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# BITOPOLOGICAL S-CLUSTER SETS

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**Abstract**

A new type of cluster set called an $S$-cluster set of functions and multifunctions between bitopological spaces has been introduced. Expressions and conditions for the degeneracy of such sets are also found. As an application, characterizations of Hausdorff and $S$-closed bitopological spaces are achieved via such cluster sets.

**Keywords:** Bitopological spaces, $ij$ - $S$-cluster sets, $ij$-semi open sets, $ij$ - $S$-closed spaces.

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## 1. Introduction

The theory of cluster sets has been studied and applied for a long time in real and complex analysis. But cluster sets for functions between topological spaces was first initiated by Wetson [15], and many other toplogists like Joseph [4] and Hamlett [3] have made valuable contributions in the area. In bitopological spaces, Nandi and Mukherjee [11] introduced and studied a type of cluster set called a $\delta$-cluster set. In this paper, our intention is to extend a type of cluster set of functions and multifunctions called an $S$-cluster set [10], to the bitopological setting. Using these cluster sets, we give a new characterization of Hausdorff and $S$-closed bitopological spaces.

Throughout the paper, $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ (or briefly, $X$ and $Y$) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let $A$ be a subset of $X$. By $i\text{-}\operatorname{int}(A)$ and $i\text{-}\operatorname{cl}(A)$, we denote respectively the interior and closure of $A$ with respect to $\tau_i$ (or $\sigma_i$) for $i = 1, 2$. Also, $i, j = 1, 2$ and $i \neq j$.

A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called *$ij$-semiopen* [2] if there exists $U \in \tau_i$ such that $U \subset A \subset j\text{-}\operatorname{cl}(U)$. Equivalently, $A$ is $ij$-semiopen if and only if $A \subset j\text{-}\operatorname{cl}(i\text{-}\operatorname{int}(A))$. The complement of an $ij$-semiopen set is called *$ij$-semiclosed*. The family of all $ij$-semiopen (resp. $\tau_i$-open) sets of $X$ containing a given subset $A$ of $X$ is denoted by $ij\text{-}SO(A)$ (resp. $\tau_i(A)$). If $A = \{x\}$, we write $ij\text{-}SO(x)$ and $\tau_i(x)$ instead of $ij\text{-}SO(\{x\})$ and $\tau_i(\{x\})$, respectively.

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A subset $A$ of a space $(X, \tau_1, \tau_2)$ is called *ij-regular open* (resp. *ij-regular closed*) [13] if $A = i\text{-}\operatorname{int}(j\text{-}\operatorname{cl}(A))$ (resp. $A = i\text{-}\operatorname{cl}(j\text{-}\operatorname{int}(A))$). We denote by $ij\text{-}\operatorname{cl}_\theta(A)$ [1] (resp. $ij\text{-}\theta_s\text{-}\operatorname{cl}(A)$ [6]) the set of all points $x$ of $X$ such that for every $\tau_i$-open (resp. $ij$-semiopen) set $U$ containing $x$ we have $j\text{-}\operatorname{cl}(U) \cap A \neq \emptyset$. Also, $A$ is called *$ij$-$\theta$-closed* (resp. *$ij$-$\theta$-semiclosed*) if $A = ij\text{-}\operatorname{cl}_\theta(A)$ (resp. $A = ij\text{-}\theta_s\text{-}\operatorname{cl}(A)$).

A nonempty collection $\Omega$ of nonempty subsets of a space $X$ is called a grill [8], if

(i) $A \in \Omega$ and $B \supset A$ implies $B \in \Omega$, and
(ii) $A \cup B \in \Omega$ implies $A \in \Omega$ or $B \in \Omega$.

A filter base $\Phi$ on $X$ is said to *$ij$-$\theta_s$-accumulate* at $x \in X$ [9], denoted as $x \in ij\text{-}\theta_s\text{-}\operatorname{ad}\Phi$, if $x \in \cap\{ij\text{-}\theta_s\text{-}\operatorname{cl}(F) \colon F \in \Phi\}$.

A grill $\Omega$ on $X$ is said to *$ij$-$\theta_s$-converge* to $x \in X$ if to each $U \in ij\text{-}SO(x)$, there corresponds some $G \in \Omega$ with $G \subset j\text{-}\operatorname{cl}(U)$.

A subset $A$ of a space $X$ is said to be *$ij$-$S$-closed* relative to $X$ if for every cover $\mathbf{U}$ of $A$ by $ij$-semiopen sets of $X$, there exists a finite subfamily $\mathbf{U}_0$ of $\mathbf{U}$ such that $A \subset \cup\{j\text{-}\operatorname{cl}(U) : U \in \mathbf{U}_0\}$. If, in addition, $A = X$, then $X$ is called an *$ij$-$S$-closed space* [9].

