ALTERING DISTANCE AND COMMON FIXED POINTS UNDER IMPLICIT RELATIONS

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Abstract

We prove a common fixed point theorem for two pairs of occasionally weakly compatible mappings satisfying an implicit relation of a new type that involves an altering distance, so generalizing a theorem of Aliouche and Djoudi (*Common fixed point theorems for mappings satisfying an implicit relation without decreasing assumption*, Hacettepe J. Math. Stat. **36** (1), 11–18, 2007). As a consequence, we obtain a fixed point theorem for two pairs of mappings satisfying an implicit relation of integral type, providing a strong generalization to a known result from Kumar, Chugh and Kumar (*Fixed point theorems for compatible mappings satisfying a contractive condition of integral type*, Soochow J. Math. **33** (2), 181–185, 2007).

1. Introduction

Let S and T be self-mappings of a metric space (X, d). Jungck [6] defined S and T to be *compatible* if $\lim_{n \to \infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t$ for some $t \in X$.

1.1. Definition. Let X be a non-empty set and S, T self-mappings of X. A point $x \in X$ is called a *coincidence point* of S and T if $Sx = Tx$. A point $w \in X$ is said to be a *point of coincidence* of S and T if there exists $x \in X$ so that $w = Sx = Tx$.

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In 1994, Pant [14] introduced the notion of pointwise R-weakly commuting mappings. It is proved in [15] that pointwise R-weak commutativity is equivalent to commutativity at coincidence points. Jungck [7] defined S and T to be *weakly compatible* if $Sx = Tx$ implies $STx = TSx$. Thus, S and T are weakly compatible if and only if S and T are pointwise R-weakly commuting.

Quite recently, Al-Thagafi and Shahzad [3] introduced the concept of occasionally weakly compatible mappings.

1.2. Definition. [3] Two self-mappings S and T of a non-empty set X are said to be *occasionally weakly compatible* (*owc*) if and only if there exists a coincidence point of S and T at which S and T commute.

1.3. Remark. The notion of weakly compatible mappings is a proper generalization of that of compatible mappings [16]. Every two weakly compatible mappings with coincidence points are owc, but the converse is not true (Example [3]).

Some fixed point theorems for owc mappings are proved in [8], [19], [13] and other papers.

1.4. Lemma. [8] *Let X be a non-empty set, and let f and g be owc self-mappings of X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

During the past decade, Banach-type contractive conditions assumed in fixed point theorems have been generalized by using, among others, implicit relations [17] and contractive conditions of integral type [4]. In [17] a general fixed point theorem for compatible mappings satisfying an implicit relation was proved and in [5] the results from [17] were improved by relaxing the compatibility to weak compatibility. Quite recently, Aliouche and Djoudi [2] proved some common fixed point theorems for two pairs of weakly compatible mappings under implicit relations.

In this paper we extend a result of [2] to owc mappings, using a more general implicit condition that involves an altering distance. Noting that the main result of [4] can be expressed in terms of an altering distance, we also obtain a common fixed point theorem for two pairs of owc mappings satisfying an implicit condition of integral type. This theorem is a strong generalization of a result of [12] and can be used to produce several new fixed point results for owc mappings satisfying conditions of integral type. The idea of reducing the study of fixed points for mappings satisfying contractive conditions of integral type to the study of fixed points for mappings satisfying contractive conditions involving an altering distance could be used further in unifying and generalizing known fixed point results involving contractive conditions of integral type.

2. Preliminaries

In [4], Branciari established the following fixed point theorem, which opened the way to the study of mappings satisfying a contractive condition of integral type.

