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COMPARISON OF ESTIMATORS FOR STRESS-STRENGTH RELIABILITY IN THE GOMPERTZ CASE

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Abstract

In this paper, the Bayesian and non Bayesian estimation problem of reliability, $R = P(Y < X)$, will be considered when X and Y are two independent but not identically random variables belonging to the Gompertz distribution. The three estimation methods applied were the maximum likelihood, uniformly minimum variance unbiased, and Bayes estimators. A numerical illustration is used to compare the three different estimators.

Keywords: Stress-strength reliability, Gompertz distribution, Minimum variance unbiased estimation, Maximum likelihood estimation, Bayes estimation, Mean square error.

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1. Introduction

The Gompertz distribution plays an important role in modeling survival times, human mortality and actuarial tables. It has many applications, particularly in medical and actuarial studies. Also, the Gompertz distribution is used as a survival model in reliability. It has an increasing hazard rate for the life of the systems. This distribution does not seem to have received enough attention, possibly because of its complicated form. Recently, this distribution has been studied by some authors. For example, see Wu *et al.* [16], Jaheen [7], Wu *et al.* [17, 18], Ismail [6] and Al-Khedhair & El-Gohary [8].

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The problem of estimating $R = P(Y < X)$ arises in the context of mechanical reliability of a system with strength X and stress Y, the reliability, R, is chosen as a measure of system reliability. In a stress strength model, the system fails if and only if, at any time, the applied stress is greater than its strength. This model, first considered by Birnbaum [2], is commonly used in many engineering applications, such as civil, mechanical and aerospace. Recently, a number of papers have dealt with the stress strength reliability problem. Several distributions have been used in the literature as failure models. For references see the book by Kotz et al. [9] or the articles by Church and Harris [4], Chao [3], Awad and Gharraf [1], Constantine and Karson [5], Surles and Padgett [15], Mokhlis [12], Raqab and Kundu [13] and Kundu and Gupta [10].

In this article, we study various estimation procedures for the reliability, R, when X and Y are two independent but not identically Gompertz random variables. This paper is organized as follows: In Section 2, the maximum likelihood estimator (MLE) for R with some of its properties is derived for the underlying Gompertz distribution. Section 3 provides the uniformly minimum variance unbiased estimator (UMVUE) for R. In section 4, the Bayes estimator for R is considered based on the mean squared error loss function. In section 5, a simulation study is performed to compare the different estimators of R.

2. Maximum likelihood estimation

Let X be the strength of a system and Y be the stress acting on it. Then X and Y will be random variables from *Gompertz* with parameters (c_1, λ_1) and (c_2, λ_2), respectively. That is, the probability density functions (pdf) and the cumulative distribution functions of X and Y are, respectively,

$$f_X(x) = \lambda_1 \exp(c_1 x) \exp\left\{-\lambda_1 c_1^{-1}[\exp(c_1 x) - 1]\right\}, \; x > 0, \; c_1 > 0, \; \lambda_1 > 0, \tag{2.1}$$

$$F(x) = 1 - \exp\left\{-\lambda_1 c_1^{-1}[\exp(c_1 x) - 1]\right\}, \tag{2.2}$$

and

$$f_Y(y) = \lambda_2 \exp(c_2 y) \exp\left\{-\lambda_2 c_2^{-1}[\exp(c_2 y) - 1]\right\}, \; y > 0, \; c_2 > 0, \; \lambda_2 > 0, \tag{2.3}$$

$$F(y) = 1 - \exp\left\{-\lambda_2 c_2^{-1}[\exp(c_2 y) - 1]\right\}, \tag{2.4}$$

where c_1 and c_2 are known parameters, and λ_1, λ_2 are unknown parameters. Then R is

$$\begin{aligned} R = P(Y < X) &= \int_0^\infty P(Y < x) f_X(x)\, dx \\ &= 1 - \exp\{\lambda_1/c_1 + \lambda_2/c_2\} \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda_2/c_2)^k (c_1/\lambda_1)^{kc_2/c_1}}{k!} \\ &\quad \times \left[\Gamma((c_2/c_1)k + 1) - \sum_{i=0}^{\infty} \frac{(-1)^i (\lambda_1/c_1)^{(c_2/c_1)k+i+1}}{((c_2/c_1)k + i + 1)\, i!}\right]. \end{aligned} \tag{2.5}$$