A bitopological space $(X, \tau_1, \tau_2)$ is called *pairwise regular* [5] (resp. *ij-almost regular* [13]) if for each $\tau_i$-closed (resp. $ij$-regular closed) set $F \subset X$ and each point $x \in X \setminus F$, there exists a *$\tau_i$-open* set $U$ and a *$\tau_j$-open* set $V$ such that $x \in U$, $F \subset V$ and $U \cap V = \emptyset$.

The space $X$ is called *pairwise Hausdorff* [5] if for every pair of distinct points $x$ and $y$ of $X$, there exist a *$\tau_i$-open* set $U$ and a *$\tau_j$-open* set $V$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

## 2. ij - S-cluster sets

**2.1. Definition.** Let $F : X \to Y$ be a multifunction and $x \in X$. The an $ij$-$S$-cluster set of $F$ at $x$, denoted by $S_{ij}(F, x)$ is defined to be the set $\bigcap\{ji\text{-}\operatorname{cl}_\theta F(j\text{-}\operatorname{cl} U) : U \in ij\text{-}SO(x)\}$. Similarly, for any function $f : X \to Y$, the $ij$-$S$-cluster set $S_{ij}(f, x)$ of $f$ at $x$ is given by $\bigcap\{ij\text{-}\operatorname{cl}_\theta f(j\text{-}\operatorname{cl} U) : U \in ij\text{-}SO(x)\}$.

**2.2. Theorem.** *For any function $f : X \to Y$, the following statements are equivalents:*

(a) $y \in S_{ij}(f, x)$.
(b) *The filterbase $f^{-1}(i\text{-}\operatorname{cl}(\sigma_j(y))$ $ij$-$\theta_s$-accumulates at $x$.*
(c) *There is a grill $\Omega$ on $X$ such that $\Omega$ $ij$-$\theta_s$-converges to $x$ and $y \in \bigcap\{ji\text{-}\operatorname{cl}_\theta f(G) :$ $G \in \Omega\}$.*

*Proof.* (a) $\Longrightarrow$ (b). Let $y \in S_{ij}(f, x)$, then for each $ij$-semiopen set $W$ containing $x$ and each $\sigma_j$-open set $V$ containing $y$, $i\text{-}\operatorname{cl}(V) \cap f(j\text{-}\operatorname{cl} W) \neq \emptyset$. Then for each $W \in ij\text{-}SO(x)$ and $V \in \sigma_j(y)$, $f^{-1}(i\text{-}\operatorname{cl} V) \cap j\text{-}\operatorname{cl} W \neq \emptyset$. This ensures that the collection $\{f^{-1}(i\text{-}\operatorname{cl} V) : V \in \sigma_j(y)\}$ is a filterbase on $X$ which $ij$-$\theta_s$-accumulates at $x$.

(b) $\Longrightarrow$ (c). Let $\Phi$ be the filter generated by the filterbase $f^{-1}(i\text{-}\operatorname{cl}(\sigma_j(y))$. Then $\Omega = \{G \subset X : G \cap F \neq \emptyset$ for each $F \in \Phi\}$ is a grill on $X$. By the hypothesis, for each $U \in ij\text{-}SO(x)$ and each $V \in \sigma_j(y)$, $j\text{-}\operatorname{cl}(U) \cap f^{-1}(i\text{-}\operatorname{cl} V) \neq \emptyset$. Hence, $F \cap j\text{-}\operatorname{cl} U \neq \emptyset$ for each $F \in \Phi$ and each $U \in ij\text{-}SO(x)$. Consequently, $j\text{-}\operatorname{cl} U \in \Omega$ for all $U \in ij\text{-}SO(X)$, which proves that $\Omega$ $ij$-$\theta_s$-converges to $x$. Now the definition of $\Omega$ yields that $f(G) \cap i\text{-}\operatorname{cl} W \neq \emptyset$ for all $W \in \sigma_j(y)$ and all $G \in \Omega$, i.e., $y \in ji\text{-}\operatorname{cl}_\theta f(G)$ for all $G \in \Omega$. Hence $y \in \bigcap\{ji\text{-}\operatorname{cl}_\theta f(G) : G \in \Omega\}$.

