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JANOWSKI STARLIKE LOG-HARMONIC UNIVALENT FUNCTIONS

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Abstract

Let $h(z)$ and $g(z)$ be analytic functions in the open unit disc $\mathbb{D} = \{z \mid |z| < 1\}$, with the normalization $h(0) = g(0) = 1$. The class of log-harmonic mappings of the form $f = zh(z)\overline{g(z)}$ is denoted by $\mathcal{S}_{lh}$. The aim of this paper is to investigate the class of Janowski starlike log-harmonic mappings, a subclass of the log-harmonic mappings.

Keywords: Log-harmonic univalent functions, Janowski starlike log-harmonic functions, Subordination principle, Distortion theorems.

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1. Introduction

Let $H(D)$ be the the linear space of all analytic functions defined on the unit disc $\mathbb{D}$. A log-harmonic mapping f is a solution of the non-linear elliptic partial differential equation

$$\frac{\overline{f_{\overline{z}}}}{\overline{f}} = w(z)\frac{f_z}{f}, \tag{1.1}$$

where the second dilatation function $w(z) \in H(\mathbb{D})$ is such that $|w(z)| < 1$ for all $z \in \mathbb{D}$. It has been shown that if f is a non-vanishing log-harmonic mapping, then f can be expressed as

$$f = h(z)\overline{g(z)} \tag{1.2}$$

where $h(z)$ and $g(z)$ are analytic functions in $\mathbb{D}$. On the other hand, if f vanishes at $z = 0$ but is not identically zero, then f admits the representation

$$f = z\,|z|^{2\beta}\,h(z)\overline{g(z)}, \tag{1.3}$$

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where $\text{Re}\beta > -1/2$, $h(z)$ and $g(z)$ are analytic functions in $\mathbb{D}$, $g(0) = 1$ and $h(0) \neq 0$, [1]. Let us denote by Ω the family of functions $\phi(z)$ which are regular in $\mathbb{D}$ and satisfy the conditions $\phi(0) = 0$ and $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. For arbitrary fixed real numbers A and B, with $-1 \leq B < A \leq 1$ we use $P(A, B)$ to denote the family of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \tag{1.4}$$

which are regular in $\mathbb{D}$, and such that $p(z)$ is in $P(A, B)$ if and only if

$$p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)} \tag{1.5}$$

for some function $\phi(z)$ and every $z \in \mathbb{D}$. Let $S^*(A, B)$ denote the class of functions $s(z) = z + c_2 z^2 + \cdots$ which are analytic in $\mathbb{D}$, such that $s(z) \in S^*(A, B)$ if and only if

$$z\frac{s'(z)}{s(z)} = p(z) \tag{1.6}$$

for every $z \in \mathbb{D}$ and for some $p(z) \in P(A, B)$ [4].

Let $f = zh(z)\overline{g(z)}$ be a univalent log-harmonic mapping. We say that f is a *Janowski starlike log-harmonic mapping* if

$$\text{Re}\left(\frac{\partial \text{Arg} f(re^{i\theta})}{\partial \theta}\right) = \text{Re}\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right) > \frac{1-A}{1-B} \tag{1.7}$$

for some $p(z)$ in $P(A, B)$ and all $z \in \mathbb{D}$ [4]. We denote by $S^*_{lh}(A, B)$ the set of all Janowski starlike log-harmonic mappings. Further, for analytic functions $S_1(z)$ and $S_2(z)$ in $\mathbb{D}$, $S_1(z)$ is said to be *subordinate* to $S_2(z)$ if there exists $\phi(z) \in \Omega$ such that $S_1(z) = S_2(\phi(z))$ for all $z \in \mathbb{D}$. We denote this subordination by $S_1(z) \prec S_2(z)$. In particular, if $S_2(z)$ is univalent in $\mathbb{D}$, then the subordination $S_1(z) \prec S_2(z)$ is equivalent to $S_1(0) = S_2(0)$ and $S_1(\mathbb{D}) \subset S_2(\mathbb{D})$ [2].