2.1. Theorem. [4] *Let (X, d) be a complete metric space, $c \in (0, 1)$ and $f : X \to X$ a mapping such that, for each $x, y \in X$,*

$$(2.1) \qquad \int_0^{d(fx, fy)} \varphi(t)dt \le c \int_0^{d(x,y)} \varphi(t)dt,$$

where $\varphi : [0, +\infty) \to [0, +\infty]$ is a Lebesgue-measurable mapping which is summable (i.e., with finite integral) on each compact subset of $[0, +\infty)$, such that, for each $\varepsilon > 0$,

$\int_0^\varepsilon \varphi(t)dt > 0$. *Then f has a unique fixed point $z \in X$ such that, for each $x \in X$,* $\lim_{n\to\infty} f^n x = z$.

Theorem 2.1 has been generalized in several papers, e.g. it has been extended to a pair of compatible mappings in [12].

2.2. Theorem. [12] *Let f and g be compatible self-mappings of a complete metric space (X, d), with g continuous, satisfying the following conditions:*

1. *$f(X) \subset g(X)$;*
2. *$\int_0^{d(fx,fy)} \varphi(t)\, dt \le c \int_0^{d(gx,gy)} \varphi(t)\, dt$ for some $c \in (0, 1)$, whenever $x, y \in X$.*

Here, $\varphi : [0, +\infty) \to [0, +\infty)$ satisfies the assumptions from Theorem 2.1. Then f and g have a unique common fixed point.

2.3. Definition. Let X be a non-empty set. A *symmetric* on X is a non-negative real-valued function D on $X \times X$ such that

(i) $D(x, y) = 0$ if and only if $x = y$, and
(ii) $D(x, y) = D(y, x)$ for all $x, y \in X$.

Some fixed point theorems in symmetric spaces for compatible and weakly compatible mappings are proved in [1], [11], [20], [17] and other papers.

Let $D(x, y) = \int_0^{d(x,y)} \varphi(t)dt$, where φ is as in Theorem 2.1. In [19] and [13] it is proved that D is a symmetric on X and the study of some fixed point problems for mappings satisfying contractive conditions of integral type in metric spaces is reduced to the study of fixed point problems in symmetric spaces.

In [10] Khan *et al.* a new type of fixed point problem was approached, where the control function is an altering distance.

2.4. Definition. An *altering distance* is a function $\psi : [0, +\infty) \to [0, +\infty)$ which is increasing, continuous and vanishes only at the origin.

Fixed point problems involving altering distances have also been studied in [18], [21] and [22].

The following lemma shows that contractive conditions of integral type can be interpreted as contractive conditions involving an altering distance.

2.5. Lemma. *Let $\varphi : [0, +\infty) \to [0, +\infty)$ be as in Theorem 2.1. Define $\Phi_0(t) = \int_0^t \varphi(\tau)d\tau$, $t \in [0, +\infty)$. Then Φ_0 is an altering distance.*

Proof. $\Phi_0 : [0, +\infty) \to [0, +\infty)$ is well-defined and increasing, since φ is Lebesgue-measurable, summable and positive. Moreover, $\Phi_0(0) = 0$ and $\Phi_0(t) > 0$ for every $t > 0$. The continuity of Φ_0 follows from the absolute continuity of the Lebesgue integral. $\square$

In the following, we will use an implicit relation involving six real non-negative arguments, that was introduced in [2].

2.6. Definition. [2]. Let $\mathcal{F}_a$ be the family of all real continuous functions $F : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(F1) There exists $h \in [0, 1)$ such that for all $u, v, w \ge 0$ with $F(u, v, v, u, w, 0) \le 0$ or $F(u, v, u, v, 0, w) \le 0$, we have $u \le hv$;
(F2) $F(u, u, 0, 0, u, u) > 0$ for every $u > 0$.

Note that, unlike in [17] and [5], $F \in \mathcal{F}_a$ is not supposed to satisfy any monotonicity conditions.