(c) $\Longrightarrow$ (a). Let $\Omega$ be a grill on $X$ such that $\Omega$ $ij$-$\theta_s$-converges to $x$ and $y \in \bigcap\{ji\text{-}\operatorname{cl}_\theta f(G) : G \in \Omega\}$. Then $\{j\text{-}\operatorname{cl} U : U \in ij\text{-}SO(x)\} \subset \Omega$ and $y \in ji\text{-}\operatorname{cl}_\theta f(G)$

for each $G \in \Omega$. Hence, in particular $y \in ji\text{-}\mathrm{cl}_\theta f(j\text{-}\mathrm{cl}\,U)$ for all $U \in ij\text{-}SO(x)$. So, $y \in \bigcap\{ji\text{-}\mathrm{cl}_\theta f(ji\text{-}\mathrm{cl}\,U) : U \in ij\text{-}SO(x)\} = S_{ij}(f,x)$. □

**2.3. Theorem.** *Let $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ be a surjection. If $S_{ij}(f,x)$ is degenerate for each $x \in X$, then $Y$ is pairwise Hausdorff.*

*Proof.* Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since $f$ is surjective, there exists $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Now, since $S_{ij}(f,x)$ is degenerate for each $x \in X$, $y_2 = f(x_2) \notin S_{ij}(f,x_1)$. Thus there are $V \in \sigma_j(y_2)$ and $U \in ij\text{-}SO(x_1)$ such that $i\text{-}\mathrm{cl}\,V \cap f(j\text{-}\mathrm{cl}\,U) = \emptyset$, i.e., $f(j\text{-}\mathrm{cl}\,U) \subset Y \setminus i\text{-}\mathrm{cl}\,V$. Then $Y \setminus i\text{-}\mathrm{cl}\,V$ is a $\sigma_i$-open set containing $y_1$, $V$ a $\sigma_j$-open set containing $y_2$ and $V \cap Y \setminus i\text{-}\mathrm{cl}\,V = \emptyset$. This shows that $Y$ is pairwise Hausdorff. □

Now, we introduce the following definition:

**2.4. Definition.** [8]. A function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is called *ij-weakly $\theta$-irresolute* if for each $x \in X$ and each $V \in ij\text{-}SO(f(x))$, there exists $U \in ij\text{-}SO(x)$ such that $f(j\text{-}\mathrm{cl}\,U) \subset V$.

**2.5. Lemma.** [7]. *Let $B$ be a subset of a space $(X,\tau_1,\tau_2)$. If $B$ is $j$-open, then $i\text{-}\mathrm{cl}\,B = ij\text{-}\mathrm{cl}_\theta B$.*

**2.6. Theorem.** *Let $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ be an ij-weakly $\theta$-irresolute function such that $Y$ is pairwise Hausdorff. Then $S_{ij}(f,x)$ is degenerate for each $x \in X$.*

*Proof.* Let $x \in X$. Since $f$ is $ij$-weakly $\theta$-irresolute, for any $V \in ij\text{-}SO(f(x))$, there exists $U \in ij\text{-}SO(x)$ such that $f(j\text{-}\mathrm{cl}\,U) \subset V$. Then

$$\begin{aligned} S_{ij}(f,x) &= \bigcap\{ji\text{-}\mathrm{cl}_\theta f(j\text{-}\mathrm{cl}\,U) : U \in ij\text{-}SO(x)\} \\ &\subset \cap[ji\text{-}\mathrm{cl}_\theta V : V \in ij\text{-}SO(f(x))\}. \end{aligned}$$

Let $y \in Y$ such that $y \neq f(x)$. Since $Y$ is pairwise Hausdorff, there are a $\sigma_j$-open set $U$ and a disjoint $\sigma_i$-open set $W$ such that $y \in U$ and $f(x) \in W$. Obviously, as $U \cap j\text{-}\mathrm{cl}\,W = \emptyset$, $y \notin j\text{-}\mathrm{cl}\,W = ji\text{-}\mathrm{cl}_\theta W$, by Lemma 2.5. Since $W \in \sigma_i(f(x)) \subset ij\text{-}SO(f(x))$, $y \notin \bigcap\{ji\text{-}\mathrm{cl}_\theta V : V \in ij\text{-}SO(f(x))\}$ and hence $y \notin S_{ij}(f,x)$. Thus $S_{ij}(f,x) = \{f(x)\}$, i.e., $S_{ij}(f,x)$ is degenerate. □

Combining the last two results, we get the following characterization for the pairwise Hausdorffness of the codomain of a certain kind of mapping via the degeneracy of its $ij\text{-}S$-cluster set.