2. Main Results

For the proof of the main theorem, we need the following lemmas which were proved by I. S. Jack [3], Kozuo Kuroki and S. Owa [5], respectively.

2.1. Lemma. *Let $\phi(z)$ be a non-constant function and analytic in $\mathbb{D}$ with $\phi(0) = 0$. If $|\phi(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathbb{D}$, then we have*

$$z_0\phi'(z_0) = k\phi(z_0) \tag{2.1}$$

for some real k with $k \geq 1$. □

2.2. Lemma. *Let $p(z)$ be an element of $P(A, B)$ then*

$$\text{Re}p(z) > \frac{1-A}{1-B} \geq 0. \tag{2.2}$$ □

The following lemma was proved by H. Silverman and E. M. Silvia [6].

2.3. Lemma. *$s(z) \in S^*(A, B)$ if and only if*

$$\left| z\frac{s'(z)}{s(z)} - \frac{1-AB}{1-B^2} \right| < \frac{A-B}{1-B^2}, \ (z \in \mathbb{D}, \ B \neq -1).$$ □

2.4. Theorem. *Let $f = zh(z)\overline{g(z)}$ be a log-harmonic mapping on $\mathbb{D}$ and $0 \notin hg(\mathbb{D})$. If*

$$\frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)} \prec \begin{cases} \dfrac{(A-B)z}{1+Bz} = F_1(z), & B \neq 0; \\ Az = F_2(z), & B = 0; \end{cases} \tag{2.3}$$

then $f \in \mathcal{S}^*_{lh}(A, B)$.

Proof. We define the function

$$\frac{h(z)}{g(z)}=\begin{cases}(1+B\phi(z))^{\frac{A-B}{B}}, & B\neq 0;\\ \mathrm{e}^{A\phi(z)}, & B=0;\end{cases} \tag{2.4}$$

where $(1+B\phi(z))^{\frac{A-B}{B}}$ has the value 1 at $z=0$ (We consider the corresponding Riemann branch). Then $\phi(z)$ is analytic in $\mathbb{D}$ and $\phi(0)=0$. If we take the logarithmic derivative

$$\frac{zh'(z)}{h(z)}-\frac{zg'(z)}{g(z)}=\begin{cases}\dfrac{(A-B)z\phi'(z)}{1+B\phi(z)}, & B\neq 0;\\ Az\phi'(z), & B=0.\end{cases} \tag{2.5}$$

Now it easy to realize that the subordination (2.3) is equivalent to $|w(z)|<1$ for all $z\in\mathbb{D}$. Indeed, assume the contrary. Then there is $z_0\in\mathbb{D}$ such that $|\phi(z_0)|=1$, so by lemma 2.1 $z_0\phi' z_0=k\phi(z_0),\ k\geq 1$, and for such $z_0\in\mathbb{D}$ we have

$$\frac{z_0h'(z_0)}{h(z_0)}-\frac{z_0g'(z_0)}{g(z_0)}=\begin{cases}\dfrac{k(A-B)\phi(z_0)}{1+B\phi(z_0)}=F_1(\phi(z_0))\notin F_1(\mathbb{D}), & B\neq 0;\\ kA\phi(z_0)=F_2(\phi(z_0))\notin F_2(\mathbb{D}), & B=0;\end{cases} \tag{2.6}$$

but this contradicts (2.3); so our assumptions is wrong, i.e, $|w(z)|<1$ for every $z\in\mathbb{D}$. By using Condition (2.3) we get

$$1+\frac{zh'(z)}{h(z)}-\frac{zg'(z)}{g(z)}=\begin{cases}\dfrac{1+A\phi(z)}{1+B\phi(z)}=p(z), & B\neq 0;\\ 1+A\phi(z)=p(z), & B=0;\end{cases} \tag{2.7}$$

and using Lemma 2.2, we have

$$\mathrm{Re}\left(1+\frac{zh'(z)}{h(z)}-\frac{zg'(z)}{g(z)}\right)=\mathrm{Re}p(z)>\frac{1-A}{1-B}. \tag{2.8}$$

