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ON A SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract

In this paper, we introduce a new subclass $\mathcal{K}_s(\lambda, A, B)$ of close-to-convex functions. Such results as inclusion relationships, coefficient estimates, distortion and covering theorems for this class are proved. The results presented here would provide extensions of those given in earlier works. Several other new results are also obtained.

Keywords: Analytic functions, Starlike functions, Close-to-convex functions, Hadamard product (or convolution), Subordination between analytic functions.

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1. Introduction

Let $\mathcal{S}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are *analytic* and *univalent* in the *open* unit disk

$$\mathbb{U} := \{z : \ z \in \mathbb{C} \ \text{ and } \ |z| < 1\}.$$

Let $\mathcal{K}$ and $\mathcal{S}^*(\alpha)$ denote the usual subclasses of $\mathcal{S}$ whose members are close-to-convex and starlike of order α $(0 \leqq \alpha < 1)$ in $\mathbb{U}$, respectively.

In a recent paper, Gao and Zhou [1] discussed a class $\mathcal{K}_s$ of analytic functions related to the starlike functions, a function $f \in \mathcal{S}$ is said to be in the class $\mathcal{K}_s$ if it satisfies the inequality:

$$\Re\left(\frac{z^2 f'(z)}{g(z)g(-z)}\right) < 0 \quad (z \in \mathbb{U}),$$

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where $g \in \mathcal{S}^*(1/2)$. More recently, Wang *et al.* [4] discussed a subclass $\mathcal{K}_s(\alpha, \beta)$ of the class $\mathcal{K}_s$, a function $f \in \mathcal{S}$ is said to be in the class $\mathcal{K}_s(\alpha, \beta)$ if it satisfies the inequality:

$$\left|\frac{z^2 f'(z)}{g(z)g(-z)} + 1\right| < \beta \left|\frac{\alpha z^2 f'(z)}{g(z)g(-z)} - 1\right| \quad (z \in \mathbb{U}),$$

where

$$0 \leqq \alpha \leqq 1, \quad 0 < \beta \leqq 1 \text{ and } g \in \mathcal{S}^*(1/2).$$

By noting that $\mathcal{K}_s(1,1) = \mathcal{K}_s$, so the class $\mathcal{K}_s(\alpha, \beta)$ is a generalization of the class $\mathcal{K}_s$.

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \text{ and } |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g\big(\omega(z)\big) \quad (z \in \mathbb{U}).$$

Indeed, it is known that

$$f(z) \prec g(z) \ (z \in \mathbb{U}) \Longrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \ (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Motivated by the classes $\mathcal{K}_s$ and $\mathcal{K}_s(\alpha, \beta)$, we now introduce and investigate the following more generalized class $\mathcal{K}_s(\lambda, A, B)$ of analytic functions, and obtain some interesting results.

1.1. Definition. A function $f \in \mathcal{S}$ is said to be in the class $\mathcal{K}_s(\lambda, A, B)$ if it satisfies the subordination condition:

$$\frac{z^2 f'(z) + \lambda z^3 f''(z)}{-g(z)g(-z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}), \tag{1.1}$$

where

$$0 \leqq \lambda \leqq 1, \quad -1 \leqq B < A \leqq 1 \text{ and } g \in \mathcal{S}^*(1/2).$$

By noting that $\mathcal{K}_s(0, \beta, -\alpha\beta) = \mathcal{K}_s(\alpha, \beta)$ (see [4]), so the class $\mathcal{K}_s(\lambda, A, B)$ is a generalization of the class $\mathcal{K}_s(\alpha, \beta)$.

In our proposed investigation of the class $\mathcal{K}_s(\lambda, A, B)$, we need the following lemmas.

1.2. Lemma. (See [1]) *Let*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*(1/2).$$

Then

$$\frac{-g(z)g(-z)}{z} = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \in \mathcal{S}^*,$$

where

$$|B_{2n-1}| = \left|2b_{2n-1} - 2b_2 b_{2n-2} + \cdots + (-1)^n 2 b_{n-1} b_{n+1} + (-1)^{n+1} b_n^2\right| \leqq 1 \quad (n = 2, 3, \ldots).$$

1.3. Lemma. *Let* $\gamma \geqq 0$ *and* $f \in \mathcal{K}$. *Then*

$$F(z) = \frac{1+\gamma}{z^\gamma}\int_0^z f(t)t^{\gamma-1}dt \in \mathcal{K}.$$

This lemma is a special case of Theorem 4 obtained by Wu [5].