If $c_1 = c_2 = c$, then R has the following form:

$$\begin{aligned} R = P(Y < X) &= \int_0^\infty P(Y < x) f_X(x)\, dx \\ &= \int_0^\infty \left[1 - \exp\left\{-\frac{\lambda_2}{c}(e^{cx} - 1)\right\}\right] \lambda_1 e^{cx} \exp\left\{-\frac{\lambda_1}{c}(e^{cx} - 1)\right\} dx \\ &= \frac{\lambda_2}{\lambda_1 + \lambda_2}. \end{aligned} \tag{2.6}$$

Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ be two independent random samples taken from the Gompertz distribution with parameters (c, λ_1) and (c, λ_2), respectively, and let c be known. To obtain the MLE for R, firstly we must obtain the MLEs for λ_1 and λ_2. Using the likelihood function for the two random samples, the MLEs for λ_1 and λ_2 are, respectively, $\widehat{\lambda}_1 = nc / \sum_{i=1}^{n} (\mathrm{e}^{cx_i} - 1)$ and $\widehat{\lambda}_2 = mc / \sum_{j=1}^{m} (\mathrm{e}^{cy_j} - 1)$. Hence $\widehat{R}_1$, the MLE of R, is written as follows;

$$\widehat{R}_1 = \frac{\widehat{\lambda}_2}{\widehat{\lambda}_1 + \widehat{\lambda}_2}. \tag{2.7}$$

The distribution of $\widehat{R}_1$ can be found as follows;

$$\begin{aligned} f_{\widehat{R}_1}(r_1) = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} \left(\frac{n\lambda_1}{m\lambda_2}\right)^n r_1^{n-1} (1-r_1)^{m-1} \\ \times \left\{1 - r_1 \left(1 - \frac{n\lambda_1}{m\lambda_2}\right)\right\}^{-(n+m)}, \end{aligned} \tag{2.8}$$

with $0 < r_1 < 1$. For $s > 0$, the s^{th} moment of $\widehat{R}_1$ is given by

$$\begin{aligned} \mathrm{E}\left(\widehat{R}_1^s\right) = \frac{\Gamma(n+s)\Gamma(n+m)}{\Gamma(n+m+s)\Gamma(n)} \left(\frac{n\lambda_1}{m\lambda_2}\right)^n \\ \times F_{2,1}\Big((n+s, n+m), n+m+s, 1 - \frac{n\lambda_1}{m\lambda_2}\Big), \end{aligned} \tag{2.9}$$

where $F_{p,q}(\mathbf{n}, \mathbf{d}, r)$ is the generalized hypergeometric function. This function is also known as Barnes's extended hypergeometric function. The definition of $F_{p,q}(\mathbf{n}, \mathbf{d}, r)$ is as follows;

$$F_{p,q}(\mathbf{n}, \mathbf{d}, r) = \sum_{k=0}^{\infty} \frac{r^k \prod_{i=1}^{p} \Gamma(n_i + k) \Gamma^{-1}(n_i)}{\Gamma(k+1) \prod_{i=1}^{q} \Gamma(d_i + k) \Gamma^{-1}(d_i)}, \tag{2.10}$$

where $\mathbf{n} = [n_1, n_2, \ldots, n_p]$, p is the number of operands of $\mathbf{n}$, $\mathbf{d} = [d_1, d_2, \ldots, d_q]$ and q is the number of operands of $\mathbf{d}$. The above generalized hypergeometric function is quickly evaluated and readily available in standard software programmes such as Maple. By substituting $s = 1$ in Eq. (2.9), the expected value of $\widehat{R}_1$ can be found as follows;

$$\mathrm{E}(\widehat{R}_1) = \frac{n}{(n+m)} \left(\frac{n\lambda_1}{m\lambda_2}\right)^n F_{2,1}\Big((n+1, n+m), n+m+1, 1 - \frac{n\lambda_1}{m\lambda_2}\Big). \tag{2.11}$$

Using Eq. (2.9), the mean squared error of the $\widehat{R}_1$ is obtained as follows

$$\begin{aligned} MSE\left(\widehat{R}_1\right) &= \mathrm{E}\left(\widehat{R}_1 - R\right)^2 \\ &= \frac{n(n+1)}{(n+m)(n+m+1)} \left(\frac{n\lambda_1}{m\lambda_2}\right)^n \\ &\quad \times F_{2,1}\Big((n+2, n+m), n+m+2, 1 - \frac{n\lambda_1}{m\lambda_2}\Big) + \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^2 \\ &\quad - \frac{2n}{(n+m)} \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right) \left(\frac{n\lambda_1}{m\lambda_2}\right)^n \\ &\quad \times F_{2,1}\Big((n+1, n+m), n+m+1, 1 - \frac{n\lambda_1}{m\lambda_2}\Big), \end{aligned} \tag{2.12}$$

see Saraçoğlu and Kaya, [14].

