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σ-REGULAR MATRICES AND A σ-CORE THEOREM FOR DOUBLE SEQUENCES

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Abstract

The famous Knopp Core of a single sequence was extended to the *P-core* of a double sequence by R. F. Patterson. Recently, the *MR-core* and *σ-core* of real bounded double sequences have been introduced and some inequalities analogues to those for Knoop's Core Theorem have been studied. The aim of this paper is to characterize a class of four-dimensional matrices, and so to obtain necessary and sufficient conditions for a new inequality related to the P- and σ-cores.

Keywords: Double sequences, Invariant means, Core theorems and matrix transformations.

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1. Introduction

A double sequence $x = [x_{jk}]_{j,k=0}^{\infty}$ is said to be *convergent to a number l in the sense of Pringsheim*, or to be *P-convergent*, if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$, the set of natural numbers, such that $|x_{jk} - l| < \varepsilon$ whenever $j, k > N$, [11]. In this case, we write $P\text{-}\lim x = l$. In what follows, we will write $[x_{jk}]$ in place of $[x_{jk}]_{j,k=0}^{\infty}$.

A double sequence x is said to be *bounded* if there exists a positive number M such that $|x_{jk}| < M$ for all j, k, i.e.,

$$\|x\| = \sup_{j,k} |x_{jk}| < \infty.$$

We note that in contrast to the case for single sequences, a convergent double sequence need not be bounded. By c_2^{∞}, we mean the space of all P-convergent and bounded double sequences.

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Let $A = [a_{jk}^{mn}]_{j,k=0}^{\infty}$ be a four dimensional infinite matrix of real numbers for all $m, n = 0, 1, \dots$. The sums

$$y_{mn} = \sum_{j}^{\infty} \sum_{k=0}^{\infty} a_{jk}^{mn} x_{jk}$$

are called the *A-transforms* of the double sequence x. We say that a sequence x is *A-summable to the limit l* if the A-transform of x exists for all $m, n = 0, 1, \dots$ and are convergent in the sense of Pringsheim, i.e.,

$$\lim_{p,q\to\infty} \sum_{j=0}^{p} \sum_{k=0}^{q} a_{jk}^{mn} x_{jk} = y_{mn}$$

and

$$\lim_{m,n\to\infty} y_{mn} = l.$$

Moricz and Rhoades [6] have defined the almost convergence of a double sequence as follows:

A double sequence $x = [x_{jk}]$ of real numbers is said to be *almost convergent to a limit l* if

$$\lim_{p,q\to\infty} \sup_{s,t\geq 0} \Big| \frac{1}{pq} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{j+s,k+t} - l \Big| = 0.$$

Note that a convergent single sequence is also almost convergent but for a double sequence this is not the case. That is, a convergent double sequence need not be almost convergent. However, every bounded convergent double sequence is almost convergent. We denote by f_2 the set of all almost convergent and bounded double sequences.

Let σ be a one-to-one mapping from $\mathbb{N}$ into itself. The almost convergence of double sequences has been generalized to the σ-convergence in [2] as follows:

A bounded double sequence $x = [x_{jk}]$ of real numbers is said to be *σ-convergent to a limit l* if

$$\lim_{p,q\to\infty} \sup_{s,t\geq 0} \frac{1}{pq} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{\sigma^j(s),\sigma^k(t)} = l.$$

In this case we write $\sigma\text{-}\lim x = l$. We denote by V_σ^2 the set of all σ-convergent and bounded double sequences.

One can see that in contrast to the case for single sequences, a convergent double sequence need not be σ-convergent. But every bounded convergent double sequence is σ-convergent. So, $c_2^\infty \subset V_\sigma^2$. In the case where $\sigma(i) = i+1$, the σ-convergence of a double sequence reduces to its almost convergence.

Let $B = (b_{nk})$ $(n, k = 1, 2, ...)$ be an infinite matrix of real numbers and $x = (x_k)$ a sequence of real numbers. We write $Bx = ((Bx)_n)$ if $B_n(x) = \sum_k b_{nk} x_k$ converges for each n. Let E and F be any two sequence spaces. If $x \in E$ implies that $Bx \in Y$, then we say that the matrix B maps E into F. We denote by (E, F) the class of matrices B which map E into F. If E and F are equipped with the limits $E - \lim$ and $F - \lim$, respectively, $B \in (E, F)$ and $F - \lim Bx = E - \lim x$ for all $x \in E$, then we write $B \in (E, F)_{\text{reg}}$. The matrix B is then said to be *regular* if $B \in (c,\ c)_{\text{reg}}$. The conditions for regularity are well-known, [3, p.4].