1.4. Lemma. (See [2]) *Let*

$$-1 \leqq B_2 \leqq B_1 < A_1 \leqq A_2 \leqq 1.$$

Then

$$\frac{1+A_1z}{1+B_1z} \prec \frac{1+A_2z}{1+B_2z}.$$

1.5. Lemma. (See [3]) *Let*

$$\mathfrak{f}(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$$

be analytic in $\mathbb{U}$ *and*

$$\mathfrak{g}(z) = 1 + \sum_{k=1}^{\infty} d_k z^k$$

be analytic and convex in $\mathbb{U}$. *If* $\mathfrak{f} \prec \mathfrak{g}$, *then*

$$|c_k| \leqq |d_1| \quad (k \in \mathbb{N} := \{1, 2, \dots\}).$$

In this paper, we aim at proving such results as inclusion relationships, coefficient estimates, distortion and covering theorems for the class $\mathcal{K}_s(\lambda, A, B)$. The results presented here would provide extensions of those given in earlier works. Several other new results are also obtained.

2. Inclusion relationships

We first prove the following inclusion relationship for the class $\mathcal{K}_s(\lambda, A, B)$, which tells us that $\mathcal{K}_s(\lambda, A, B)$ is a subclass of close-to-convex functions.

2.1. Theorem. *Let*

$$0 \leqq \lambda \leqq 1 \text{ and } -1 \leqq B < A \leqq 1.$$

Then

$$\mathcal{K}_s(\lambda, A, B) \subset \mathcal{K} \subset \mathcal{S}.$$

Proof. Suppose that

$$F(z) := (1-\lambda)f(z) + \lambda z f'(z) \text{ and } G(z) := \frac{-g(z)g(-z)}{z} \tag{2.1}$$

with $f \in \mathcal{K}_s(\lambda, A, B)$. Then the condition (1.1) can be written as follows:

$$\frac{zF'(z)}{G(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}).$$

By Lemma 1.2, we know that $G \in \mathcal{S}^*$, thus we have

$$F(z) = (1-\lambda)f(z) + \lambda z f'(z) \in \mathcal{K}.$$

We now split it into two cases to prove.

(1) When $\lambda = 0$. It is obvious that $f = F \in \mathcal{K}$.

(2) When $0 < \lambda \leqq 1$. By noting that $F = (1-\lambda)f + \lambda z f'$, we find that

$$f(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z F(t) t^{\frac{1}{\lambda}-2} dt.$$

Since

$$\gamma = \frac{1}{\lambda} - 1 \geqq 0,$$

by Lemma 1.3, we know that $f \in \mathcal{K}$. Therefore,

$$\mathcal{K}_s(\lambda, A, B) \subset \mathcal{K} \subset \mathcal{S}.$$

The proof of Theorem 2.1 is evidently completed. □

2.2. Theorem. *Let*

$$0 \leqq \lambda \leqq 1 \ \textit{and}\ -1 \leqq B_2 \leqq B_1 < A_1 \leqq A_2 \leqq 1.$$

Then

$$\mathcal{K}_s(\lambda, A_1, B_1) \subset \mathcal{K}_s(\lambda, A_2, B_2).$$

Proof. Suppose that $f \in \mathcal{K}_s(\lambda, A_1, B_1)$. Then

$$\frac{z^2 f'(z) + \lambda z^3 f''(z)}{-g(z)g(-z)} \prec \frac{1 + A_1 z}{1 + B_1 z}.$$

Since $-1 \leqq B_2 \leqq B_1 < A_1 \leqq A_2 \leqq 1$, by Lemma 1.4, we have

$$\frac{z^2 f'(z) + \lambda z^3 f''(z)}{-g(z)g(-z)} \prec \frac{1 + A_1 z}{1 + B_1 z} \prec \frac{1 + A_2 z}{1 + B_2 z},$$

that is $f \in \mathcal{K}_s(\lambda, A_2, B_2)$, which implies that

$$\mathcal{K}_s(\lambda, A_1, B_1) \subset \mathcal{K}_s(\lambda, A_2, B_2).$$

We thus complete the proof of Theorem 2.2. □

3. Coefficient estimates

In this section, we give the coefficient estimates of functions belonging to the class $\mathcal{K}_s(\lambda, A, B)$.