It is easy to see that $F : \mathbb{R}_+^6 \to \mathbb{R}$ belongs to $\mathcal{F}_a$ in each of the following cases:

2.7. Example. $F(t_1, \ldots, t_6) = t_1 - ct_2 - a(t_3 + t_4) - b\sqrt{t_5 t_6}$, where $c > 0$, $a, b \geq 0$ and $c + \max\{2a, b\} < 1$;

2.8. Example. $F(t_1, \ldots, t_6) = t_1 (t_1 - at_2 - bt_3 - ct_4) - dt_5 t_6$, where $a > 0$, $b, c, d \geq 0$ and $a + \max\{b + c, d\} < 1$;

2.9. Example. $F(t_1, \ldots, t_6) = t_1 - at_2 - bt_3 - ct_4 - d\min\{t_5, t_6\}$, where $a > 0$, $b, c, d \geq 0$ and $a + \max\{b + c, d\} < 1$;

2.10. Example. $F(t_1, \ldots, t_6) = t_1^2 - at_2^2 - b\frac{1}{1+t_3+t_4}\min\{t_5^2, t_6^2\}$, where $a > 0$, $b \geq 0$ and $a + b < 1$;

2.11. Example. $F(t_1, \ldots, t_6) = t_1^2 - at_2^2 + b\frac{t_3 t_4}{1+t_5+t_6}$, where $0 < a < 1$ and $b \geq 0$.

For other examples see [2].

Condition (F1) is satisfied with $h = \dfrac{a+c}{1-a}$ in Example 2.7, $h = \max\left\{\dfrac{a+c}{1-b}, \dfrac{a+b}{1-c}\right\}$ in Examples 2.8 and 2.9, and $h = \sqrt{a}$ in Examples 2.10 and 2.11. In Examples 2.7–2.11, Condition (F2) is satisfied, since $F(u, u, 0, 0, u, u)$ is equal respectively to $(1 - b - c)u$, $(1 - a - d)u^2$, $(1 - a - d)u$, $(1 - a - b)u^2$, and $(1 - a)u^2$.

3. Main results

We prove a common fixed point for two pairs of owc mappings satisfying an implicit relation involving an altering distance.

3.1. Theorem. *Let A, B, S and T be self-mappings of a metric space (X, d) satisfying*

(3.1) $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$,

and

(3.2) $F\left(\begin{array}{l} \psi(d(Ax, By)),\ \psi(d(Sx, Ty)),\ \psi(d(Sx, Ax)), \\ \quad \psi(d(Ty, By)),\ \psi(d(Sx, By)),\ \psi(d(Ty, Ax)) \end{array}\right) \leq 0$

for all $x, y \in X$, where $F \in \mathcal{F}_a$ and ψ is an altering distance. Assume that at least one of the sets $A(X)$, $B(X)$, $S(X)$ and $T(X)$ is a complete subspace of X. Then each of the pairs (A, S) and (B, T) has a unique point of coincidence. Moreover, if each of the pairs (A, S) and (B, T) is owc, then A, B, S and T have a unique common fixed point.

Proof. Pick $x_0 \in X$. Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$, we can define two sequences $\{x_n\}$, $\{y_n\}$ in X such that

(3.3) $y_{2n+1} = Tx_{2n+1} = Ax_{2n}$ and $y_{2n+2} = Sx_{2n+2} = Bx_{2n+1}$,

for every non-negative integer n. Let $\Psi(x, y) = \psi(d(x, y))$ for $(x, y) \in X \times X$. The function Ψ is continuous on $X \times X$.

Taking $x = x_{2n}$ and $y = x_{2n+1}$ in (3.2), we get by (3.3)

$$F\left(\begin{array}{l} \Psi(y_{2n+1},\ y_{2n+2}),\ \Psi(y_{2n}, y_{2n+1}),\ \Psi(y_{2n}, y_{2n+1}), \\ \quad \Psi(y_{2n+1}, y_{2n+2}),\ \Psi(y_{2n}, y_{2n+2}),\ 0 \end{array}\right) \leq 0.$$

Since $F \in \mathcal{F}_a$ and $F(u, v, v, u, w, 0) \leq 0$ for $u = \Psi(y_{2n+1}, y_{2n+2})$, $v = \Psi(y_{2n}, y_{2n+1})$ and $w = \Psi(y_{2n}, y_{2n+2})$, it follows that there exists $h \in [0, 1)$ such that

(3.4) $\Psi(y_{2n+1}, y_{2n+2}) \leq h\Psi(y_{2n}, y_{2n+1})$.

