1$ and $C = \|J\|(\|J\|+1)\,\|f\|$. □

Let Ω be the generalized spectral function of the matrix J, defined above in Section 2. The following theorem describes the structure of Ω.

3.3. Theorem. *Let $\lambda_1,\dots,\lambda_p$ be all the distinct eigenvalues of the matrix J and $m_1,\dots,m_p$ their multiplicities, respectively, as roots of the characteristic polynomial (2.4). There exist complex numbers β_{kj}, $(j=1,\dots,m_k,\ k=1,\dots,p)$ uniquely determined by the matrix J such that for any polynomial $G(\lambda)\in\mathbb{C}_{2N}[\lambda]$ the formula*

$$\langle\Omega,G\rangle = \sum_{k=1}^{p}\sum_{j=1}^{m_k}\frac{\beta_{kj}}{(j-1)!}G^{(j-1)}(\lambda_k), \tag{3.8}$$

holds, where $G^{(n)}(\lambda)$ denotes the n th order derivative of $G(\lambda)$ with respect to λ.

Proof. Let f be an arbitrary element (column vector) of $\mathbb{C}^N$, with components $f_0,f_1,\dots,f_{N-1}$. Writing (3.6) for this vector f and then integrating both sides, we obtain for each $n\in\{0,1,\dots,N-1\}$,

$$f_n = -\frac{1}{2\pi i}\oint_{\Gamma_r}\left\{\sum_{m=0}^{N-1}R_{nm}(\lambda)f_m\right\}d\lambda+\frac{1}{2\pi i}\oint_{\Gamma_r}r_n(\lambda)d\lambda, \tag{3.9}$$

where r is a sufficiently large positive number and Γ_r is the circle in the λ-plane of radius r centered at the origin.

Denote by $\lambda_1, \ldots, \lambda_p$ all the distinct roots of the polynomial $P_N(\lambda)$ (which coincides by (2.4) with the characteristic polynomial of the matrix J up to a constant factor) and by $m_1, \ldots, m_p$ their multiplicities, respectively:

$$P_N(\lambda) = c(\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_p)^{m_p}, \tag{3.10}$$

where c is a constant. We have $1 \leq p \leq N$ and $m_1 + \cdots + m_p = N$. By (3.10), we can rewrite the rational function $Q_N(\lambda)/P_N(\lambda)$ as the sum of partial fractions:

$$\frac{Q_N(\lambda)}{P_N(\lambda)} = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \frac{\beta_{kj}}{(\lambda - \lambda_k)^j}, \tag{3.11}$$

where β_{kj} are some uniquely determined complex numbers depending on the matrix J. Substituting (3.5) in (3.9) and taking into account (3.4), (3.11), and (3.7), we get, applying the residue theorem and passing then to the limit as $r \to \infty$,

$$f_n = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \frac{\beta_{kj}}{(j-1)!} \left\{ \frac{d^{j-1}}{d\lambda^{j-1}} \left[F(\lambda) P_n(\lambda) \right] \right\}_{\lambda = \lambda_k}, \ n \in \{0, 1, \ldots, N-1\}, \tag{3.12}$$

where

$$F(\lambda) = \sum_{m=0}^{N-1} f_m P_m(\lambda). \tag{3.13}$$

Now define on $\mathbb{C}_{2N}[\lambda]$ the functional Ω by the formula

$$\langle \Omega, G \rangle = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \frac{\beta_{kj}}{(j-1)!} G^{(j-1)}(\lambda_k), \ G(\lambda) \in \mathbb{C}_{2N}[\lambda]. \tag{3.14}$$

Then formula (3.12) can be written in the form

$$f_n = \langle \Omega, F P_n \rangle, \ n \in \{0, 1, \ldots, N-1\}. \tag{3.15}$$

From here by (3.13) and the arbitrariness of $\{f_m\}_{m=0}^{N-1}$ it follows that the "orthogonality" relation

$$\langle \Omega, P_m P_n \rangle = \delta_{mn}, \ m, n \in \{0, 1, \ldots, N-1\}, \tag{3.16}$$

holds. Further, in virtue of (3.10) and (3.14) we have also

$$\langle \Omega, P_m P_N \rangle = 0, \ m \in \{0, 1, \ldots, N\}. \tag{3.17}$$

These mean, by Theorem 2.1, that the generalized spectral function of the matrix J has the form (3.14). $\square$

3.4. Definition. The collection of quantities

$$\{\lambda_k, \beta_{kj} \ (j = 1, \ldots, m_k, \ k = 1, \ldots, p)\},$$

determining the structure of the generalized spectral function of the matrix J according to Theorem 3.3, we call the *spectral data of the matrix* J. For each $k \in \{1, \ldots, p\}$ the sequence

$$\{\beta_{k1}, \ldots, \beta_{km_k}\}$$

we call the *normalizing chain* (of the matrix J) associated with the eigenvalue λ_k.