3.1. Theorem. *Let*

$$0 \leqq \lambda \leqq 1 \ \textit{and}\ -1 \leqq B < A \leqq 1.$$

If

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{K}_s(\lambda, A, B),$$

then

$$|a_{2n}| \leqq \frac{n(A-B)}{2n[1 + (2n-1)\lambda]} \quad (n \in \mathbb{N} := \{1, 2, \ldots\}), \tag{3.1}$$

and

$$|a_{2n+1}| \leqq \frac{n(A-B)+1}{(2n+1)(1+2n\lambda)} \quad (n \in \mathbb{N}). \tag{3.2}$$

In particular, if we set $A = 1$ *and* $B = -1$ *in* (3.1) *and* (3.2)*, respectively, then*

$$|a_n| \leqq \frac{1}{1 + (n-1)\lambda} \quad (n \in \mathbb{N}), \tag{3.3}$$

and inequality (3.3) *is sharp for*

$$f(z) = \frac{z}{1-z} \in \mathcal{K}_s(0,1,-1) \quad (\lambda = 0),$$

and

$$f(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \frac{t}{1-t} t^{\frac{1}{\lambda}-2} dt \in \mathcal{K}_s(\lambda,1,-1) \quad (0 < \lambda \leqq 1).$$

Proof. From the definition of $\mathcal{K}_s(\lambda, A, B)$ and (2.1), we know that there exists a function with positive real part

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (z \in \mathbb{U})$$

such that

$$p(z) = \frac{zf'(z) + \lambda z^2 f''(z)}{[-g(z)g(-z)]/z} = \frac{zF'(z)}{G(z)} \prec \frac{1+Az}{1+Bz}. \tag{3.4}$$

By Lemma 1.5, we know that

$$|p_n| \leqq A - B \quad (n \in \mathbb{N}). \tag{3.5}$$

At the same time, by Lemma 1.2, we have

$$G(z) = \frac{-g(z)g(-z)}{z} = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1}$$

and

$$|B_{2n-1}| \leqq 1 \quad (n \in \mathbb{N} \setminus \{1\}). \tag{3.6}$$

Upon substituting the series expressions of functions $f(z)$, $G(z)$, $p(z)$ into equality (3.4), and comparing the coefficients of two sides of this equation, we get

$$2n[1 + (2n-1)\lambda]a_{2n} = p_{2n-1} + p_{2n-3}B_3 + p_{2n-5}B_5 + \cdots + p_1 B_{2n-1} \quad (n \in \mathbb{N}), \tag{3.7}$$

and

$$(2n+1)(1+2n\lambda)a_{2n+1} = p_{2n} + p_{2n-2}B_3 + p_{2n-4}B_5 + \cdots + p_2 B_{2n-1} + B_{2n+1} \quad (n \in \mathbb{N}). \tag{3.8}$$

Combining (3.5), (3.6), (3.7) and (3.8), we have

$$2n[1 + (2n-1)\lambda]\,|a_{2n}| \leqq n(A-B) \quad (n \in \mathbb{N}), \tag{3.9}$$

and

$$(2n+1)(1+2n\lambda)\,|a_{2n+1}| \leqq n(A-B) + 1 \quad (n \in \mathbb{N}). \tag{3.10}$$

The assertions (3.1) and (3.2) of Theorem 3.1 can now easily be derived from (3.9) and (3.10). □

3.2. Remark. Setting $\lambda = 0$, $A = 1$ and $B = -1$ in (3.3) in Theorem 3.1, we get the corresponding result obtained by Gao and Zhou [1].

4. Covering theorem

In this section, we give the covering theorem for the class $\mathcal{K}_s(\lambda, A, B)$.