**2.7. Corollary.** *Let $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ be an ij-weakly $\theta$-irresolute surjection. Then the space $Y$ is pairwise Hausdorff if and only if $S_{ij}(f,x)$ is degenerate for each $x \in X$.*

We have just seen that the degeneracy of the $ij\text{-}S$-cluster set of an arbitrary function is a sufficient condition for the pairwise Hausdorffness of the codomain space. We thus like to examine some other situations when $ij\text{-}S$-cluster sets are degenerate, thereby ensuring the Hausdorffness of the codomain space of the function concerned. To this end, consider the following lemma and definition:

**2.8. Lemma.** [12]. *In an ij-almost regular space $X$, for each $A \subset X$, $ij\text{-}\mathrm{cl}_\theta A$ is $ij\text{-}\theta$-closed.*

**2.9. Definition.** A function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is called an *$ij\text{-}\theta$-closed function* if $f(F)$ is $ij\text{-}\theta$-closed in $Y$ for each $ij\text{-}\theta$-closed set $F$ of $X$.

**2.10. Theorem.** *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a $ji$-$\theta$-closed function from a $ji$-almost regular space $X$ into a space $Y$. If $f^{-1}(y)$ is $ij$-$\theta$-closed in $X$ for all $y \in Y$, then $S_{ij}(f, x)$ is degenerate for each $x \in X$.*

*Proof.* We have

$$\begin{aligned} S_{ij}(f,x) &= \bigcap\{ji\text{-}\operatorname{cl}_\theta f(j\text{-}\operatorname{cl} U) : U \in ij\text{-}SO(x)\} \\ &\subset \bigcap\{ji\text{-}\operatorname{cl}_\theta f(ji\text{-}\operatorname{cl}_\theta U) : U \in ij\text{-}SO(x)\}. \end{aligned}$$

Since $X$ is $ji$-almost regular, $ji\text{-}\operatorname{cl}_\theta U$ is $ji$-$\theta$-closed for all $U \in ij\text{-}SO(x)$. Now, since $f$ is $ji$-$\theta$-closed and by Lemma 2.8, $ji\text{-}\operatorname{cl}_\theta f(ji\text{-}\operatorname{cl}_\theta U) = f(ji\text{-}\operatorname{cl}_\theta U)$ for each $U \in ij\text{-}SO(x)$. Thus $S_{ij}(f,x) \subset \bigcap\{f(ji\text{-}\operatorname{cl}_\theta U) : U \in ij\text{-}SO(x)\}$. Now, let $y \in Y$ such that $y \neq f(x)$. Then, since $f^{-1}(y)$ is $ij$-$\theta$-closed and $x \notin f^{-1}(y)$, there is some $G \in \tau_i(x)$ such that $j\text{-}\operatorname{cl} G \cap f^{-1}(y) = \emptyset$. So, $y \notin f(j\text{-}\operatorname{cl} G) = f(ji\text{-}\operatorname{cl}_\theta G)$ (as $G$ is $\tau_i$-open) and hence, $y \notin \bigcap\{f(ji\text{-}\operatorname{cl}_\theta U) : U \in ij\text{-}SO(x)\}$. In view of what we have deduced above, we conclude that $y \notin S_{ij}(f,x)$, which proves that $S_{ij}(f,x)$ is degenerate. □

**2.11. Theorem.** *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a $ji$-$\theta$-closed injection on a $ji$-almost regular pairwise Hausdorff space $X$ into $Y$. Then $S_{ij}(f, x)$ is degenerate for each $x \in X$.*

*Proof.* Since $X$ is $ji$-almost regular and $f$ is a $ji$-$\theta$-closed function, by Lemma 2.8, we have $ji\text{-}\operatorname{cl}_\theta f(ji\text{-}\operatorname{cl}_\theta U) = f(ji\text{-}\operatorname{cl}_\theta U)$ for any $U \in ij\text{-}SO(x)$, and hence

$$\begin{aligned} S_{ij}(f,x) &= \bigcap\{ji\text{-}\operatorname{cl}_\theta f(j\text{-}\operatorname{cl} U) : U \in ij\text{-}SO(x)\} \\ &\subset \bigcap\{ji\text{-}\operatorname{cl}_\theta f(ji\text{-}\operatorname{cl}_\theta U) : U \in ij\text{-}SO(x)\} \\ &= \bigcap\{f(ji\text{-}\operatorname{cl}_\theta U) : U \in ij\text{-}SO(x)\}. \end{aligned}$$