If we delete the first row and the first column of the matrix J given in (1.1), then we get the new matrix

$$J^{(1)} = \begin{bmatrix} b_0^{(1)} & a_0^{(1)} & 0 & \cdots & 0 & 0 & 0 \\ a_0^{(1)} & b_1^{(1)} & a_1^{(1)} & \cdots & 0 & 0 & 0 \\ 0 & a_1^{(1)} & b_2^{(1)} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{N-4}^{(1)} & a_{N-4}^{(1)} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-4}^{(1)} & b_{N-3}^{(1)} & a_{N-3}^{(1)} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-3}^{(1)} & b_{N-2}^{(1)} \end{bmatrix},$$

where

$$a_n^{(1)} = a_{n+1},\ n \in \{0, 1, \dots, N-3\},$$
$$b_n^{(1)} = b_{n+1},\ n \in \{0, 1, \dots, N-2\}.$$

The matrix $J^{(1)}$ is called the *first truncated matrix* (with respect to the matrix J).

3.5. Theorem. *The normalizing numbers β_{kj} of the matrix J can be calculated by decomposing the rational function*

$$-\frac{\det(J^{(1)} - \lambda I)}{\det(J - \lambda I)}$$

into partial fractions.

Proof. Let us denote the polynomials of the first and the second kinds, corresponding to the matrix $J^{(1)}$, by $P_n^{(1)}(\lambda)$ and $Q_n^{(1)}(\lambda)$, respectively. It is easily seen that

$$P_n^{(1)}(\lambda) = a_0 Q_{n+1}(\lambda),\ n \in \{0, 1, \dots, N-1\}, \tag{3.18}$$

$$Q_n^{(1)}(\lambda) = \frac{1}{a_0}\{(\lambda - b_0)Q_{n+1}(\lambda) - P_{n+1}(\lambda)\},\ n \in \{0, 1, \dots, N-1\}. \tag{3.19}$$

Indeed, both sides of each of these equalities are solutions of the same difference equation

$$a_{n-1}^{(1)} y_{n-1} + b_n^{(1)} y_n + a_n^{(1)} y_{n+1} = \lambda y_n,\ n \in \{0, 1, \dots, N-2\},\ a_{N-2}^{(1)} = 1,$$

and the sides coincide for $n = -1$ and $n = 0$. Therefore the equalities hold by the uniqueness theorem for solutions.

Consequently, taking into account (2.4) and using (3.18), we have

$$\begin{aligned} \det(J^{(1)} - \lambda I) &= (-1)^{N-1} a_0^{(1)} a_1^{(1)} \cdots a_{N-3}^{(1)} P_{N-1}^{(1)}(\lambda) \\ &= (-1)^{N-1} a_1 \cdots a_{N-2} a_0 Q_N(\lambda). \end{aligned}$$

Comparing this with (2.4), we get

$$\frac{Q_N(\lambda)}{P_N(\lambda)} = -\frac{\det(J^{(1)} - \lambda I)}{\det(J - \lambda I)},$$

so that the statement of the theorem follows from (3.11). □

4. Inverse problem for the spectral data

By the inverse spectral problem is meant the problem of recovering the matrix J, i.e. its entries a_n and b_n, from the spectral data.