On the other hand

$$\begin{aligned} f=zh(z)\overline{g(z)}\implies \frac{zf_z-\overline{z}f_{\overline{z}}}{f}&=1+\frac{zh'(z)}{h(z)}-\frac{\overline{z}\overline{g'(z)}}{\overline{g(z)}}\\ \implies \mathrm{Re}\left(\frac{zf_z-\overline{z}f_{\overline{z}}}{f}\right)&=\mathrm{Re}\left(1+\frac{zh'(z)}{h(z)}-\frac{\overline{z}\overline{g'(z)}}{\overline{g(z)}}\right)\\ &=\mathrm{Re}\left(1+\frac{zh'(z)}{h(z)}-\frac{zg'(z)}{g(z)}\right). \end{aligned} \tag{2.9}$$

Considering the relations (2.7), (2.8), (2.9) and Lemma 2.3 together, we obtain that $f\in\mathcal{S}^*_{lh}(A,B)$. □

The corollary below is a simple consequence of Theorem 2.3. It is known as the *Marx-Strohhacker Inequality of f*.

2.5. Corollary.

$$\begin{cases}\left|\left(\frac{h(z)}{g(z)}\right)^{\frac{B}{A-B}}-1\right|<|B|, & B\neq 0;\\ \left|\log\left(\frac{h(z)}{g(z)}\right)\right|<|A|, & B=0.\end{cases} \tag{2.10}$$

Proof. Using (2.7) we have:

$$\left(\frac{h(z)}{g(z)}\right)^{\frac{B}{A-B}}-1=B\phi(z),\ B\neq 0 \implies \left|\left(\frac{h(z)}{g(z)}\right)^{\frac{B}{A-B}}-1\right|<|B|,\ B\neq 0,$$

$$\log\left(\frac{h(z)}{g(z)}\right)=A\phi(z),\ B=0 \implies \left|\log\left(\frac{h(z)}{g(z)}\right)\right|<|A|,\ B=0.$$

This completes the proof. □

2.6. Theorem. *If $f\in \mathcal{S}^*_{lh}(A,B)$ then*

$$(2.11)\qquad \begin{cases}(1-Br)^{\frac{A-B}{B}}\leq\left|\dfrac{h(z)}{g(z)}\right|\leq(1+Br)^{\frac{A-B}{B}}, & B\neq 0;\\ \mathrm{e}^{-Ar}\leq\left|\dfrac{h(z)}{g(z)}\right|\leq \mathrm{e}^{Ar}, & B=0.\end{cases}$$

Proof. Using Theorem 2.3 we have

$$(2.12)\qquad \begin{cases}\left|\left(z\dfrac{h'(z)}{h(z)}-z\dfrac{g'(z)}{g(z)}\right)-\dfrac{B(B-A)r^2}{1-B^2r^2}\right|\leq\dfrac{(A-B)r}{1-B^2r^2}, & B\neq 0,\\ \left|z\dfrac{h'(z)}{h(z)}-z\dfrac{g'(z)}{g(z)}\right|\leq Ar, & B=0.\end{cases}$$

The inequalities (2.12) can be written in the following form:

$$(2.13)\qquad \begin{cases}\dfrac{-(A-B)}{1+Br}\leq\dfrac{\partial}{\partial r}\log|h(z)|-\dfrac{\partial}{\partial r}\log|g(z)|\leq\dfrac{(A-B)}{1-Br}, & B\neq 0,\\ -A\leq\dfrac{\partial}{\partial r}\log|h(z)|-\dfrac{\partial}{\partial r}\log|g(z)|\leq A, & B=0.\end{cases}$$