The concept of regularity has been defined for four-dimensional matrices in the same way, (see [4] and [12]). Moricz and Rhoades [6] have determined necessary and sufficient conditions for a four-dimensional matrix A to be strongly regular. In [9], necessary and sufficient conditions have been given for a four dimensional matrix A to belong to the class $(c_2^\infty, f_2)_{\mathrm{reg}}$.

Recall that Knopp's Core of a bounded sequence x is the closed interval $[\liminf x, \limsup x]$, [3, p. 138]. Recently, on analogy with Knopp's Core, the P-core of a double sequence was introduced by Patterson as the closed interval $[-L(-x), L(x)]$, where $-L(-x) = P - \liminf x$ and $L(x) = P - \limsup x$, [10]. Some inequalities related to the these concepts have been studied in [10] and [1].

Let us write

$$L^\star(x) = \limsup_{p,q\to\infty} \sup_{s,t} \frac{1}{pq} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{j+s,k+t}$$

and

$$C_\sigma(x) = \limsup_{p,q\to\infty} \sup_{s,t} \frac{1}{pq} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{\sigma^j(s),\sigma^k(t)}.$$

Then the MR- and σ-core of a double sequence have been introduced as the closed intervals $[-L^*(-x), L^*(x)]$ and $[-C_\sigma(-x), C_\sigma(x)]$, and the inequalities

$$L(Ax) \le L^*(x),\ L^*(Ax) \le L(x),\ L^*(Ax) \le L^*(x),\ L(Ax) \le C_\sigma(x)$$

have also been studied for all $x \in \ell_\infty^2$ in [8], [9], [7] and [2], respectively; where ℓ_∞^2 is the space of all bounded double sequences.

In this paper, we investigate necessary and sufficient conditions for the inequality

$$C_\sigma(Ax) \le L(x) \tag{1.1}$$

for all $x \in \ell_\infty^2$. We should note that in the case $\sigma(i) = i + 1$, the inequality in (1.1) reduces to $L^*(Ax) \le L(x)$.

2. The Main Results

One can expect that in order for (1.1) to be satisfied, first of all $A = [a_{jk}^{mn}]$ must be in the class $(c_2^\infty, V_\sigma^2)_{\mathrm{reg}}$. So, we need to characterize this class of four dimensional matrices. For convenience, a matrix $A \in (c_2^\infty, V_\sigma^2)_{\mathrm{reg}}$ will be called a *σ-regular matrix* in what follows.

2.1. Theorem. *A matrix $A = [a_{jk}^{mn}]$ is σ-regular if and only if*

$$\|A\| = \sup_{m,n} \sum_j \sum_k \left|a_{jk}^{mn}\right| < \infty, \tag{2.1}$$

$$\lim_{p,q\to\infty} \alpha(p,q,j,k,s,t) = 0, \tag{2.2}$$

$$\lim_{p,q\to\infty} \sum_j \sum_k \alpha(p,q,j,k,s,t) = 1, \tag{2.3}$$

$$\lim_{p,q\to\infty} \sum_j \left|\alpha(p,q,j,k,s,t)\right| = 0, (k \in \mathbb{N}), \tag{2.4}$$

$$\lim_{p,q\to\infty} \sum_k \left|\alpha(p,q,j,k,s,t)\right| = 0, (j \in \mathbb{N}), \tag{2.5}$$

$$\lim_{p,q\to\infty} \sum_j \sum_k \left|\alpha(p,q,j,k,s,t)\right| \text{ exists}, \tag{2.6}$$

where the limits are uniform in s, t and

$$\alpha(p,q,j,k,s,t) = \frac{1}{pq} \sum_{m=0}^{p} \sum_{n=0}^{q} a_{jk}^{\sigma^m(s),\sigma^n(t)}.$$

Proof. Firstly, suppose that the conditions (2.1)-(2.6) hold. Take a sequence $x \in c_2^\infty$ with $P-\lim_{j,k} x_{jk} = L$, say. Then, by the definition of P-limit, for any given $\varepsilon > 0$, there exists a $N > 0$ such that $|x_{jk}| < |L| + \varepsilon$ whenever $j,k > N$.