Similarly, taking $x = x_{2n+2}$ and $y = x_{2n+1}$ in (3.2), we get

$$F\left(\begin{array}{l} \Psi(y_{2n+3}, y_{2n+2}),\ \Psi(y_{2n+2}, y_{2n+1}),\ \Psi(y_{2n+2}, y_{2n+3}), \\ \quad \Psi(y_{2n+1}, y_{2n+2}),\ 0,\ \Psi(y_{2n+1}, y_{2n+3}) \end{array}\right) \leq 0.$$

Since $F \in \mathcal{F}_a$ and $F(u, v, u, v, 0, w) \leq 0$ for $u = \Psi(y_{2n+3}, y_{2n+2})$, $v = \Psi(y_{2n}, y_{2n+1})$ and $w = \Psi(y_{2n+1}, y_{2n+3})$, we infer that

(3.5) $\Psi(y_{2n+2}, y_{2n+3}) \leq h\Psi(y_{2n+1}, y_{2n+2})$ for every $n \geq 0$.

From (3.4) and (3.5) we conclude that $\Psi(y_{n+1}, y_{n+2}) \leq h\Psi(y_{n+1}, y_n)$, $n \geq 1$, hence

$$\Psi(y_n, y_{n+1}) \leq h^{n-1}\Psi(y_1, y_2) \text{ for every } n \geq 1.$$

The sequence $a_n := \psi(d(y_n, y_{n+1}))$, $n \geq 1$, of non-negative real numbers, converges to zero. Then there exists a bijection $\sigma : \mathbb{N}^* \to \mathbb{N}^*$ such that the sequence $\left\{a_{\sigma(n)}\right\}$ is non-increasing. Since ψ is increasing, it follows that the sequence $\left\{d(y_{\sigma(n)}, y_{\sigma(n)+1})\right\}$ of non-negative real numbers is non-increasing, hence it is convergent. Using the continuity of ψ we obtain $0 = \lim_{n \to \infty} \psi(d(y_{\sigma(n)}, y_{\sigma(n)+1})) = \psi(\lim_{n \to \infty} d(y_{\sigma(n)}, y_{\sigma(n)+1}))$. Since $\psi(t) = 0$ if and only if $t = 0$, it follows that $\lim_{n \to \infty} d(y_{\sigma(n)}, y_{\sigma(n)+1}) = 0$, therefore

(3.6) $\lim_{n \to \infty} d(y_n, y_{n+1}) = 0.$

Since (3.6) holds, it suffices to know that the subsequence $\{y_{2n}\}$ is Cauchy in order to prove that the sequence $\{y_n\}$ is Cauchy. Suppose that $\{y_{2n}\}$ is not Cauchy. Then there is $\varepsilon_0 > 0$ such that for each positive integer k there exist positive integers $m(k)$ and $n(k)$ with $k < n(k) < m(k)$, satisfying

(3.7) $d(y_{2n(k)}, y_{2m(k)-2}) < \varepsilon_0$ and $d(y_{2n(k)}, y_{2m(k)}) \geq \varepsilon_0.$

Moreover, we may assume that $m(k) < n(k+1)$ for each k.