For $x, x_1 \in X$ with $x \neq x_1$, $f(x) \neq f(x_1)$, since $f$ is injective. By the pairwise Hausdorffness of $X$, there are a $\tau_i$-open set $U$ and a disjoint $\tau_j$-open set $V$ such that $x_1 \in V$ and $x \in U$. Obviously, $U \cap i\text{-}\operatorname{cl} V = \emptyset$, so $x_1 \notin ji\text{-}\operatorname{cl}_\theta U$ and hence $f(x_1) \notin f(ji\text{-}\operatorname{cl}_\theta U)$. Since $U \in \tau_i(x) \subset ij\text{-}SO(x)$, therefore in view of what we have deduced above, we conclude that $f(x_1) \notin S_{ij}(f,x)$. Thus $S_{ij}(f,x)$ is degenerate for each $x \in X$. □

The above theorem is equivalent to the following apparently weaker result when $X$ is pairwise regular:

**2.12. Theorem.** *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a $ji$-$\theta$-closed injection on a pairwise $T_3$ space $X$ into $Y$. Then $S_{ij}(f, x)$ is degenerate for each $x \in X$ ($X$ is pairwise $T_3$ if it is pairwise $T_1$ and pairwise regular).*

*Proof.* It is well known that in a pairwise regular space, $ji\text{-}\operatorname{cl}_\theta U = j\text{-}\operatorname{cl} U$ for any $U \subset X$. Since $X$ is pairwise $T_3$ and $f$ is a $ji$-$\theta$-closed injection, $\{f(x)\} \subset S_{ij}(f,x) = \bigcap\{f(j\text{-}\operatorname{cl} U) : U \in ij\text{-}SO(x)\} \subset \bigcap\{f(j\text{-}\operatorname{cl} U) : U \in \tau_i(x)\} = \{f(x)\}$. □

A sort of degeneracy condition for the $ij$-$S$-cluster set of a multifunction with an $ij$-$\theta$-closed graph is now obtained.

**2.13. Definition.** [9]. Let $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ be two bitopological spaces. The cross product of the spaces $X$ and $Y$ is defined to be the space $(X \times Y, T_1, T_2)$, where $T_i = \tau_i \times \sigma_j$ is the product topology.

**2.14. Theorem.** *For a multifunction $F : X \to Y$, if $F$ has an $ij$-$\theta$-closed graph, then $S_{ij}(F, x) = F(x)$.*

*Proof.* For any $y \in S_{ij}(F,x)$, $j\text{-}\operatorname{cl} W \cap F(i\text{-}\operatorname{cl} U) \neq \emptyset$ and hence $F^{-}(j\text{-}\operatorname{cl} W) \cap i\text{-}\operatorname{cl} U \neq \emptyset$, for each $U \in ij\text{-}SO(x)$ and each $W \in \sigma_j(y)$, where $F^{-}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$ for any subset $B$ of $Y$. Then for any $i$-open set $M \times N$ in $X \times Y$ containing $(x,y)$, we have $F^{-}(j\text{-}\operatorname{cl} N) \cap i\text{-}\operatorname{cl} M \neq \emptyset$. So, $(i\text{-}\operatorname{cl} M \times j\text{-}\operatorname{cl} N) \cap G(F) \neq \emptyset$ and hence $j$-$\operatorname{cl}(M \times N) \times G(F) \neq \emptyset$, where $G(F) = \{(x,y) \in X \times Y : y \in F(x)\}$ denotes the graph of $F$. Hence $(x,y) \in ij\text{-}\operatorname{cl}_{\theta} F = F$ (as $G(F)$ is $ij$-$\theta$-closed). Therefore $(x,y) \in [G(F) \cap (\{x\} \times Y)]$ so that $y \in \rho_2[(\{x\} \times Y) \cap G(F)] = F(x)$, where $\rho_2 : X \times Y \to Y$ is the second projection map. It is obvious that $F(x) \subset S_{ij}(F,x)$ for each $x \in X$. Hence $S_{ij}(F,x) = F(x)$ holds for all $x \in X$. □

**2.15. Lemma.** *Let $(X \times Y, T_1, T_2)$ denote the cross product of two spaces $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$. Then a subset $W = A \times B$ of $X \times Y$ is $ij$-semiopen in $X \times Y$ if and only if $A$ is an $ij$-semiopen subset of $X$ and $B$ is a $ji$-semiopen subset of $Y$.*

*Proof.* Let $W = A \times B$ where $A$ is an $ij$-semiopen subset of $X$ and $B$ is a $ji$-semiopen subset of $Y$. Then $A \subset \tau_j\text{-}\operatorname{cl}(\tau_i\text{-}\operatorname{int}(A))$ and $B \subset \sigma_i\text{-}\operatorname{cl}(\sigma_j\text{-}\operatorname{int}(B))$. Therefore,

$$\begin{aligned} A \times B &\subset \tau_j\text{-}\operatorname{cl}(\tau_i\text{-}\operatorname{int}(A)) \times \sigma_i\text{-}\operatorname{cl}(\sigma_j\text{-}\operatorname{int}(B)) \\ &= T_j\text{-}\operatorname{cl}(T_i\text{-}\operatorname{int}(A \times B)), \end{aligned}$$

i.e., $W = A \times B$ is $ij$-semiopen in $X \times Y$.