Then, after integration we obtain (2.11). □

2.7. Corollary. *If $f=h(z)\overline{g(z)}\in\mathcal{S}^*_{lh}(A,B)$, then*

$$(2.14)\qquad \begin{cases}\dfrac{||b_1|-|a_1|\,r|}{|a_1|-|b_1|\,r}\cdot\dfrac{1}{(1+Br)^{\frac{A-B}{B}}}\leq\left|\dfrac{g'(z)}{h'(z)}\right|\leq\dfrac{|b_1|+|a_1|\,r}{|a_1|+|b_1|\,r}.\dfrac{1}{(1-Br)^{\frac{A-B}{B}}}, & B\neq 0,\\ \dfrac{||b_1|-|a_1|\,r|}{|a_1|-|b_1|\,r}\cdot\mathrm{e}^{-Ar}\leq\left|\dfrac{g'(z)}{h'(z)}\right|\leq\dfrac{|b_1|+|a_1|\,r}{|a_1|+|b_1|\,r}.\mathrm{e}^{-Ar}, & B=0.\end{cases}$$

Proof. Since $f=h(z)\overline{g(z)}$ is the solution of the non-linear elliptic partial differential equation $\dfrac{\overline{f_{\overline{z}}}}{\overline{f}}=w(z).\dfrac{f_z}{f}$, we have

$$(2.15)\qquad w(z)=\frac{\overline{f_{\overline{z}}}.f}{\overline{f}\cdot f_z}=\frac{\dfrac{g'(z)}{h'(z)}}{\dfrac{g(z)}{h(z)}}=\frac{b_1}{a_1}+\frac{1}{a_1}.\left(2b_2-b_1^2-\frac{b_1}{a_1}\cdot(2a_2-a_1^2)\right)z+\cdots.$$

Therefore we define the function

$$(2.16)\qquad \phi(z)=\frac{w(z)-w(0)}{1-\overline{w(0)}w(z)}$$

that satisfies the assumptions of Schwarz's Lemma. Using the estimate of Schwarz's Lemma we have $|\phi(z)|\leq r$, which gives

$$(2.17)\qquad |w(z)-w(0)|\leq r\left|1-\overline{w(0)}\,w(z)\right|.$$

This inequality is equivalent to

$$\left| w(z) - \frac{(1-r^2)\left|\frac{b_1}{a_1}\right|}{1-\left|\frac{b_1}{a_1}\right|^2 r^2} \right| \leq \frac{\left(1-\left|\frac{b_1}{a_1}\right|\right)}{1-\left|\frac{b_1}{a_1}\right|^2 r^2}, \tag{2.18}$$

and equality holds only for the function

$$w(z) = \frac{z + \frac{b_1}{a_1}}{1 + \overline{\left(\frac{b_1}{a_1}\right)} z}. \tag{2.19}$$

From the inequality (2.18) we obtain

$$|w(z)| \geq \left| \frac{(1-r^2)\left|\frac{b_1}{a_1}\right|}{1-\left|\frac{b_1}{a_1}\right|^2 r^2} - \frac{\left(1-\left|\frac{b_1}{a_1}\right|^2\right)}{1-\left|\frac{b_1}{a_1}\right|^2} \right| = \frac{\left|\left|\frac{b_1}{a_1}\right| - r\right|}{1-\left|\frac{b_1}{a_1}\right| r},$$

$$|w(z)| \leq \frac{(1-r^2)\left|\frac{b_1}{a_1}\right|}{1-\left|\frac{b_1}{a_1}\right|^2 r^2} + \frac{\left(1-\left|\frac{b_1}{a_1}\right|^2\right) r}{1-\left|\frac{b_1}{a_1}\right|^2 r^2} = \frac{\left|\left|\frac{b_1}{a_1}\right| + r\right|}{1+\left|\frac{b_1}{a_1}\right| r}$$

Therefore we have

$$\frac{\left|\left|\frac{b_1}{a_1}\right| - r\right|}{1-\left|\frac{b_1}{a_1}\right| r} \leq |w(z)| \leq \frac{\left|\left|\frac{b_1}{a_1}\right| + r\right|}{1+\left|\frac{b_1}{a_1}\right| r}. \tag{2.20}$$

Considering (2.15), (2.18) and Theorem 2.6 together we obtain (2.14). □

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