As in [9, Theorem 2.2] we deduce that the sequences

$$\left\{d(y_{2n(k)}, y_{2m(k)})\right\}, \ \left\{d(y_{2n(k)}, y_{2m(k)-1})\right\} \text{ and } \left\{d(y_{2n(k)+1}, y_{2m(k)-1})\right\}$$

are convergent, and

$$\lim_{k \to \infty} d(y_{2n(k)}, y_{2m(k)}) = \varepsilon_0, \text{ and,}$$

(3.8) $\lim_{k \to \infty} d(y_{2n(k)}, y_{2m(k)-1}) = \lim_{k \to \infty} d(y_{2n(k)+1}, y_{2m(k)})$
$$= \lim_{k \to \infty} d(y_{2n(k)+1}, y_{2m(k)-1})$$
$$= \varepsilon_0$$

Indeed, (3.7) and the triangle inequality imply

$$\varepsilon_0 \leq d(y_{2n(k)}, y_{2m(k)}) < \varepsilon_0 + d(y_{2m(k)-2}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)})$$

for each $k \geq 1$. But

$$\lim_{k \to \infty} d(y_{2m(k)-2}, y_{2m(k)-1}) = \lim_{k \to \infty} d(y_{2m(k)-1}, y_{2m(k)}) = 0$$

by (3.6), hence $\lim_{k \to \infty} d(y_{2n(k)}, y_{2m(k)}) = \varepsilon_0$. Then $\lim_{k \to \infty} d(y_{2n(k)}, y_{2m(k)-1}) = \varepsilon_0$ since

$$\left|d(y_{2n(k)}, y_{2m(k)}) - d(y_{2n(k)}, y_{2m(k)-1})\right| \leq d(y_{2m(k)-1}, y_{2m(k)})$$

for all $k \geq 1$. Similarly, $\lim_{k \to \infty} d(y_{2n(k)+1}, y_{2m(k)}) = \varepsilon_0$ and $\lim_{k \to \infty} d(y_{2n(k)+1}, y_{2m(k)}) = \varepsilon_0$.

Setting $x = x_{2n(k)}$ and $y = x_{2m(k)-1}$ in (3.2) and taking account of (3.3) we obtain

$$F\left(\begin{matrix} \Psi(y_{2n(k)+1}, y_{2m(k)}), \ \Psi(y_{2n(k)}, y_{2m(k)-1}), \ \Psi(y_{2n(k)}, y_{2n(k)+1}), \\ \Psi(y_{2m(k)-1}, y_{2m(k)}), \ \Psi(y_{2n(k)}, y_{2m(k)}), \ \Psi(y_{2m(k)-1}, y_{2n(k)}) \end{matrix}\right) \leq 0.$$

Letting k tend to infinity in the above inequality we get

$$F(\psi(\varepsilon_0), \psi(\varepsilon_0), 0, 0, \psi(\varepsilon_0), \psi(\varepsilon_0)) \leq 0.$$

We used (3.8) and the continuity of F and ψ. The last inequality leads to a contradiction, by (F2) and $\psi(\varepsilon_0) > 0$. It follows that $\{y_{2n}\}$ is a Cauchy sequence, hence $\{y_n\}$ is Cauchy.

Assume first that at least one of the sets $A(X)$ and $T(X)$ is a complete subspace of X. Since $y_{2n+1} \in A(X) \subset T(X)$ and $\{y_{2n+1)}\}$ is a Cauchy sequence, there exists $u \in T(X)$ such that $\lim_{n \to \infty} y_{2n+1} = u$. The sequence $\{y_n\}$ converges to u, since it is Cauchy and has the subsequence $\{y_{2n+1}\}$ convergent to u. Let $v \in X$ so that $u = Tv$. Setting $x = x_{2n}$ and $y = v$ in (3.2) we get

$$F\left(\begin{array}{c}\psi(d(y_{2n+1}, Bv)), \psi(d(y_{2n}, u)), \ \psi(d(y_{2n}, y_{2n+1})), \\ \psi(d(u, Bv)), \ \psi(d(y_{2n}, Bv)), \ \psi(d(u, y_{2n}))\end{array}\right) \le 0.$$

Letting n tend to infinity in the above inequality we obtain

$$F\left(\psi(d(u, Bv)), \ 0, \ 0, \ \psi(d(u, Bv)), \ \psi(d(u, Bv)), \ 0\right) \le 0.$$