4.1. Theorem. *Let an arbitrary collection of complex numbers*

$$\{\lambda_k, \beta_{kj} \ (j = 1, \dots, m_k, \ k = 1, \dots, p)\} \tag{4.1}$$

be given, where $\lambda_1, \lambda_2, \dots, \lambda_p$, $(1 \le p \le N)$, *are distinct,* $1 \le m_k \le N$ *and* $m_1 + \dots + m_p = N$. *In order for this collection to be the spectral data for some Jacobi matrix* J *of the form (1.1) with entries belonging to the class (1.2), it is necessary and sufficient that the following two conditions be satisfied:*

(i) $\sum_{k=1}^{p} \beta_{k1} = 1$;
(ii) $D_n \neq 0$, *for* $n \in \{1, 2, \dots, N-1\}$ *and* $D_N = 0$, *where* D_n *is defined by (2.8) in which*

$$s_l = \sum_{k=1}^{p} \sum_{j=1}^{n_{kl}} \binom{l}{j-1} \beta_{kj} \lambda_k^{l-j+1}, \tag{4.2}$$

$n_{kl} = \min\{m_k, l+1\}$ *and* $\binom{l}{j-1}$ *is a binomial coefficient.*

Proof. The necessity of conditions of the theorem follows from Theorem 2.4, because the generalized spectral function of the matrix J is defined by the spectral data according to formula (3.8) and therefore the quantity (4.2) coincides with $\langle \Omega, \lambda^l \rangle$. Besides,

$$\sum_{k=1}^{p} \beta_{k1} = \langle \Omega, 1 \rangle = s_0 = D_0.$$

Note that the condition (iii) of Theorem 2.3 holds with

$$T(\lambda) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_p)^{m_p}. \tag{4.3}$$

Let us prove the sufficiency. Assume that we have a collection of quantities (4.1) satisfying the conditions of the theorem. Using these data we construct the functional Ω on $\mathbb{C}_{2N}[\lambda]$ by formula (3.8). Then this functional Ω satisfies the conditions of Theorem 2.4 and therefore there exists a matrix J of the form (1.1), (1.2) for which Ω is the generalized spectral function. Now we have to prove that the collection (4.1) is the spectral data for the recovered matrix J.

For this purpose we define the polynomials $P_{-1}(\lambda), P_0(\lambda), \dots, P_N(\lambda)$ as the solution of equation (3.1), constructed by means of the matrix J, under the initial conditions (3.2). Then the relations (2.5), (2.6) and the equalities

$$a_n = \langle \Omega, \lambda P_n P_{n+1} \rangle, \ n \in \{0, 1, \dots, N-2\}, \tag{4.4}$$

$$b_n = \langle \Omega, \lambda P_n^2 \rangle, \ n \in \{0, 1, \dots, N-1\} \tag{4.5}$$

hold. We show that (3.8) holds, which will mean, in particular, that $\lambda_1, \dots, \lambda_p$ are eigenvalues of the matrix J with multiplicities $m_1, \dots, m_p$, respectively.

Let $T(\lambda)$ be defined by (4.3). Let us show that there exists a constant c such that

$$a_{N-2} P_{N-2}(\lambda) + b_{N-1} P_{N-1}(\lambda) + c T(\lambda) = \lambda P_{N-1}(\lambda) \tag{4.6}$$

for all $\lambda \in \mathbb{C}$. If we prove this, then from here and (3.1) with $y_k = P_k(\lambda)$ and $n = N-1$, we get that $P_N(\lambda) = c T(\lambda)$.

Since $\deg P_n(\lambda) = n$, $(0 \le n \le N-1)$ and $\deg T(\lambda) = m_1 + \cdots + m_p = N$, the polynomials $P_0(\lambda), \ldots, P_{N-1}(\lambda), T(\lambda)$ form a basis of the linear space of all polynomials of degree $\le N$. Therefore we have the decomposition

$$\lambda P_{N-1}(\lambda) = cT(\lambda) + \sum_{n=0}^{N-1} c_n P_n(\lambda), \tag{4.7}$$

where $c, c_0, c_1, \ldots, c_{N-1}$ are some constants. By (4.3) and (3.8) it follows that

$$\langle \Omega, TP_n \rangle = 0, \quad n \in \{0, 1, \ldots, N\}.$$

Hence, taking into account the relations (2.5), (2.6) and (4.4), (4.5), we find from (4.7) that

$$c_n = 0 \ (0 \le n \le N-3), \quad c_{N-2} = a_{N-2}, \ c_{N-1} = b_{N-1}.$$

So (4.6) is shown.