3. Uniformly minimum variance unbiased estimation

Let $\theta_0, \in \Theta$, be an arbitrary value of θ. Denote by

$$g(T \mid \theta_0) \text{ and } g_{\theta_0}(T \mid X_1 = x_1, \ldots, X_n = x_k,\ Y_1 = y_1, \ldots, Y_k = y_k)$$

the pdf of $T(\underline{X}, \underline{Y})$ and the conditional density of T for given $X_j = x_j$, $Y_j = y_j$, $j = 1, \ldots, k$, respectively, at $\theta = \theta_0$. Then the UMVUE of $f(x_1, \ldots, x_k;\ y_1, \ldots, y_k \mid \theta)$ is of the form

$$\hat{f}(x_1, \ldots, x_k;\ y_1, \ldots, y_k) = \prod_{j=1}^{k} f(x_j, y_j) \frac{g_{\theta_0}(T \mid X_1 = x_1, \ldots, X_k = x_k, Y_1 = y_1, \ldots, Y_k = y_k)}{g(T \mid \theta_0)}. \tag{3.1}$$

The UMVUE of R is in the form

$$\widehat{R}_2 = \int I(y < x) \widehat{f}(x, y)\, dx\, dy, \tag{3.2}$$

(Lumelski and Sapoznikov [11]).

Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ be the two independent random samples from the Gompertz distribution, with parameters (c, λ_1) and (c, λ_2), respectively, and let c is known. To find the UMVUE $\hat{R}_2$ for R, firstly we must find $\hat{f}(x_1, \ldots, x_k;\ y_1, \ldots, y_k)$, which is the UMVUE for $f(x_1, \ldots, x_k;\ y_1, \ldots, y_k)$. Now $W = c^{-1} \sum_{i=1}^{n} (\mathrm{e}^{cx_i} - 1)$ and $V = c^{-1} \sum_{i=1}^{m} (\mathrm{e}^{cy_i} - 1)$ have Gamma distributions with parameters (n, λ_1) and (m, λ_2), respectively. If c is known, the Gompertz family is an exponential family. Therefore W and V are sufficient and complete. The probability density functions of W and V are respectively

$$f_W(w) = \frac{\lambda_1^n}{\Gamma(n)} w^{n-1} \mathrm{e}^{-\lambda_1 w},\ w > 0,\ \lambda_1 > 0 \tag{3.3}$$

$$f_V(v) = \frac{\lambda_2^m}{\Gamma(m)} v^{m-1} \mathrm{e}^{-\lambda_2 v},\ v > 0,\ \lambda_2 > 0. \tag{3.4}$$

Since $\mathbf{X}$ and $\mathbf{Y}$ are independent, Eq. (3.1) can be written as follows,

$$\hat{f}(x_1, \ldots, x_k;\ y_1, \ldots, y_k) = \hat{f}(x_1, \ldots, x_k) \hat{f}(y_1, \ldots, y_k). \tag{3.5}$$

Then

$$\begin{aligned} \hat{f}(x_1, \ldots, x_k;\ y_1, \ldots, y_k) &= \prod_{j=1}^{k} f(x_j) \times \frac{g(w \mid X_1 = x_1, \ldots, X_k = x_k)}{g(w)} \\ &\quad \times \prod_{j=1}^{k} f(y_j) \times \frac{g(v \mid Y_1 = y_1, \ldots, Y_k = y_k)}{g(v)} \end{aligned} \tag{3.6}$$

with

$$\begin{aligned} g(w \mid X_1 = x_1, \ldots, X_k = x_k) &= \frac{\lambda_1^{n-k}}{\Gamma(n-k)} \times \left(w - \sum_{i=1}^{k} \left(\frac{\mathrm{e}^{cx_i} - 1}{c} \right) \right)^{n-k-1} \\ &\quad \times \exp\left\{ -\lambda_1 \left(w - \sum_{i=1}^{k} \left(\frac{\mathrm{e}^{cx_i} - 1}{c} \right) \right) \right\} \\ &\quad \times I \left(w - \sum_{i=1}^{k} \left(\frac{\mathrm{e}^{cx_i} - 1}{c} \right) \geq 0 \right) \end{aligned} \tag{3.7}$$

and

$$g(v \mid Y_1 = y_1, \ldots, Y_k = y_k) = \frac{\lambda_2^{m-k}}{\Gamma(m-k)} \times \left(v - \sum_{i=1}^{k} \left(\frac{e^{cy_i}-1}{c}\right)\right)^{m-k-1} \times \exp\left\{-\lambda_2 \left(v - \sum_{i=1}^{k}\left(\frac{e^{cy_i}-1}{c}\right)\right)\right\} \times I\left(v - \sum_{i=1}^{k}\left(\frac{e^{cy_i}-1}{c}\right) \geq 0\right). \tag{3.8}$$