Now, we can write

$$\begin{aligned}\sum_j \sum_k \alpha(p,q,j,k,s,t)x_{jk} = &\sum_{j=0}^{N}\sum_{k=0}^{N} \alpha(p,q,j,k,s,t)x_{jk} \\ &+ \sum_{j=N}^{\infty}\sum_{k=0}^{N-1} \alpha(p,q,j,k,s,t)x_{jk} \\ &+ \sum_{j=0}^{N-1}\sum_{k=N}^{\infty} \alpha(p,q,j,k,s,t)x_{jk} \\ &+ \sum_{j=N+1}^{\infty}\sum_{k=N+1}^{\infty} \alpha(p,q,j,k,s,t)x_{jk}.\end{aligned}$$

Hence,

$$\begin{aligned}\Big|\sum_j \sum_k \alpha(p,q,j,k,s,t)x_{jk}\Big| \leq &\|x\| \sum_{j=0}^{N}\sum_{k=0}^{N} \big|\alpha(p,q,j,k,s,t)\big| \\ &+ \|x\| \sum_{j=N}^{\infty}\sum_{k=0}^{N-1} \big|\alpha(p,q,j,k,s,t)\big| \\ &+ \|x\| \sum_{j=0}^{N-1}\sum_{k=N}^{\infty} \big|\alpha(p,q,j,k,s,t)\big| \\ &+ (|L|+\varepsilon)\Big|\sum_j \sum_k \alpha(p,q,j,k,s,t)\Big|.\end{aligned}$$

Therefore, by letting $p,q \to \infty$ and considering the conditions (2.1)-(2.6), we have

$$\Big|\lim_{p,q\to\infty} \sum_j \sum_k \alpha(p,q,j,k,s,t)x_{jk}\Big| \leq |L| + \varepsilon,$$

i.e., $|\sigma - \lim Ax| \leq |L| + \varepsilon$. Since ε is arbitrary, this implies the σ-regularity of $A = [a_{jk}^{mn}]$.

For the converse, suppose that A is σ-regular. Then, by the definition, the A-transform of x exists and $Ax \in V_\sigma^2$ for each $x \in c_2^\infty$. Therefore, Ax is also bounded. So, there exists a positive number M such that

$$\sup_{m,n} \sum_j \sum_k \big|a_{jk}^{mn} x_{jk}\big| < M < \infty \tag{2.7}$$

for each $x \in c_2^\infty$. Now, let us choose a sequence $y = [y_{jk}]$ with

$$y_{jk} = \begin{cases} \operatorname{sgn} a_{jk}^{mn}, & 0 \leq j \leq r,\ 0 \leq k \leq r, \\ 0, & \text{otherwise.} \end{cases} \quad (m,n = 1,2,\ldots).$$

Then, the necessity of the condition (2.1) follows by considering the sequence y in (2.7).

For the necessity of (2.6), define a sequence $v = [v_{jk}]$ by $y = [y_{jk}]$, with $\alpha(p,q,j,k,s,t)$ in place of a_{jk}^{mn}. Then, $P - \lim Av$ implies (2.6).

Let us define the sequence e^{il} as follows:

$$e_{jk}^{il} = \begin{cases} 1, & \text{if } (j,k) = (i,l), \\ 0, & \text{otherwise;} \end{cases} \tag{2.8}$$

and denote the pointwise sums by $s^l = \sum_i e^{il}$ $(l \in \mathbb{N})$ and $r^i = \sum_l e^{il}$ $(i \in \mathbb{N})$. Then, the necessity of the condition (2.2) follows from $\sigma - \lim Ae^{il}$. Also,

$$\sigma - \lim Ar^j = \lim_{p,q\to\infty} \sum_j |\alpha(p,q,j,k,s,t)| = 0, \ (k \in \mathbb{N})$$

and

$$\sigma - \lim As^k = \lim_{p,q\to\infty} \sum_k |\alpha(p,q,j,k,s,t)| = 0, \ (j \in \mathbb{N}).$$