It remains to show that for each $k \in \{1, \ldots, p\}$, the sequence $\{\beta_{k1}, \ldots, \beta_{km_k}\}$ is the normalizing chain of the matrix J associated with the eigenvalue λ_k. Since we have already shown that λ_k is an eigenvalue of the matrix J of multiplicity m_k, the normalizing chain of J associated with the eigenvalue λ_k has the form $\{\widetilde{\beta}_{k1}, \ldots, \widetilde{\beta}_{km_k}\}$. Therefore for $\langle \Omega, G \rangle$ we have an equality of the form (3.8) in which β_{kj} is replaced by $\widetilde{\beta}_{kj}$. Subtracting these two equalities for $\langle \Omega, G \rangle$ term by term we get that

$$\sum_{k=1}^{p} \sum_{j=1}^{m_k} \frac{\beta_{kj} - \widetilde{\beta}_{kj}}{(j-1)!} G^{(j-1)}(\lambda_k) = 0 \text{ for all } G(\lambda) \in \mathbb{C}_{2N}[\lambda].$$

Since the values $G^{(j-1)}(\lambda_k)$ can be arbitrary numbers (by virtue of the Hermite general interpolation problem), we get that $\beta_{kj} = \widetilde{\beta}_{kj}$ for all k and j. $\square$

Under the conditions of Theorem 4.1, the entries a_n and b_n of the matrix J for which the collection (4.1) is spectral data, are recovered from the formulas (2.10), (2.11).

References

[1] Akhiezer, N. I. *The Classical Moment Problem and Some Related Questions in Analysis* (Hafner, New York, 1965).

[2] Beckermann, B. *Complex Jacobi matrices*, J. Comput. Appl. Math. **127**, 17–65, 2001.

[3] Berezanskii, Yu. M. *Expansion in Eigenfunctions of Selfadjoint Operators* (Translations of Mathematical Monographs, Vol. 17, American Mathematical Society, Providence, R.I., 1968).

[4] Berezanskii, Yu. M. *The integration of semi-infinite Toda chain by means of inverse spectral problem*, Rep. Math. Phys. **24**, 21–47, 1986.

[5] Brüning, J., Chelkak, D. and Korotyaev, E. *Inverse spectral analysis for finite matrix-valued Jacobi operators*, preprint, arXiv: math/0607809.

[6] Charris, J. A. and Soriano, F. H. *Complex and distributional weights for sieved ultraspherical polynomials*, Internat. J. Math. and Math. Sci. **19**, 229–242, 1996.

[7] Chihara, T. S. *An Introduction to Orthogonal Polynomials* (Gordon and Breach, New York, 1978).

[8] Chu, M. T. and Golub, G. H. *Inverse Eigenvalue Problems: Theory, Algorithms, and Applications* (Oxford University Press, New York, 2005).

[9] Guseinov, G. Sh. *Determination of an infinite non-selfadjoint Jacobi matrix from its generalized spectral function*, Mat. Zametki **23**, 237–248, 1978; English transl. in Math. Notes **23**, 130–136, 1978.

[10] Guseinov, G. Sh. *An inverse spectral problem for complex Jacobi matrices*, Communications in Nonlinear Science and Numerical Simulation. (To appear)

[11] Korotyaev, E. and Kutsenko, A. *Lyapunov functions of periodic matrix-valued Jacobi operators,* in *Spectral Theory of Differential Operators: M. Sh. Birman 80 th Anniversary Collection* (American Mathematical Society Translations - Series 2, Vol. 225, Amer. Math. Soc., Providence, RI, 2008).

[12] Korotyaev, E. and Kutsenko, A. *Borg-type uniqueness Theorems for periodic Jacobi operators with matrix-valued coefficients,* Proc. Amer. Math. Soc. **137**, 1989–1996, 2009.

[13] Marcellan, F. and Alvarez-Nodarse, R. *On the "Favard theorem" and its extensions*, J. Comput. Appl. Math. **127**, 231–254, 2001.

[14] Moser, J. K. *Various aspects of integrable Hamiltonian systems*, Dynamical Systems, Progr. Math., Vol. 8, Birkhäuser, Boston, 233–289, 1980.

[15] Nikishin, E. M. and Sorokin, V. N. *Rational Approximations and Orthogonality* (Translations of Mathematical Monographs, Vol. 92, American Mathematical Society, Providence, R.I., 1991).

[16] Teschl, G. *Jacobi Operators and Completely Integrable Nonlinear Lattices* (Mathematical Surveys and Monographs, Vol. 72, American Mathematical Society, Providence, R.I., 2000).

[17] Toda, M. *Theory of Nonlinear Lattices* (Springer-Verlag, New York, 1981).

[18] Wall, H. S. *Analytic Theory of Continued Fractions* (Chelsea, New York, 1973).