Substituting Eq.(2.1), Eq.(2.3), Eq.(3.7) and Eq.(3.8) in Eq. (3.6), $\hat{f}(x_1, \ldots, x_k;\ y_1, \ldots, y_k)$ is obtained as

$$\begin{aligned} \hat{f}(x_1, \ldots, x_k;\ y_1, \ldots, y_k) &= \frac{\Gamma(n)\Gamma(m)}{\Gamma(n-k)\Gamma(m-k)w^{n-1}v^{m-1}} \exp\left\{c\left(\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i\right)\right\} \\ &\quad \times \left(w - \sum_{i=1}^{k}\left(\frac{e^{cx_i}-1}{c}\right)\right)^{n-k-1} \left(v - \sum_{i=1}^{k}\left(\frac{e^{cy_i}-1}{c}\right)\right)^{m-k-1} \\ &\quad \times I\left(w - \sum_{i=1}^{k}\left(\frac{e^{cx_i}-1}{c}\right) \geq 0\right) I\left(v - \sum_{i=1}^{k}\left(\frac{e^{cy_i}-1}{c}\right) \geq 0\right). \end{aligned} \tag{3.9}$$

From Eq.(3.2), the UMVUE for R is

$$\begin{aligned} \hat{R}_2 &= \iint_G f(x_1, y_1/w, v)dx_1dy_1 \\ &= \frac{(n-1)(m-1)}{w^{n-1}v^{m-1}} \\ &\quad \times \iint_G e^{c(x+y)} \left(w - \frac{e^{cx}-1}{c}\right)^{n-2} \left(v - \frac{e^{cy}-1}{c}\right)^{m-2} d_y\, d_x, \end{aligned} \tag{3.10}$$

where

$$G = \left\{(x, y) : 0 < x < \frac{\ln(cw+1)}{c},\ 0 < y < \frac{\ln(cv+1)}{c},\ y < x\right\}.$$

From Eq.(3.10), for $w < v$,

$$\begin{aligned} \hat{R}_2 &= \frac{(n-1)(m-1)}{w^{n-1}v^{m-1}} \\ &\quad \times \int_{x=0}^{\frac{\ln(cw+1)}{c}} \int_{y=0}^{x} e^{c(x+y)} \left(w - \frac{e^{cx}-1}{c}\right)^{n-2}\left(v - \frac{e^{cy}-1}{c}\right)^{m-2} d_y\, d_x \\ &= 1 - \frac{n-1}{w^{n-1}v^{m-1}} \int_{u=0}^{w} (v-u)^{m-1}(w-u)^{n-2}\, du \end{aligned} \tag{3.11}$$

$$(v-u)^{m-1} = \sum_{k=0}^{m-1} (-1)^k \binom{m-1, k}{u}^k v^{m-1-k} \tag{3.12}$$

fulfil Eq.(3.12) to Eq.(3.11), so the UMVUE for R is obtained as follows:

$$\hat{R}_2 = 1 - \sum_{k=0}^{m-1} (-1)^k \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+k)\Gamma(m-k)} \left(\frac{w}{v}\right)^k, \ w < v. \tag{3.13}$$

For $v < w$ the UMVUE for R is obtained as follows

$$\begin{aligned} \hat{R}_2 &= \frac{(n-1)(m-1)}{w^{n-1}v^{m-1}} \\ &\times \int_{y=0}^{\frac{\ln(cv+1)}{c}} \int_{x=y}^{\frac{\ln(cw+1)}{c}} e^{c(x+y)} \left(w - \frac{e^{cx}-1}{c}\right)^{n-2} \left(v - \frac{e^{cy}-1}{c}\right)^{m-2} d_x \, d_y \\ &= \sum_{k=0}^{n-1} (-1)^k \frac{\Gamma(n)\Gamma(m)}{\Gamma(n-k)\Gamma(m+k)} \left(\frac{v}{w}\right)^k, \ v < w. \end{aligned} \tag{3.14}$$

Thus,

$$\widehat{R}_2 = \begin{cases} 1 - \sum\limits_{i=0}^{m-1} d_i \left(\dfrac{W}{V}\right)^i, & W < V \\ \sum\limits_{j=0}^{n-1} c_j \left(\dfrac{V}{W}\right)^j, & W \geq V \end{cases} \tag{3.15}$$

where

$$c_j = (-1)^j \frac{\Gamma(n)\Gamma(m)}{\Gamma(n-j)\Gamma(m+j)} \tag{3.16}$$

and

$$d_i = (-1)^i \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+i)\Gamma(m-i)}. \tag{3.17}$$