To verify the conditions (2.4) and (2.5), we need to prove that these limits are uniform in s,t. So, let us suppose that (2.5) does not hold, i.e., for any $j_o \in \mathbb{N}$,

$$\lim_{p,q} \sup_{s,t} \sum_k |\alpha(p,q,j_0,k,s,t)| \neq 0.$$

Then, there exists an $\varepsilon > 0$ and index sequences $(p_i),(q_i)$ such that

$$\sup_{s,t} \sum_k |\alpha(p_i,q_i,j_0,k,s,t)| \geq \varepsilon \quad (i \in \mathbb{N}).$$

Therefore, for every $i \in \mathbb{N}$, we can choose $s_i, t_i \in \mathbb{N}$ such that

$$\sum_k |\alpha(p_i,q_i,j_0,k,s_i,t_i)| \geq \varepsilon.$$

Since

$$\sum_k |\alpha(p_i,q_i,j_0,k,s_i,t_i)| \leq \sup_{m,n} \sum_{j,k} |a_{jk}^{mn}| < \infty,$$

and (2.2) holds, we may find an index sequence (k_i) such that

$$\sum_{k=1}^{k_i} |\alpha(p_i,q_i,j_0,k,s_i,t_i)| \leq \frac{\varepsilon}{8}$$

and

$$\sum_{k=k_{i+1}+1}^{\infty} |\alpha(p_i,q_i,j_0,k,s_i,t_i)| \leq \frac{\varepsilon}{8}, \quad (i \in \mathbb{N}).$$

So,

$$\sum_{k=k_i+1}^{k_{i+1}} |\alpha(p_i,q_i,j_0,k,s_i,t_i)| \geq \frac{3\varepsilon}{4}, \quad (i \in \mathbb{N}).$$

Now, define a sequence $x = [x_{jk}]$ by

$$x_{jk} = \begin{cases} (-1)^i \alpha(p_i,q_i,j_0,k,s_i,t_i), & \text{if } k_i + 1 \leq k \leq k_{i+1} \ (i \in \mathbb{N}); j = j_0, \\ 0, & \text{if } j \neq j_0. \end{cases}$$

Then, clearly $x \in c_2^\infty$ with $\|x\|_\infty \leq 1$. But, for even i, we have

$$\begin{aligned}\frac{1}{p_i q_i} \sum_{m=s_i}^{s_i+p_i-1} \sum_{n=t_i}^{t_i+q_i-1} (Ax)_{mn} &= \sum_k \alpha(p_i, q_i, j_0, k, s_i, t_i) x_{j_0 k} \\ &\geq \sum_{k=k_i+1}^{k_{i+1}} \alpha(p_i, q_i, j_0, k, s_i, t_i) x_{j_0 k} \\ &\quad - \sum_{k=1}^{k_i} |\alpha(p_i, q_i, j_0, k, s_i, t_i)| \\ &\quad - \sum_{k=k_{i+1}+1}^{\infty} |\alpha(p_i, q_i, j_0, k, s_i, t_i)| \\ &\geq \sum_{k=k_i+1}^{k_{i+1}} |\alpha(p_i, q_i, j_0, k, s_i, t_i)| - \frac{\varepsilon}{8} - \frac{\varepsilon}{8} \\ &\geq \frac{3\varepsilon}{4} - \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.\end{aligned}$$

Analogously, for odd i, one can show that

$$\frac{1}{p_i q_i} \sum_{m=s_i}^{s_i+p_i-1} \sum_{n=t_i}^{t_i+q_i-1} (Ax)_{mn} \leq -\frac{\varepsilon}{2}.$$

Hence, the sequence

$$\left(\frac{1}{pq} \sum_{m=s}^{s+p-1} \sum_{n=t}^{t+q-1} (Ax)_{mn} \right)$$

does not converge uniformly in $s, t \in \mathbb{N}$ as $p, q \to \infty$. This means that $Ax \notin V_\sigma^2$, which is a contradiction. So, (2.5) holds. In the same way, we get the necessity of (2.4).

On the other hand, for the necessity of the condition (2.3) it is enough to take the sequence $e_{jk} = 1$ for each j, k.

This completes the proof of the theorem. □

We should mention that in the case $\sigma(i) = i+1$, Theorem 2.1 gives a characterization of the class $(c_2^\infty, f_2)_{\text{reg}}$.

Now, we are ready to formulate our main theorem.