According to (F1) it follows that $\psi(d(u, Bv)) \le h \cdot 0$, hence $u = Bv$, therefore u is a point of coincidence of T and B. Since $u = Bv \in B(X) \subset S(X)$, there exists $w \in X$ such that $u = Sw$. Using a similar argument as above we obtain $u = Aw$, hence $u = Tv = Bv = Sw = Aw$. Indeed, setting $x = w$ and $y = x_{2n+1}$ in (3.2) we have

$$F\left(\begin{array}{c}\psi(d(Aw, y_{2n+2})), \ \psi(d(u, y_{2n+1})), \ \psi(d(u, Aw)), \\ \psi(d(y_{2n+1}, y_{2n+2})), \ \psi(d(u, y_{2n+2})), \ \psi(d(y_{2n+1}, Aw))\end{array}\right) \le 0,$$

and letting n tend to infinity we get

$$F\left(\psi(d(Aw, u)), \ 0, \ \psi(d(Aw, u)), \ 0, \ 0, \ \psi(d(Aw, u))\right) \le 0.$$

By (F1) it follows that $\psi(d(Aw, u)) \le h \cdot 0$, hence $u = Aw$.

We prove that u is the unique point of coincidence of A and S. Let $z \in X$ be such that $Az = Sz$. Setting $x = z$ and $y = v$ in (3.2) we obtain

$$F\left(\Psi(Az, u), \ \Psi(Az, u), \ 0, \ 0, \ \Psi(Az, u), \ \Psi(Az, u)\right) \le 0.$$

By (F2), $\Psi(Az, u) = 0$, hence $u = Az = Sz$. Similarly, u is the unique point of coincidence of B and T: if $Bt = Tt$, setting $x = w$ and $y = t$ in (3.2) we get

$$F\left(\Psi(u, Bt), \ \Psi(u, Bt), \ 0, \ 0, \ \Psi(u, Bt), \ \Psi(u, Bt)\right) \le 0,$$

hence $u = Bt = Tt$.

If at least one of the sets $B(X)$ and $S(X)$ is a complete subspace of X, an analogous argument shows that each of the pairs (A, S) and (B, T) has a unique point of coincidence u.

If the pair (A, S) is owc (respectively, (B, T) is owc), then by Lemma 1.4 u is the unique common fixed point of A and S (respectively, of B and T), and the proof is completed. $\square$

Letting the altering distance ψ be the identity in Theorem 3.1, we obtain

3.2. Corollary. *Let A, B, S and T be self-mappings of a metric space (X, d) satisfying $A(X) \subset T(X)$ and $B(X) \subset S(X)$. Suppose that*

$$F\left(\begin{array}{c}d(Ax, By), \ d(Sx, Ty), \ d(Sx, Ax), \\ d(Ty, By), \ d(Sx, By), \ d(Ty, Ax)\end{array}\right) \le 0$$

for all $x, y \in X$, where $F \in \mathcal{F}_a$. Assume that at least one of the sets $A(X)$, $B(X)$, $S(X)$ and $T(X)$ is a complete subspace of X. Then each of the pairs (A, S) and (B, T) has a unique point of coincidence.

If each of the pairs (A, S) and (B, T) is owc, then A, B, S and T have a unique common fixed point.

3.3. Remark. The above corollary yields Theorem 4.1 of [2], with a different notation.

Next we show that we may assume in Theorem 3.1, without loss of generality, that both of the pairs (A, S) and (B, T) are weakly compatible.

Suppose that A, B, S and T satisfy conditions (3.1) and (3.2). If the pairs (A, S) and (B, T) are owc, then these pairs are weakly compatible. Indeed, if $Ax = Sx$ and $By = Ty$, then by condition (3.2) we have $F(\delta, \delta, 0, 0, \delta, \delta) \leq 0$, where $\delta := \psi(d(Ax, By))$, hence $Ax = By$. By the owc property, there exist $x_0, y_0 \in X$ such that $Ax_0 = Sx_0$, $ASx_0 = SAx_0$, $By_0 = Ty_0$ and $BTy_0 = TBy_0$. The above argument shows that $Ax = Sx$ implies $Ax = By_0$, but $Ax_0 = By_0$, hence $ASx = ASx_0 = SAx_0 = SAx$, therefore A and S are weakly commuting. Similarly, B and T are weakly commuting.