The MSE of $\widehat{R}_2$ is

$$\text{MSE}\left(\widehat{R}_2\right) = \text{E}\left(\widehat{R}_2 - R\right)^2 = \text{E}\left(\widehat{R}_2^2\right) - R^2. \tag{3.18}$$

The second moment of $\widehat{R}_2$ in (3.18) could be written as

$$\begin{aligned} \text{E}\left(\widehat{R}_2^2\right) &= \sum_{j=0}^{n-1} \sum_{j'=0}^{n-1} c_j c_{j'} \text{E}\left\{ \left(\frac{V}{W}\right)^{j+j'} \mid V \leq W \right\} P(V \leq W) \\ &\quad + P(V > W) - 2 \sum_{i=0}^{m-1} d_i \text{E}\left\{ \left(\frac{W}{V}\right)^i \mid V > W \right\} P(V > W) \\ &\quad + \sum_{i=0}^{m-1} \sum_{i'=0}^{m-1} d_i d_{i'} \text{E}\left\{ \left(\frac{W}{V}\right)^{i+i'} \mid V > W \right\} P(V > W), \end{aligned} \tag{3.19}$$

where the probability density functions of the random variables $T = W/V$ and $S = V/W$ are as follows:

$$f_T(t) = \frac{\Gamma(n+m)\, t^{n-1}}{\Gamma(m)\Gamma(n)} \left(\frac{\lambda_1}{\lambda_2}\right)^n \left(1 + \frac{\lambda_1}{\lambda_2} t\right)^{-(n+m)}, \ t > 0, \tag{3.20}$$

$$f_S(s) = \frac{\Gamma(n+m)\, s^{m-1}}{\Gamma(m)\Gamma(n)} \left(\frac{\lambda_2}{\lambda_1}\right)^m \left(1 + \frac{\lambda_2}{\lambda_1} s\right)^{-(n+m)}, \ s > 0. \tag{3.21}$$

Using (3.20) and (3.21),

$$
\begin{aligned}
&\mathrm{E}\left(\left(\frac{V}{W}\right)^{j} \mid V \leq W\right) P\left(V \leq W\right) \\
&\quad = \mathrm{E}\left(\left(\frac{V}{W}\right)^{j} \mid \frac{V}{W} \leq 1\right) P\left(\frac{V}{W} \leq 1\right) \\
&\quad = \int_{0}^{1} s^{j} f_{S}(s)\, ds \qquad (3.22) \\
&\quad = \frac{\Gamma\left(n+m\right)}{\Gamma\left(m\right)\Gamma\left(n\right)} \left(\frac{\lambda_2}{\lambda_1}\right)^{m} \int_{0}^{1} s^{m+j-1} \left(1+\frac{\lambda_2}{\lambda_1}s\right)^{-(n+m)} ds \\
&\quad = \frac{\Gamma\left(n+m\right)}{\Gamma\left(n\right)\Gamma\left(m\right)\left(m+j\right)} \left(\frac{\lambda_2}{\lambda_1}\right)^{m} F_{2,1}\left(\left(n+m, m+j\right), m+j+1, -\frac{\lambda_2}{\lambda_1}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{E}\left(\left(\frac{W}{V}\right)^{i} \mid V > W\right) P\left(V > W\right) \\
&\quad = \mathrm{E}\left(\left(\frac{W}{V}\right)^{i} \mid \frac{W}{V} < 1\right) P\left(\frac{W}{V} < 1\right) \\
&\quad = \int_{0}^{1} t^{i} f_{T}(t)\, dt \qquad (3.23) \\
&\quad = \frac{\Gamma\left(n+m\right)}{\Gamma\left(m\right)\Gamma\left(n\right)} \left(\frac{\lambda_1}{\lambda_2}\right)^{n} \int_{0}^{1} t^{n+i-1} \left(1+\frac{\lambda_1}{\lambda_2}t\right)^{-(n+m)} dt \\
&\quad = \frac{\Gamma\left(n+m\right)}{\Gamma\left(n\right)\Gamma\left(m\right)\left(n+i\right)} \left(\frac{\lambda_1}{\lambda_2}\right)^{n} F_{2,1}\left(\left(n+m, n+i\right), n+i+1, -\frac{\lambda_1}{\lambda_2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
P\left(V > W\right) &= P\left(\frac{W}{V} < 1\right) \\
&= \int_{0}^{1} f_{T}(t)\, dt \\
&= \frac{\Gamma\left(n+m\right)}{\Gamma\left(m\right)\Gamma\left(n\right)} \left(\frac{\lambda_1}{\lambda_2}\right)^{n} \int_{0}^{1} t^{n-1} \left(1+\frac{\lambda_1}{\lambda_2}t\right)^{-(n+m)} dt \qquad (3.24) \\
&= \left(\frac{\lambda_1}{\lambda_2}\right)^{n} \frac{\Gamma(n+m)}{m\lambda_1\Gamma(n)\Gamma(m)} \Big\{ \lambda_2 F_{2,1}\left(\left(n-1, m+n'\right), n, -\frac{\lambda_1}{\lambda_2}\right) \\
&\qquad - \lambda_2^{-m-n}\left(\lambda_1+\lambda_2\right)^{-m-n+1} \Big\}
\end{aligned}
$$