4.1. Theorem. *Let $f \in \mathcal{K}_s(\lambda, A, B)$. Then the unit disk $\mathbb{U}$ is mapped by f on a domain that contain the disk $|w| < r_1$, where*

$$r_1 := \frac{2(1+\lambda)}{A - B + 4(1+\lambda)}.$$

Proof. Suppose that $f \in \mathcal{K}_s(\lambda, A, B)$, and let w_0 be any complex number such that $f(z) \neq w_0$ for $z \in \mathbb{U}$. Then $w_0 \neq 0$ and

$$\frac{w_0 f(z)}{w_0 - f(z)} = z + \left(a_2 + \frac{1}{w_0}\right) z^2 + \cdots$$

is univalent in $\mathbb{U}$ by Theorem 2.1. This leads to

$$\left|a_2 + \frac{1}{w_0}\right| \leqq 2. \tag{4.1}$$

On the other hand, from Theorem 3.1, we know that

$$|a_2| \leqq \frac{A - B}{2(1+\lambda)}. \tag{4.2}$$

Combining (4.1) and (4.2), we deduce that

$$|w_0| \geqq \frac{1}{|a_2| + 2} \geqq \frac{2(1+\lambda)}{A - B + 4(1+\lambda)} = r_1.$$

We thus complete the proof of Theorem 4.1. □

5. Distortion theorem

Finally, we prove the distortion theorem for the class $\mathcal{K}_s(\lambda, A, B)$.

5.1. Theorem. *Let*

$$0 \leqq \lambda \leqq 1, \ -1 \leqq B < A \leqq 1 \text{ and } f \in \mathcal{K}_s(\lambda, A, B).$$

(1) *If $\lambda = 0$, then for $|z| = r < 1$, we have*

$$\int_0^r \frac{1 - At}{(1 - Bt)(1 + t^2)} dt \leqq |f(z)| \leqq \int_0^r \frac{1 + At}{(1 + Bt)(1 - t^2)} dt. \tag{5.1}$$

(2) *If $0 < \lambda \leqq 1$, then for $|z| = r < 1$, we have*

$$\begin{aligned} &\frac{1}{\lambda} r^{1-\frac{1}{\lambda}} \int_0^r \int_0^s \frac{1 - At}{(1 - Bt)(1 + t^2)} dt s^{\frac{1}{\lambda}-2} ds \\ &\qquad \leqq |f(z)| \leqq \frac{1}{\lambda} r^{1-\frac{1}{\lambda}} \int_0^r \int_0^s \frac{1 + At}{(1 + Bt)(1 - t^2)} dt s^{\frac{1}{\lambda}-2} ds. \end{aligned} \tag{5.2}$$

Proof. Suppose that $f \in \mathcal{K}_s(\lambda, A, B)$. Then from the definition of subordination between analytic functions, we deduce that

$$\begin{aligned} \frac{1 - Ar}{1 - Br} &\leqq \frac{1 - A|\omega(z)|}{1 - B|\omega(z)|} \leqq \left|\frac{z^2 f'(z) + \lambda z^3 f''(z)}{-g(z)g(-z)}\right| \\ &= \left|\frac{z f'(z) + \lambda z^2 f''(z)}{[-g(z)g(-z)]/z}\right| = \left|\frac{zF'(z)}{G(z)}\right| \\ &= \left|\frac{1 + A\omega(z)}{1 + B\omega(z)}\right| \leqq \frac{1 + A|\omega(z)|}{1 + B|\omega(z)|} \leqq \frac{1 + Ar}{1 + Br} \quad (|z| = r < 1), \end{aligned} \tag{5.3}$$

where ω is a Schwarz function with

$$\omega(0)=0 \text{ and } |\omega(z)|<1 \quad (z\in\mathbb{U}).$$

Since

$$G(z)=\frac{-g(z)g(-z)}{z} \quad (z\in\mathbb{U})$$

is an odd starlike function, it is well known that

$$\frac{r}{1+r^2}\leqq |G(z)|\leqq \frac{r}{1-r^2} \quad (|z|=r<1). \tag{5.4}$$

It now follows from (5.3) and (5.4) that

$$\frac{1-Ar}{(1-Br)(1+r^2)}\leqq \left|F'(z)\right|\leqq \frac{1+Ar}{(1+Br)(1-r^2)} \quad (|z|=r<1). \tag{5.5}$$

Upon integrating (5.5) from 0 to r, we have

$$\int_0^r \frac{1-At}{(1-Bt)(1+t^2)}dt\leqq \left|(1-\lambda)f(z)+\lambda zf'(z)\right|\leqq \int_0^r \frac{1+At}{(1+Bt)(1-t^2)}dt. \tag{5.6}$$

To complete the proof we consider the following two cases:

(1) When $\lambda=0$. From (5.6), we easily get (5.1).

(2) When $0<\lambda\leqq 1$. From the proof of Theorem 2.1 together with (5.6), we readily arrive at (5.2).

□

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