Since it involves an altering distance, Theorem 3.1 is a proper generalization of Theorem 4.1 of [2], as the following example shows.

3.4. Example. Let $F \in \mathcal{F}_a$ and $\psi : [0, +\infty) \to [0, +\infty)$ be an altering distance. Consider the function $F_\psi : \mathbb{R}_+^6 \to \mathbb{R}$ defined by $F_\psi(t_1, \ldots, t_6) = F(\psi(t_1), \ldots, \psi(t_6))$. Then F_ψ satisfies condition (F2) from Definition 2.6. If F_ψ were to satisfy condition (F1) from Definition 2.6, then Theorem 3.1 could be derived from Theorem 4.1 of [2], but this does not happen in general. If $F_\psi(u, v, v, u, w, 0) \leq 0$ or $F_\psi(u, v, u, v, 0, w) \leq 0$, then we have $\psi(u) \leq h\psi(v)$. The last inequality does not imply in general $u \leq h'v$ for some constant $h' \in [0, 1)$. Take for example $\psi(t) = e^t - 1$, $t \in [0, +\infty)$ and $h \in (0, 1)$. Then $\psi(u) \leq h\psi(v)$ is equivalent to $u \leq \ln(1 + h(e^v - 1))$. Since $\lim\limits_{v \to \infty} \frac{\ln(1 + h(e^v - 1))}{v} = 1$, there is no constant $h' \in [0, 1)$ such that $u \leq h'v$ whenever $u \leq \ln(1 + h(e^v - 1))$ and $u, v \geq 0$.

Considering a certain altering distance in Theorem 3.1 we obtain a common fixed point theorem for two pairs of owc mappings satisfying an implicit condition of integral type.

3.5. Theorem. *Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying $A(X) \subset T(X)$, $B(X) \subset S(X)$ and*

$$
(3.9) \quad F \begin{pmatrix} \int\limits_0^{d(Ax,By)} \varphi(t)\,dt, & \int\limits_0^{d(Sx,Ty)} \varphi(t)\,dt, & \int\limits_0^{d(Sx,Ax)} \varphi(t)\,dt, \\ \int\limits_0^{d(Ty,By)} \varphi(t)\,dt, & \int\limits_0^{d(Sx,By)} \varphi(t)\,dt, & \int\limits_0^{d(Ty,Ax)} \varphi(t)\,dt \end{pmatrix} \leq 0
$$

for all $x, y \in X$, where $F \in \mathcal{F}_a$ and φ is as in Theorem 2.1. Assume that at least one of the sets $A(X)$, $B(X)$, $S(X)$ and $T(X)$ is a complete subspace of X. Then each of the pairs (A, S) and (B, T) has a unique point of coincidence. Moreover, if each of the pairs (A, S) and (B, T) is owc, then A, B, S and T have a unique common fixed point.

Proof. By Lemma 2.5, the function $\Phi_0 : [0, \infty) \to \mathbb{R}$, $\Phi_0(t) = \int_0^t \varphi(\tau)\,d\tau$ is an altering distance. By (3.9), inequality (3.2) holds with the altering distance $\psi = \Phi_0$. The claim now follows using Theorem 3.1. $\square$