are obtained. The MSE of $\widehat{R}_2$ is given by:

(3.25)

$$MSE\left(\widehat{R}_2\right) = \sum_{j=0}^{n-1}\sum_{\acute{j}=0}^{n-1} \frac{c_j c_{\acute{j}} \left(\frac{\lambda_2}{\lambda_1}\right)^m \Gamma(n+m)}{\Gamma(n)\Gamma(m)(m+j+\acute{j})} \times F_{2,1}\left((n+m, m+j+\acute{j}), m+j+\acute{j}+1, -\frac{\lambda_2}{\lambda_1}\right)$$
$$+ \frac{\left(\frac{\lambda_1}{\lambda_2}\right)^n \Gamma(n+m)}{m\lambda_1\Gamma(n)\Gamma(m)} \left\{\lambda_2 F_{2,1}\left((n-1, m+n), n, -\frac{\lambda_1}{\lambda_2}\right) - \frac{\lambda_2^{n+m}}{(\lambda_1+\lambda_2)^{n+m-1}}\right\}$$
$$- 2\sum_{i=0}^{m-1} \frac{d_i \left(\frac{\lambda_1}{\lambda_2}\right)^n \Gamma(n+m)}{\Gamma(n)\Gamma(m)(n+i)} \times F_{2,1}\left((n+m, n+i), n+i+1, -\frac{\lambda_1}{\lambda_2}\right)$$
$$- \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^2 + \sum_{i=0}^{m-1}\sum_{i'=0}^{m-1} \frac{d_i d_{\acute{i}} \left(\frac{\lambda_1}{\lambda_2}\right)^n \Gamma(n+m)}{\Gamma(n)\Gamma(m)(n+i+\acute{i})} \times F_{2,1}\left((n+m, n+i+\acute{i}), n+i+\acute{i}+1, -\frac{\lambda_1}{\lambda_2}\right),$$

where $F_{p,q}(\mathbf{n}, \mathbf{d}, r)$ is the generalized hypergeometric function in Eq.(2.10), c_j and d_i are defined in (3.16) and (3.17), respectively.

4. Bayes estimation

In this section we consider the Bayes estimator $\widehat{R}_3$ for R with respect to the mean square error. Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ be the two independent random samples from a Gompertz distribution with parameters (c, λ_1) and (c, λ_2), respectively. Assuming c is known, the likelihood functions are

$$L_1(x_1, \ldots, x_n) = \lambda_1^n \exp\left(c\sum_{i=1}^n x_i\right) \exp\left(-\frac{\lambda_1}{c}\sum_{i=1}^n (\mathrm{e}^{cx_i} - 1)\right) \tag{4.1}$$

and

$$L_2(y_1, \ldots, y_n) = \lambda_2^m \exp\left(c\sum_{i=1}^m y_i\right) \exp\left(-\frac{\lambda_2}{c}\sum_{i=1}^m (\mathrm{e}^{cy_i} - 1)\right). \tag{4.2}$$

Let λ_1 and λ_2 be independent with priors

$$\pi(\lambda_1) = \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)} \lambda_1^{\alpha_1 - 1} \mathrm{e}^{-\lambda_1\beta_1},\ \alpha_1 > 0,\ \beta_1 > 0,\ \lambda_1 > 0, \tag{4.3}$$

$$\pi(\lambda_2) = \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_1)} \lambda_2^{\alpha_2 - 1} \mathrm{e}^{-\lambda_2\beta_2},\ \alpha_2 > 0,\ \beta_2 > 0,\ \lambda_2 > 0, \tag{4.4}$$

respectively. The posterior distributions of λ_1 and λ_2 are as follows.