Now let $W = A \times B$ be an $ij$-semiopen set in $X \times Y$. Then

$$\begin{aligned} A \times B = W &\subset T_j\text{-}\operatorname{cl}(T_i\text{-}\operatorname{int}(W)) \\ &= T_j\text{-}\operatorname{cl}(T_i\text{-}\operatorname{int}(A \times B)) \\ &= T_j\text{-}\operatorname{cl}(\tau_i\text{-}\operatorname{int}(A) \times \sigma_j\text{-}\operatorname{int}(B)) \\ &= \tau_j\text{-}\operatorname{cl}(\tau_i\text{-}\operatorname{int}(A)) \times \sigma_i\text{-}\operatorname{cl}(\sigma_j\text{-}\operatorname{int}(B)). \end{aligned}$$

Thus $A \subset \tau_j\text{-}\operatorname{cl}(\tau_i\text{-}\operatorname{int}(A))$ and $B \subset \sigma_i\text{-}\operatorname{cl}(\sigma_j\text{-}\operatorname{int}(B))$. This shows that $A$ is an $ij$-semiopen subset of $X$ and $B$ is a $ji$-semiopen subset of $Y$. □

**2.16. Corollary.** *Let $(X \times Y, T_1, T_2)$ denote the cross product of the two spaces $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$. If $A$ is $ij$-semiopen in $X$ and $B$ is $j$-open in $Y$, then $A \times B$ is $ij$-semiopen in $X \times Y$.*

**2.17. Theorem.** *For a multifunction $F : X \to Y$, if $S_{ij}(F,x) = F(x)$ for each $x \in X$, then the graph $G(F)$ of $F$ is $ij\text{-}\theta$-semiclosed (and hence $ij$-semiclosed).*

*Proof.* Let $(x,y) \in X \times Y \setminus G(F)$. Now, $y \notin F(x) = S_{ij}(F,x)$, there exist some $W \in ij\text{-}SO(x)$ and some $V \in \sigma_j(y)$ such that $i\text{-}\operatorname{cl} V \cap F(j\text{-}\operatorname{cl} W) = \emptyset$. Then $(j\text{-}\operatorname{cl} W \times \text{i}-\operatorname{cl} V) \cap G(F) = \emptyset$ and so $j\text{-}\operatorname{cl}(W \times V) \cap G(F) = \emptyset$. As, by Corollary 2.16, $W \times V$ is an $ij$-semiopen set in $X \times Y$ containing $(x,y)$ and so $(x,y) \notin ij\text{-}\theta_s\text{-}\operatorname{cl} G(F)$. Hence $G(F)$ is $ij\text{-}\theta$-semiclosed. □

We now turn our attention to the characterizations of $ij\text{-}S$-closedness in terms of $ij\text{-}S$-cluster sets. For this purpose, we need the following lemmas.

**2.18. Lemma.** *A subset $A$ of a bitopological space $X$ is $ij\text{-}S$-closed relative to $X$ if and only if for every filterbase $\Phi$ on $X$ with $F \cap C \neq \emptyset$ for all $F \in \Phi$ and $C \in ij\text{-}SO(A)$, we have $A \cap ij\text{-}\theta_s\text{-}\operatorname{ad} \Phi \neq \emptyset$.*

*Proof.* Let $A$ be an $ij\text{-}S$-closed set relative to $X$ and let $\Phi$ be a filterbase on $X$ with the stated property. If possible, suppose that $A \cap ij\text{-}\theta_s\text{-}\operatorname{ad} \Phi = \emptyset$. Then for all $x \in A$, there is an $ij$-semiopen set $V_x$ in $X$ containing $x$ such that $j\text{-}\operatorname{cl}(V_x) \cap F_x = \emptyset$ for some $F_x \in \Phi$. Now $\{V_x : x \in A\}$ is a cover of $A$ by $ij$-semiopen sets of $X$. By the $ij\text{-}S$-closedness of $A$ relative to $X$, there is a finite subset $A^*$ of $A$ such that $A \subset \bigcup\{j\text{-}\operatorname{cl} V_x : x \in A^*\}$.