2.2. Theorem. *The inequality in* (1.1) *holds for all* $x \in \ell_\infty^2$ *if and only if the matrix* $A = [a_{jk}^{mn}]$ *is* σ*-regular and*

$$\limsup_{p,q\to\infty} \sup_{s,t} \sum_j \sum_k \left|\alpha(p, q, j, k, s, t)\right| \leq 1. \tag{2.9}$$

Proof. Firstly, let (1.1) hold for all $x \in \ell_\infty^2$. Then, since $c_2^\infty \subset \ell_\infty^2$, (1.1) also holds for any convergent sequence $x = [x_{jk}]$ with $\lim_{j,k} x_{jk} = L$, say. In this case, since $-L(-x) = L(x) = \lim_{j,k} x_{jk} = L$, by (1.1) one has that

$$L = -L(-x) \leq -C_\sigma(-Ax) \leq C_\sigma(Ax) \leq L(x) = L, \tag{2.10}$$

where

$$-C_\sigma(-Ax) = \liminf_{p,q\to\infty} \sup_{s,t} \sum_j \sum_k \alpha(p, q, j, k, s, t) x_{jk}.$$

Therefore, it follows from (2.10) that $-C_\sigma(-Ax) = C_\sigma(Ax) = \sigma - \lim Ax = L$, which gives the σ-regularity of A.

To show the necessity of (2.9) we note first that, by Patterson [10, Lemma 3.1], there exists a $y \in \ell^2_\infty$ with $\|y\| \leq 1$ such that

$$C_\sigma(Ay) = \limsup_{p,q\to\infty} \sup_{s,t} \sum_j \sum_k \big|\alpha(p,q,j,k,s,t)\big|.$$

Now, let us consider the sequence e^{il} defined by (2.8). Then, since $\|e^{il}\| \leq 1$, we have from (1.1) that

$$C_\sigma(Ae^{il}) = \limsup_{p,q\to\infty} \sup_{s,t} \sum_j \sum_k \big|\alpha(p,q,j,k,s,t)\big| \leq L(e^{il}) \leq \|e^{il}\| \leq 1,$$

which is the condition (2.9).

Conversely, suppose that A is σ-regular and (2.9) holds. Let $x = [x_{jk}]$ be an arbitrary bounded sequence. Then, for any $\varepsilon > 0$, there exists $M, N > 0$ such that $x_{jk} \leq L(x) + \varepsilon$ whenever $j, k \geq M, N$.

Now, we can write

$$\begin{aligned}
\sum_j \sum_k \alpha(p,q,j,k,s,t)x_{jk} &\leq \Bigg| \sum_j \sum_k \Bigg(\frac{\big|\alpha(p,q,j,k,s,t)\big| + \alpha(p,q,j,k,s,t)}{2} \\
&\qquad + \frac{\big|\alpha(p,q,j,k,s,t)\big| - \alpha(p,q,j,k,s,t)}{2} \Bigg) x_{jk} \Bigg| \\
&\leq \|x\| \sum_{j=0}^{M} \sum_{k=0}^{N} \big|\alpha(p,q,j,k,s,t)\big| \\
&\quad + \Bigg| \sum_{j=M+1}^{\infty} \sum_{k=N+1}^{\infty} \alpha(p,q,j,k,s,t)x_{jk} \Bigg| \\
&\quad + \|x\| \sum_j \sum_k \big(\big|\alpha(p,q,j,k,s,t)\big| - \alpha(p,q,j,k,s,t) \big) \\
&\leq \|x\| \sum_{j=0}^{M} \sum_{k=0}^{N} \big|\alpha(p,q,j,k,s,t)\big| \\
&\quad + (L(x) + \varepsilon) \sum_j \sum_k \big|\alpha(p,q,j,k,s,t)\big| \\
&\quad + \|x\| \sum_j \sum_k \big(\big|\alpha(p,q,j,k,s,t)\big| - \alpha(p,q,j,k,s,t) \big).
\end{aligned}$$

Applying the operator $\limsup_{p,q\to\infty} \sup_{s,t}$ and taking the conditions into consideration, we get that $C_\sigma(Ax) \leq L(x) + \varepsilon$, which is the inequality in (1.1) since ε is arbitrary. □

Here, we should note that our Theorem 2.2 is an extension of [5, Theorem 2] to the double sequences.

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