$$\begin{aligned}\pi(\lambda_1,\lambda_2 \mid \underline{X},\underline{Y}) &= \frac{f(\underline{x},\underline{y} \mid \lambda_1,\lambda_2)\,\Pi(\lambda_1,\lambda_2)}{\int_{\lambda_1}\int_{\lambda_2} f(\underline{x},\underline{y} \mid \lambda_1,\lambda_2)\,\Pi(\lambda_1,\lambda_2)\,d\lambda_2\,d\lambda_1} \\ &= \frac{(\beta_1+k_1)^{n+\alpha_1}(\beta_2+k_2)^{m+\alpha_2}\lambda_1^{n+\alpha_1-1}\lambda_2^{m+\alpha_2-1}}{\Gamma(n+\alpha_1)\,\Gamma(m+\alpha_2)} \\ &\quad \times \exp\{-\lambda_1(\beta_1+k_1)-\lambda_2(\beta_2+k_2)\},\end{aligned} \tag{4.5}$$

where

$$k_1 = c^{-1}\sum_{i=1}^{n}\left(e^{cx_i}-1\right), \tag{4.6}$$

$$k_2 = c^{-1}\sum_{i=1}^{m}\left(e^{cy_i}-1\right). \tag{4.7}$$

Let $r = \lambda_2/(\lambda_1+\lambda_2)$ and $u = \lambda_1+\lambda_2,\ u>0,\ 0<r<1$. Then

$$\begin{aligned}\pi(r,u \mid \underline{X},\underline{Y}) &= S(k_1,k_2,r)u^{n+m+\alpha_1+\alpha_2-1} \\ &\quad \times \exp\{-u(1-r)(\beta_1+k_1)-ru(\beta_2+k_2)\}\end{aligned} \tag{4.8}$$

where

$$S(k_1,k_2,r) = \frac{(\beta_1+k_1)^{n+\alpha_1}(\beta_2+k_2)^{m+\alpha_2}r^{m+\alpha_2-1}(1-r)^{n+\alpha_1-1}}{\Gamma(n+\alpha_1)\,\Gamma(m+\alpha_2)}. \tag{4.9}$$

The marginal posterior distribution of R is

$$\begin{aligned}\pi_R(r \mid \underline{X},\underline{Y}) &= \int_{u=0}^{\infty} u^{n+m+\alpha_1+\alpha_2-1}S(k_1,k_2,r) \\ &\quad \times \exp\{-u(1-r)(\beta_1+k_1)-ru(\beta_2+k_2)\}\,du \\ &= \frac{S(k_1,k_2,r)\,\Gamma(n+m+\alpha_1+\alpha_2)}{[(1-r)(\beta_1+k_1)+r(\beta_2+k_2)]^{n+m+\alpha_1+\alpha_2}}.\end{aligned} \tag{4.10}$$

Hence the Bayes Estimator $\widehat{R}_3$ of R with respect to the mean squared error loss function is the posterior mean

$$\begin{aligned}\widehat{R}_3 &= \int_{r=0}^{1} r.\pi_R(r \mid \underline{x},\underline{y})\,dr \\ &= \frac{v_1^{\delta_1}v_2^{\delta_2}\,\Gamma(\delta_1+\delta_2)}{\Gamma(\delta_1)\,\Gamma(\delta_2)}\int_{r=0}^{1}\frac{r^{\delta_2}(1-r)^{\delta_1-1}}{[(1-r)v_1+rv_2]^{\delta_1+\delta_2}}\,dr \\ &= \begin{cases}\left(\frac{v_2}{v_1}\right)^{\delta_2}\frac{\delta_2}{\delta_1+\delta_2}F_{2,1}\left((\delta_1+\delta_2,\delta_2+1),\delta_1+\delta_2+1,\frac{v_2-v_1}{v_1}\right), & v_2\le v_1,\\ \left(\frac{v_1}{v_2}\right)^{\delta_1}\frac{\delta_2}{\delta_1+\delta_2}F_{2,1}\left((\delta_1+\delta_2,\delta_1),\delta_1+\delta_2+1,\frac{v_2-v_1}{v_2}\right), & v_2>v_1,\end{cases}\end{aligned} \tag{4.11}$$

where

$$\delta_1 = n+\alpha_1\ \delta_2 = m+\alpha_2,\ v_1=\beta_1+k_1,\ v_2=\beta_2+k_2. \tag{4.12}$$

5. Comparison of the estimators

In this section, the software package MathCAD (2001) was used for the simulation study. The numerical illustration was carried out to obtain and study the properties of the MLE, UMVUE and Bayes Estimators for R. An extensive numerical investigation was used to make a comparison among the three estimators, MLE, UMVUE and Bayes estimators for R. The following steps were considered when obtaining these numerical results:

Step 1. First, 10000 uniform (0,1) random samples were generated. The usual transformation technique was used to get the corresponding Gompertz random samples. In this way, samples $X_1, \ldots, X_n$ from the Gompertz distribution with sample sizes $n = 5$, 20 and 50, and scale parameter $\lambda_1 = 4$ were obtained.