Choose $F^* \in \Phi$ such that $F^* \subset \bigcap\{F_x : x \in A^*\}$. Then $F^* \cap (\bigcup\{j\text{-}\mathrm{cl}\, V_x : x \in A^*\}) = \emptyset$, i.e., $F^* \cap j\text{-}\mathrm{cl}\,(\bigcup\{V_x : x \in A^*\}) = \emptyset$. Now, as $\bigcup\{V_x : x \in A^*\}$ is an $ij$-semiopen set in $X$, $\bigcup\{j\text{-}\mathrm{cl}\, V_x : x \in A^*\} \in ij\text{-}SO(A)$. This is a contradiction.

Conversely, assume that $A$ is not $ij\text{-}S$-closed relative to $X$. Then for some cover $\{U_\alpha : \alpha \in \Lambda\}$ of $A$ by $ij$-semiopen sets of $X$, $A \not\subset \bigcup\{j\text{-}\mathrm{cl}\, U_\alpha : \alpha \in \Lambda_0\}$ for each finite subset $\Lambda_0$ of $\Lambda$. So,

$$\Phi = \Big\{A \setminus \bigcup \{j\text{-}\mathrm{cl}\, U_\alpha : \alpha \in \Lambda_0\} : \Lambda_0 \text{ is a finite subset of } \Lambda\Big\}$$

is a filterbase on $X$, with $F \cap C \neq \emptyset$ for each $F \in \Phi$ and each $C \in ij\text{-}SO(A)$. But $A \cap ij\text{-}\theta_s\text{-}\mathrm{ad}\, \Phi = \emptyset$. □

**2.19. Lemma.** [9]. *A bitopological space $X$ is $ij\text{-}S$-closed if and only if every filterbase $ij\text{-}\theta_s$-accumulates to a point in $X$.* □

**2.20. Lemma.** *Any $ij\text{-}\theta$-semiclosed subset of an $ij\text{-}S$-closed space $X$ is $ij\text{-}S$-closed relative to $X$.*

*Proof.* Let $K$ be an $ij\text{-}\theta$-semiclosed subset of an $ij\text{-}S$-closed space $X$. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a cover of $K$ by $ji$-regular closed sets of $X$. For each $x \in X \setminus K$, there exists $U_x \in ij\text{-}SO(x)$ such that $j\text{-}\mathrm{cl}\, U_x \subset X \setminus K$. By [9, Lemma 2.12], $j\text{-}\mathrm{cl}\, U_x$ is a $ji$-regular closed set of $X$. Then the family $\{j\text{-}\mathrm{cl}\, U_x : x \in X \setminus K\} \bigcup \{V_\alpha : \alpha \in \Lambda\}$ is a cover of $X$ by $ji$-regular closed sets of $X$. Since $X$ is $ij\text{-}S$-closed, by [9, Theorem 2.14], there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $K \subset \bigcup\{V_\alpha : \alpha \in \Lambda_0\}$. Thus, by [9, Theorem 2.14], $K$ is $ij\text{-}S$-closed relative to $X$. □

**2.21. Definition.** For a function or a multifunction $F : X \to Y$ and a set $A \subset X$, the notation $S_{ij}(F, A)$ stands for the set $\bigcup\{S_{ij}(F, x) : x \in A\}$.

**2.22. Theorem.** *For any bitopological space $(X, \tau_1, \tau_2)$, the following statements are equivalent:*

(a) *$X$ is $ij\text{-}S$-closed.*
(b) *$S_{ij}(F, A) \supset \bigcap\{ji\text{-}\mathrm{cl}_\theta F(U) : U \in ij\text{-}SO(A)\}$, for each $ij\text{-}\theta$-semiclosed subset $A$ of $X$, for each bitopological space $Y$ and each multifunction $F : X \to Y$.*
(c) *$S_{ij}(F, A) \supset \bigcap\{ji\text{-}\theta_s\text{-}\mathrm{cl}\, F(U) : U \in ij\text{-}SO(A)\}$, for each $ij\text{-}\theta$-semiclosed subset $A$ of $X$, for each bitopological space $Y$ and each multifunction $F : X \to Y$.*