3.6. Corollary. *Let A, B, S and T be self-mappings of a metric space (X, d) satisfying (3.1) and*

$$
\int\limits_0^{d(Ax,By)} \varphi(t)\,dt \leq c \int\limits_0^{d(Sx,Ty)} \varphi(t)\,dt + a \left(\int\limits_0^{d(Sx,Ax)} \varphi(t)\,dt + \int\limits_0^{d(Ty,By)} \varphi(t)\,dt \right)
$$

$$
+ b \left(\int\limits_0^{d(Sx,By)} \varphi(t)\,dt \cdot \int\limits_0^{d(Ty,Ax)} \varphi(t)\,dt \right)^{1/2}
$$

for all $x, y \in X$, where $c > 0$, $a, b \geq 0$, $c + \max\{2a, b\} < 1$ and φ is as in Theorem 2.1. If at least one of the sets $A(X)$, $B(X)$, $S(X)$ and $T(X)$ is a complete subspace of X, then each of the pairs (A, S) and (B, T) has a unique point of coincidence. Moreover, if each of the pairs (A, S) and (B, T) is owc, then A, B, S and T have a unique common fixed point.

Proof. The claim follows by Theorem 3.5 and Example 2.7. $\qquad\square$

3.7. Remark. Taking $A = B = f$, $S = T = g$ and $a = b = 0$ in Corollary 3.6, we obtain a generalization of Theorem 2.2. This special case of Corollary 3.6 generalizes Theorem 2.2 in three respects, since we assume neither that (X, d) is complete, nor that g is continuous, while the notion of owc mappings is a proper generalization of the notion of compatible mappings having at least one coincidence point.

3.8. Remark. Note that we do not obtain a generalization of Theorem 2.2 if we simply replace in that theorem the assumption "*f and g are compatible*" by "*f and g are owc*". Suppose that f and g are owc, g is continuous and conditions 1 and 2 of Theorem 2.2 are fulfilled. We show that f and g are compatible. The continuity of g and the contractive condition 2 of Theorem 2.2 imply the continuity of f, due to the properties of φ. Let a be a coincidence point at which f and g commute. Assume that $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$. Taking $x = x_n$ and $y = a$ in condition 2 and letting n tend to infinity, we get $\Phi_0(d(t, a)) \leq c\Phi_0(d(t, a))$, hence $t = a$. Since f and g are continuous, $\lim_{n\to\infty} fgx_n = fga = gfa = \lim_{n\to\infty} gfx_n$. We proved that $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, therefore f and g are compatible. Moreover, assuming only that f and g are owc and that condition 2 of Theorem 2.2 is satisfied, it follows that f and g have a unique point of coincidence, hence f and g are weakly compatible.

There are cases when Theorem 2.2 is not applicable, while Corollary 3.6 with $A = B = f$, $S = T = g$ and $a = c = 0$ can be applied, as the following Example shows.

3.9. Example. Let $X = \mathbb{R}$ with the usual metric. Consider the self-mappings of X defined by $f(x) = \frac{1}{4}\sin^2(\min\{x, 0\})$, $x \in \mathbb{R}$ and $g(x) = \begin{cases} x & \text{if } x \leq 0 \\ x+1 & \text{if } x > 0 \end{cases}$. Then f and g have a unique coincidence point, namely $x = 0$, and $fg0 = gf0 = 0$, hence f and g are owc and have a unique common fixed point.

The functions f and g satisfy condition 2 of Theorem 2.2 with φ being the identity and $c = \frac{1}{2}$. Let $x_n < 0$, $n \geq 1$, be such that $\lim_{n\to\infty} x_n = 0$. Then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0$. On the other hand, $\lim_{n\to\infty} fgx_n = 0$ and $\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} (1 + \frac{1}{4}\sin^2 x_n) = 1$, since $\sin x_n \neq 0$ for n sufficiently great. Since $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n \in \mathbb{R}$ and $\lim_{n\to\infty} d(fgx_n, gfx_n) = 1 \neq 0$, it follows that f and g are not compatible. In conclusion, the existence of a unique common fixed point of f and g follows in this case by Corollary 3.5 with $A = B = f$, $S = T = g$ and $a = b = 0$, but not by Theorem 2.2.

By Theorem 5.1 and Examples 2.8–2.11 we may obtain new fixed point theorems for owc mappings satisfying implicit relations of integral type.

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