Table 1. The MSE values of the MLE, UMVUE and Bayes Estimator

samp.		param.		reliability	MLE		UMVUE		Bayes	
n	m	λ_1	λ_2	R	$\widehat{R}_1$	MSE($\widehat{R}_1$)	$\widehat{R}_2$	MSE($\widehat{R}_2$)	$\widehat{R}_3$	MSE($\widehat{R}_3$)
			2	0.3333	0.3443	0.0278	0.3331	0.0324	0.3516	0.0185
5	3	4	4	0.5000	0.4994	0.0297	0.4992	0.0380	0.4994	0.0240
			6	0.6000	0.6111	0.0271	0.6224	0.0346	0.5877	0.0202
			2	0.3333	0.3265	0.0194	0.3378	0.0223	0.3609	0.0155
5	5	4	4	0.5000	0.5125	0.0227	0.5013	0.0275	0.5180	0.0172
			6	0.6000	0.6101	0.0215	0.6176	0.0256	0.6043	0.0150
			2	0.3333	0.3318	0.0139	0.3432	0.0158	0.3323	0.0142
5	10	4	4	0.5000	0.5177	0.0176	0.5001	0.0203	0.5250	0.0150
			6	0.6000	0.6019	0.0173	0.6042	0.0192	0.6151	0.0125
			2	0.3333	0.3339	0.0233	0.3443	0.0242	0.3482	0.0091
20	3	4	4	0.5000	0.5203	0.0232	0.5007	0.0272	0.5290	0.0043
			6	0.6000	0.6189	0.0199	0.6055	0.0240	0.6194	0.0120
			2	0.3333	0.3331	0.0140	0.3339	0.0143	0.3392	0.0047
20	5	4	4	0.5000	0.4919	0.0152	0.5038	0.0169	0.5013	0.0090
			6	0.6000	0.6015	0.0134	0.6033	0.0152	0.6003	0.0069
			2	0.3333	0.3341	0.0077	0.3345	0.0079	0.3291	0.0101
20	10	4	4	0.5000	0.5013	0.0091	0.5014	0.0098	0.5100	0.0060
			6	0.6000	0.6003	0.0083	0.6008	0.0089	0.5924	0.0077
			2	0.3333	0.3328	0.0020	0.3338	0.0020	0.3374	0.0033
50	50	4	4	0.5000	0.5005	0.0025	0.5012	0.0025	0.5001	0.0040
			6	0.6000	0.6007	0.0023	0.6002	0.0023	0.5966	0.0017

Step 2. Similarly, 10000 random samples $Y_1, \ldots, Y_m$ were generated from the Gompertz distribution with sample sizes $m = 3, 5$ and 10, and scale parameters $\lambda_2 = 2$, 4 and 6. The MLE for the parameters were obtained using

$$\widehat{\lambda}_1 = nc/\sum_{i=1}^{n} (\mathrm{e}^{cx_i} - 1) \text{ and } \widehat{\lambda}_2 = mc/\sum_{j=1}^{m} (\mathrm{e}^{cy_j} - 1)\,.$$

Step 3. Using the results for the MLE of the two unknown parameters of the Gompertz distribution and using Eq. (2.7), the MLE $\widehat{R}_1$ of R was obtained. Then using Eq. (2.12) the MSE of UMVUE was calculated.

Step 4. The MathCAD program and Eq. (3.15) was used for computing the UMVUE $\widehat{R}_2$ of R. Then using Eq.(3.25) the MSE of UMVUE was computed.

Step 5. To obtain the Bayes estimator of R under the Gompertz distribution, the simulated random samples with different values of the parameters and sample sizes were used with Eq. (4.12), the values of the prior parameters being $\beta_1 = 2$, $\alpha_1 = 3$, $\beta_2 = 2$ and $\alpha_2 = 3$. Then the MSE for the bayes estimator was computed using simulation.

Table (1) shows the simulation results for the MSEs of the MLE, UMVUE and Bayes estimators for R, for different values of the parameters and sample sizes.

From Table 1, we note that the mean square error of Bayes is smaller than the mean square error for maximum likelihood, and the minimum variance unbiased, except for small sample size and with $R = .3333$. Also, maximum likelihood has mean square error less than UMVUE.

It can be noted that the MSE of the three estimators decrease with increasing sample size n and fixed sample size m, also MSE increases with increase sample size m and fixed n.

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