*Proof.* (a) $\Longrightarrow$ (b) Let $A$ be any $ij\text{-}\theta$-semiclosed subset of $X$, where $X$ is $ij\text{-}S$-closed. Then by Lemma 2.18, $A$ is $ij\text{-}S$-closed relative to $X$. Now, let $z \in \bigcap\{ji\text{-}\mathrm{cl}_\theta F(W) : W \in ij\text{-}SO(A)\}$. Then for all $W \in \sigma_j(z)$ and for each $U \in ij\text{-}SO(A)$, $i\text{-}\mathrm{cl}\, W \cap F(U) \neq \emptyset$, i.e., $F^-(i\text{-}\mathrm{cl}\, W) \cap U \neq \emptyset$, where $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$ for any subset $B$ of $Y$. Thus $\Phi = \{F^-(i\text{-}\mathrm{cl}\, W) : W \in \sigma_j(z)\}$ is clearly a filterbase on $X$, satisfying the condition of Lemma 2.18. Hence $x \in A \cap ij\text{-}\theta_s\text{-}\mathrm{ad}\, \Phi$. Then $x \in A$ and for all $U \in ij\text{-}SO(x)$ and each $W \in \sigma_j(z)$, $j\text{-}\mathrm{cl}\, U \cap F^-(i\text{-}\mathrm{cl}\, W) \neq \emptyset$, i.e., $F(j\text{-}\mathrm{cl}\, U) \cap i\text{-}\mathrm{cl}\, W \neq \emptyset$. This shows that $z \in S_{ij}(F, x) \subset S_{ij}(F, A)$.

(b) $\Longrightarrow$ (c) Obvious.

(c) $\Longrightarrow$ (a) In order to show that $X$ is $ij\text{-}S$-closed, it is enough to show, by virtue of Lemma 2.20, that every filterbase $\Phi$ on $X$ $ij\text{-}\theta_s$-accumulates at some $x \in X$. Take $y_0 \notin X$ and construct $Y = X \cup \{y_0\}$. Define $\sigma_Y = \{U \subset X : y_0 \notin U\} \bigcup \{U \subset Y : y_0 \in U, F \subset U \text{ for some } F \in \Phi\}$. Then $\sigma_Y$ is a topology on $Y$. Consider the bitopological space $(Y, \sigma_1, \sigma_2)$, where $\sigma_1 = \sigma_2 = \sigma_Y$, and let $I : X \to Y$ be the identity map. In order to avoid possible confusion, let us denote the closure of a set $A$ in $X$ (resp. $Y$) by $\mathrm{cl}_X A$ (resp. $\mathrm{cl}_Y A$). Since $X$ is $ij$-$\theta$-semiclosed in $X$, by the given condition,

$$\begin{aligned} S_{ij}(I,X) &\supset \bigcap\{ji\text{-}\theta_s\text{-}\mathrm{cl}_Y I(U) : U \in ij\text{-}SO(x)\} \\ &= \bigcap\{ji\text{-}\theta_s\text{-}\mathrm{cl}_Y U : U \in ij\text{-}SO(X)\} \\ &= ji\text{-}\theta_s\text{-}\mathrm{cl}_Y X. \end{aligned}$$

We consider $y_0 \in Y$ and $G_0 \in ij\text{-}SO(y_0)$. There is some $W \in \sigma_Y$ such that $W \subset G_0 \subset j\text{-}\mathrm{cl}_Y W$. If $y_0 \notin W$, then $W \subset X$ and hence $j\text{-}\mathrm{cl}_Y W \cap X \neq \emptyset$. If, on the other hand, $y_0 \in W$, then there is some $F \in \Phi$ such that $F \subset W$, i.e., $j\text{-}\mathrm{cl}_Y F \subset j\text{-}\mathrm{cl}_Y W$. So, $X \cap j\text{-}\mathrm{cl}_Y W \neq \emptyset$. So, in any case, $X \cap j\text{-}\mathrm{cl}_Y W \neq \emptyset$ and consequently, as $j\text{-}\mathrm{cl}_Y W = j\text{-}\mathrm{cl}_Y G_0$, $X \cap j\text{-}\mathrm{cl}_Y G_0 \neq \emptyset$. Thus $y_0 \in ij\text{-}\theta_s\text{-}\mathrm{cl}_Y X$, so $y_0 \in S_{ij}(I,x)$ for some $x \in X$.

Consider any $V \in ij\text{-}SO(x)$ and $F \in \Phi$. Then $F \cup \{y_0\} \in \sigma_Y$. Again, $Y \setminus (F \cup \{y_0\})$ is a subset of $Y$ not containing $y_0$. Thus $Y \setminus (F \cup \{y_0\})$ is $\sigma_Y$-open in $Y$, which proves that $j\text{-}\mathrm{cl}_Y(F \cup \{y_0\}) = F \cup \{y_0\}$. Now, $i\text{-}\mathrm{cl}_X V \cap F = I(i\text{-}\mathrm{cl}_X V) \cap j\text{-}\mathrm{cl}_Y(F \cup \{y_0\}) \neq \emptyset$. Thus $x \in ij\text{-}\theta_s\text{-}\mathrm{ad}\,\Phi$. □

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